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ADAPTIVE OPTIMAL ESTIMATION CONTROL STRATEGIES FOR SYSTEMS OF SIMULTANEOUS EQUATIONS

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Abstract—The choice of control strategies to improve estimation of the parameters in a model of a simultaneous equations system with time-varying parameters is considered. Open-loop feedback (OLF) sequential procedures for handling nonlinear restrictions on reduced form parameters implied by the structural form are suggested, and the combination of sequential estimation and design control strategies feature a marked improvement in the behavior of estimates over the nonsequential [open-loop (OL)] formulation. The maximum accuracy control problem considered in this paper can also be treated as an initial phase of a forecasting and/or stochastic control problem. This will avoid solution to a more difficult problem such as the dual control problem.

1. INTRODUCTION

It seems a little time ago that economists viewed experimentation as a tool available to physical scientists and psychologists, educators and sociologists, but not to them. Economists are beginning to see, however, that experiences generated from simple controlled settings can be used as criteria for determining the relative acceptability of general theories and related models of complex economic systems. Although there still are economists who think that experimental methods are in principle not applicable in economics, controlled experimentation in economics is becoming more and more common and scientific thinking is shifting to a qualified acknowledgment that the experimental methods are applicable when the economic problems are carefully defined.

Several large-scale experiments in negative income taxation have been conducted, including the New Jersey–Pennsylvania (Orcutt and Orcutt[1], Orr[2], Conlick and Watts[3], Watts[4, 5], Kershaw and Fair[6] and Watts and Rees[7, 8]), Rural (Bawden[9, 10]); Gary (Kelly and Singer[11]) and Seattle and Denver Income Maintenance Experiments (Kurz and Spiegelman[12, 13]).

Other examples of real-world experiments, attempting to measure responsiveness to various types of economic incentive programs, are (1) the housing demand experiment, which was designed to find out how household expenditures for housing were related to various forms and levels of housing allowance (Abt Associates[14]); (2) the health insurance experiments, whose concern centered on finding out how individual use of medical care relates to the coinsurance and deductible features of health insurance policies (Newhouse[15]) and (3) the peak-load pricing of electricity experiments, whose principal goal was (or is) the measurement of residential customer responsiveness to charging higher rates during hours and seasons of higher demand (Wenders and Taylor[16], Ferbert and Hirsh[17], Manning and others[18] and Aigner and Hausman[19]). In the environmental area, the experimental method is being explored as a tool to elicit individual preferences about environmental variables (Brookshire and others[20]), and experiments on the effectiveness of various pollution taxes could now be laying the foundation of new antipollution laws (Papakyriazis[21]). Finally, game experiments (Rousseas and Hart[22], Yaari[23], MacCrimmon and Toda[24], Friedman[25–27], Shubik[28], Smith[29–33], Carson[34], Dolbear and others[35], Frahm and Schrander[36], Fisher[37] and *Review of Economic Studies*[38]) and computer simulation experiments (Naylor, Balintfy, Burdick and Chu[39], Naylor, Burdick and Sasser[40] and Naylor[41, 42]) are other economic contexts in which experimentation is applicable.

A properly designed and executed social experiment can provide the strongest evidence that a certain intervention (policy action) actually causes or, if implemented, would cause, a given result. The great cost, size and administrative complexity of such experiments, however, make them different from the classical experiments in physical sciences, agriculture or psychology where they were developed for simpler situations. Thus the problems of controlled experimentation in economics provide an opportunity for economists to extend design theory to handle peculiarities of economic experimentation and hence have them contribute to the literature of experimental design.

When the purpose of the experiment is to generate data for estimation of a behavioral relation in the context of a linear regression model where the experimenter determines the settings of the control variables, and then observations are made on a single response or dependent variable, the analysis of experimental design is straightforward and has been discussed by Kiefer[43, 44], Conlick and Watts[3] and Aigner[45]. Also the book by Fedorov[46] summarizes many of the key results. The first systematic attempt at obtaining an optimal design outside the context of linear regression model seems to have been that of Box and Jenkins ([47], pp. 416–420). Recently, the Box–Jenkins approach has been utilized by Papakyriazis[48] to obtain optimal design processes in terms of certain functions of their moments for various time series models.

Extension of the design theory to the multivariate case, where the experimenter is faced with more than one dependent variable or response for each experimental setting, has been discussed in Conlisk and Watts[3] and Papakyriazis[49]. For example, in the New Jersey negative tax experiment, the experimenter might observe work response, savings response and different types of expenditures (spending on food, durables, travel, entertainment and so on). Similarly, in a study of the effectiveness of advertising on sales, the experimenter might observe demand for good one, demand for good two, and so on. It is not difficult conceptually, although the method suggested requires a guessed value for the variance–covariance matrix of the disturbance terms.

In designing experiments in the context of multiple response models, emphasis is placed on the overall effect of treatments on the response of dependent variables. In many experimental situations, however, especially in economics, apart from the direct effects of the control variables on the response variables, there is the question of indirect or structural effects (the simultaneous equations problem). A primary source of potential simultaneity in the context of the widely discussed New Jersey negative tax experiment, for example, lies in the fact that interrelations of labor supply decisions among family members can affect the labor supply quantities. In particular, suppose the objective of the experiment is to estimate the aggregate (family) labor supply response of a negative income tax. The main reason for interest in family labor supply response is that, in reality, any negative income tax or similar income subsidy is likely to be defined in terms of family income and need. The Federal Income Tax essentially applies to the family as a unit, and certainly the experimental treatments in the New Jersey negative income tax experiment were so defined. This is a simultaneous equations design problem, since there are theoretical reasons for expecting the labor supply choices of family members to be a simultaneous decision rather than a collection of independent choices. Another source of potential simultaneity in the labor supply model involves the wage variable. In particular, it is likely that the amount of time offered by a worker will partly determine the wage the worker receives. Full-time workers, for example, may be able to command a higher wage than part-time workers. Hence, both the decision about the wage earned and the decision about hours of work may be simultaneously determined.

Finally, simultaneous equations experimental design may be exemplified also by pollution control and monitoring in the environmental area, where the purpose of a (pollution producing) firm (electric power plant, for example) is to discover the functional relation between the various amounts of pollution it generates and the level of different control techniques. In particular, suppose that a model for pollution control exists that is capable of describing atmospheric diffusion and chemical reaction, thus relating the various amounts of pollution (measured in terms of concentrations at certain monitoring points) to the level of control techniques and realizations of stochastic noise sequences denoting the effects of uncontrollable exogenous factors such as climate conditions and geography. There is one equation for each monitored point and pollution type existing at that point. Although it seems reasonable to assume that the different types of pollution are uncoupled from one receptor to another, local attenuation of pollution due to chemical reactions and diffusion as well as description of the movement of pollutants from one receptor location to another should be allowed.

The above situations, and undoubtedly many others, represent typical situations in which the design for simultaneous equations is important. The purpose of this paper is to extend the design theory in the context of simultaneous equations and hence help remedy deficiencies of existing design theory.

The design theory for estimation of structural effects has been considered only by Conlisk[50] and Papakyriazis[51, 52]. Conlisk[50] has discussed the problem of designing experiments for parameter estimation in a structural regression model; his proposed criterion is based on the three-stage least-squares/full information maximum-likelihood (3SLS/FIML) asymptotic variance–covariance matrix. But, as Conlisk has pointed out, the optimal design cannot be derived, because the design criterion function depends upon the unknown parameters to be estimated. A possible approach that can be used to gain insight into the experimental design problem is to assume knowledge of the true parameters. However, in practice, the true parameter values are unknown and "prior beliefs" are therefore used. But one may be hesitant to design all observations on the basis of questionable "prior beliefs." The need for prior beliefs of the unknown parameters is of less concern when a sequential approach is feasible. Then we can improve the design as we find out more about the unknown parameters. Papakyriazis[51, 52] formulates the single structural equation design problem adaptively, and thus helps resolve the above conceptual difficulty.

The choice of estimation control strategies which is optimal from the viewpoint of estimating our structural equation, however, may not be optimal from the viewpoint of estimating other structural equations. Extensions of the design theory to cover such complications would be useful. This paper is concerned with extension of the single equation sequential design theory of Papakyriazis[51, 52] to the whole system of structural equations with time-varying parameters.

The organization of the paper is as follows: In Section 2 a brief statement of the experimental design problem in the context of a model of a simultaneous equations system is presented. Extensions of the design theory to the simultaneous equations econometric models follow in Section 3, and the paper ends with some concluding remarks.

2. PROBLEM STATEMENT

Let

$$y_{ti} = \mathbf{y}_{ti}\boldsymbol{\beta}_{\cdot i} + \mathbf{x}_{ti}\boldsymbol{\gamma}_{\cdot i} + \epsilon_{ti}$$
$$= \mathbf{x}^0_{ti}\boldsymbol{\delta}_{\cdot i} + \epsilon_{ti} \qquad (i = 1, 2, \ldots, N; t = 1, 2, \ldots, T) \tag{1}$$

be the ith of N structural equations in a simultaneous equations model where $\mathbf{y}_{ti}$ is an $(1 \times N_i)$ vector of endogenous variables whose elements are endogenous variables other than y_{ti}; $\mathbf{x}_{ti}$ is a $(1 \times K_i)$ subvector of the $(1 \times K)$ vector of exogenous variables $\mathbf{x}_t$; $\boldsymbol{\beta}_{\cdot i}$ and $\boldsymbol{\gamma}_{\cdot i}$ are $(N_i \times 1)$ and $(K_i \times 1)$ vectors of parameters, respectively; ϵ_{ti} is a zero mean error term; and $\mathbf{x}^0_{ti} = (\mathbf{y}_{ti}, \mathbf{x}_{ti})$, $\boldsymbol{\delta}_{\cdot i} = (\boldsymbol{\beta}'_{\cdot i}, \boldsymbol{\gamma}'_{\cdot i})'$. Assume the absence of serial correlation in the sense that, if $\boldsymbol{\epsilon}_t = (\epsilon_{t1} \cdots \epsilon_{tN})'$, then $E(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}'_{t'}) = \boldsymbol{\Omega}_t\delta_{tt'}$ where $\delta_{tt'}$ is the Kronecker delta. Identifiability of all equations is assumed. Further, let

$$\mathbf{y}_{ti} = \mathbf{x}_t\Pi_i + \mathbf{v}_{ti} \qquad (i = 1, 2, \ldots, N; \quad t = 1, 2, \ldots, T), \tag{2}$$

where Π_i is the $(K \times N_i)$ reduced form coefficients matrix and $\mathbf{v}_{ti}$ is an $(1 \times N_i)$ reduced form disturbances.

In this paper we investigate the class of control strategies that are optimal for estimation, where optimality is defined in some appropriate sense. Even though no control aspect is included in the analysis, the estimation control problem can be treated as an initial phase of a general control problem. Before the control policies are implemented, a fast and efficient estimation phase, by means of carefully designing experiments, is preceded to determine the unknown structural parameters up to the desirable accuracy and the complete learning will eventually take place during the control phase. Such a procedure will avoid solution to a costly and time-consuming dual control problem.

Once the structural form simultaneous equations model (1) has been selected and the experimental variables $\mathbf{x}_t$ may be set by the experimenter subject to constraints dictated by the nature of the experiment (such as sample size constraints, budget constraints and so on), the general experimental design problem discussed in this paper can be stated as follows: Find $\{\mathbf{x}_t\}_{t=1}^T$ such that a suitable criterion function related to the (asymptotic) variance–covariance matrix of the structural parameters is optimized. In the sequel, in order to use generally accepted notions of optimality, the trace criterion will be used for optimal experimental design. The trace criterion is discussed, along with the determinant and other (classical and Bayesian) criteria, in Papakyriazis[48, 51].

3. EXTENSION OF ESTIMATION CONTROL THEORY TO THE SIMULTANEOUS EQUATIONS CASE

3.1. *Prior information problem*

A variety of techniques are available for estimating the parameters in models of simultaneous equations systems. However, the achievable estimation accuracy for all methods is a function of the experimental conditions which are chosen so that the information provided by the experiment is maximal in some sense. This paper is concerned with the problem of designing the experimental conditions for maximum estimation accuracy.

For the most part, those interested in using the results of large-scale public policy experiments plan to use them as parameters in models which have broad policy implications. This means, of course, that the design of experiments itself must be developed in a manner which reflects the underlying economic theory. Thus any systematic design procedure for precise parameter estimation in the context of an econometric model in general, and in the simultaneous equations context in particular, requires a heavy dose of prior information. The requirement that the experimenter knows something ahead of time above the very phenomenon (model) which it is the purpose of the experiment to discover is a vexing problem, but a common one in design contexts (see, for example, Box and Lucas[53], Box and Jenkins[47] (pp. 416–420), Conlisk[50, 54], Papakyriazis[48] and Aigner[45]).

At the heart of the matter of designing experiments in the context of a simultaneous equation model is functional form specification. Note that the problem here is one of specifying *a priori* the appropriate structural form for the equations of a model. As Conlisk[54] and Aigner[55] demonstrate in the context of a single equation regression model, significant efficiencies of design can be obtained if the model intended for ultimate use during the data analysis phase is used at the experimental stage. On the other hand, the efficiency cost of designing for one model when some other is the "true" one may be very high. It is apparent that in a public policy experiment the potential benefits and risks of exploiting prior structural form specification in the sample design are related to both the internal and external validity of the experiment. Reducing the risk associated with incorrectly specifying the structural form in the experimental design phase (by means of noninformative analysis of covariance structure, perhaps), may increase the risk of incorrect extrapolation to the nonexperimental situation by reducing the information needed by our methodological strategy for extrapolating the results. Hence, designing an experiment under the assumption that no prior knowledge of structural relationships exists introduces the risk of losing information in the event some specific structure is correct in exactly the same way that designing an experiment according to an "inappropriate" criterion function, for example, introduces risk. It is apparent, therefore, that a sensitive compromise among risks must be struck. Various approaches for making the compromise have been suggested. In particular, Conlisk[54] has proposed three such approaches, including a decision theoretic one in which the experimenter assigns probabilities to functional forms and then minimizes expected loss. Presumably this information is often available. Information from other studies, for example, might be available. If it is not, a sequential approach is a possibility worth further investigation.

Once an appropriate model is selected, the objective of the experiment changes. In particular, the experimenter now wishes to estimate the model parameters according to an experimental design he is free to choose. It is the main purpose of this paper to show how, in the context of a simultaneous equation model, the experimental conditions for maximum estimation accuracy can be obtained.

A second example of prior information in the context of a simultaneous equation model is knowledge of the model's true parameters. In particular, in the usual regression case, the choice of optimal design does not depend on the values of the unknown parameters, and the independent variables can be set to the values required by the design without knowledge of the unknown values being estimated. The determination of optimal designs for simultaneous equations models, however, is more similar to the nonlinear estimation case than to the classical linear regression, and the parameters also appear in the solution to the design problem. Thus the optimal design cannot be implemented without knowledge of the model parameters. On the other hand, designing under a wrong assumption about parameter values can lead to a disastrous design (efficiency-wise).

Intuitively, the need for prior knowledge of the unknown parameters in the context of the simultaneous equations model (1) arises from the fact that to solve the design problem, the experimenter must specify the values of Π_i ($i = 1, 2, \ldots, N$), which depend on the structural parameters and Ω_{ϵ_i}. In particular, in a typical structural equation, the dependent variables $\mathbf{y}_i$ cannot be controlled directly, but only indirectly by controlling the exogenous variables. Since the unknown parameters Π_i determine how the exogenous variables influence the endogenous variables $\mathbf{y}_i$, it is not possible to get the optimal relationships among the regressors unless Π_i is known. Finally, the choice of the design variables, $\{\mathbf{x}_t\}_{t=1}^T$, which is optimal from the point of view of estimating a particular structural equation may not be optimal from the point of view of estimating other structural equations. It is in compromising these conflicting goals that the contemporaneous variance–covariance matrix Ω_{ϵ_t} enters.

Since the efficiency cost of designing under a wrong assumption about parameter values may be very high, a sequential approach, where we improve the design as we find more about the parameters, immediately suggests itself. In this paper we formulate the simultaneous equations design problems sequentially, and hence help resolve the "unknown parameters" conceptual difficulty. But first we formulate the experimental design problem as an optimal control problem.

3.2. *Open-loop (OL) estimation control*

Combining (1) and (2) we obtain

$$\begin{aligned} y_{ti} &= \mathbf{x}_t \Pi_i \boldsymbol{\beta}_{\cdot i} + \mathbf{x}_{ti} \boldsymbol{\gamma}_{\cdot i} + v_{ti}^0 \\ &= \mathbf{x}_t H_i \boldsymbol{\delta}_{\cdot i} + v_{ti}^0, \end{aligned} \tag{3}$$

$$\mathbf{y}_{ti} = \mathbf{x}_t \Pi_i + \mathbf{v}_{ti} \qquad (i = 1, 2, \ldots, N; \quad t = 1, 2, \ldots, T), \tag{4}$$

where $H_i = (\Pi_i : J_i)$; J_i is a matrix of unit vectors such that $\mathbf{x}_{ti} = \mathbf{x}_t J_i$; and v_{ti}^0 is the error term in the reduced form equation for y_{ti} (i.e. $v_{ti}^0 = \epsilon_{ti} + \mathbf{v}_{ti}\boldsymbol{\beta}_{\cdot i}$). In a more compact form (3) and (4) can be written as follows:

$$\mathbf{Y}_t = (I \otimes \mathbf{x}_t)\mathbf{H}_t \boldsymbol{\delta}_t + \mathbf{v}_t^0, \tag{5}$$

$$\mathbf{Y}_t = (I \otimes \mathbf{x}_t)\boldsymbol{\Pi}_t + \mathbf{v}_t, \tag{6}$$

where $\mathbf{Y}_t = (y_{t1} y_{t2} \cdots y_{tN})'$, $H_t = \mathrm{diag}(H_{t1}, H_{t2} \cdots H_{tN})$, $\boldsymbol{\delta}_t = (\boldsymbol{\delta}'_{\cdot 1t} \boldsymbol{\delta}'_{\cdot 2t} \cdots \boldsymbol{\delta}'_{\cdot NT})'$, $\mathbf{v}_t^0 = (v_{t1}^0 v_{t2}^0 \cdots v_{tN}^0)'$, $\boldsymbol{\Pi}_t = (\boldsymbol{\Pi}'_{\cdot t1} \boldsymbol{\Pi}'_{\cdot t2} \cdots \boldsymbol{\Pi}'_{\cdot tN})'$, $\boldsymbol{\Pi}_{\cdot ti} = (\Pi_{ti1} \Pi_{ti2} \cdots \Pi_{tiK})$, $\mathbf{v}_t = (v_{t1} v_{t1} \cdots v_{tN})'$ and $(I \otimes \mathbf{x}_t)$ is the Kronecker product of the identity matrix I, and the vector of design variables, $\mathbf{x}_t$. Alternatively, the systems corresponding to (5) and (6) can be written as

$$\mathbf{Y}_t = H_{(SF),t} \boldsymbol{\delta}_t + \mathbf{v}_t^0, \tag{7}$$

$$\mathbf{Y}_t = H_{(RF),t} \boldsymbol{\Pi}_t + \mathbf{v}_t, \tag{8}$$

where $H_{(SF),t} = (I \otimes \mathbf{x}_t) H_t$ and $H_{(RF),t} = (I \otimes \mathbf{x}_t)$. Finally,

$$E(\mathbf{v}_t^0) = E(\mathbf{v}_t) = 0, \tag{9a}$$

$$E(\mathbf{v}_t^0 \mathbf{v}_{t'}^{0\prime}) = \Omega_{\mathbf{v}_t^0} \delta_{tt'}, \tag{9b}$$

$$E(\mathbf{v}_t \mathbf{v}'_{t'}) = \Omega_{\mathbf{v}_t} \delta_{tt'}. \tag{9c}$$

The "state" equation system corresponding to (7) and (8) is assumed to evolve according to

$$\boldsymbol{\delta}_t = \Phi_t^{\delta} \boldsymbol{\delta}_{t-1} + \mathbf{a}_t^{\delta}; \qquad \mathbf{a}_t^{\delta} \sim N(\mathbf{O}; \ \Omega_{\mathbf{a}_t^{\delta}}); \qquad \boldsymbol{\delta}_0 \sim N(\hat{\boldsymbol{\delta}}_0; \hat{\Sigma}_{\boldsymbol{\delta}_0}), \tag{10}$$

where $\{\Phi_t^{\delta}, \Omega_{\mathbf{a}_t^{\delta}}, \hat{\boldsymbol{\delta}}_0, \hat{\Sigma}_{\boldsymbol{\delta}_0}\}_{t=1}^T$ are assumed known and the sequences $\mathbf{v}_t^0$ and $\mathbf{a}_t^{\delta}$ are assumed uncorrelated.

For any given $H_{(SF),t}$ and $\Omega_{\mathbf{v}_t^0}$ the "reduced form" model (7) and the "state" equation system (10) are in the standard discrete dynamic linear form and the Kalman filter (Kalman[56]) can be constructed. The Kalman filter will generate the estimate $\hat{\boldsymbol{\delta}}_t$ of the "state" $\boldsymbol{\delta}_t$ and the variance–covariance matrix of the error vector $\mathbf{e}_t = (\boldsymbol{\delta}_t - \hat{\boldsymbol{\delta}}_t)$, recursively. It is well known that the estimate $\hat{\boldsymbol{\delta}}_t$ is the minimum variance (and unbiased under appropriate assumptions) estimate of $\boldsymbol{\delta}_t$. Hence,

$$\hat{\boldsymbol{\delta}}_t = (\Phi_t^\delta \hat{\boldsymbol{\delta}}_{t-1} + (\Phi_t^\delta V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^\delta})H'_{[SF],t}\,(\Omega_{\mathbf{v}_t^0} + H_{[SF],t}[\Phi_t^\delta V(\boldsymbol{\delta}_{t-1})\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t}]H'_{[SF]\cdot t})^{-1}\,(\mathbf{Y}_t - H_{[SF],t}\Phi_t^\delta\hat{\boldsymbol{\delta}}_{t-1}), \quad (11)$$

$$V(\hat{\boldsymbol{\delta}}_t) = (\Phi_t^\delta V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t}) - (\Phi_t^\delta V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^\delta})H'_{[SF],t}(\Omega_{\mathbf{v}_t^0} + H_{[SF],t}[\Phi_t^\delta V(\hat{\boldsymbol{\delta}}_{t-1})\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^\delta}]H'_{[SF],t})^{-1}H_{[SF],t}(\Phi_t^\delta V[\boldsymbol{\delta}_{t-1}]\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^\delta}). \quad (12)$$

Now suppose that attention is limited to l admissible design points which have been chosen by the experimenter so that the relevant region of the design space is adequately covered and that, at each period of time t, the experimenter is able to choose only one out of the l admissible design points. Limiting our attention to l admissible design points may be viewed as a two-part assumption: first, that observations must be restricted to a given region in the design space (a matter of necessity); second, that within that region they must fall at only l points (a matter of convenience). It should be clear, however, that actual determination of the number of design points and their exact specification is not a trivial task and depends on many factors, including a prior knowledge of the appropriate range of variation for policy purposes of each design variable (see, for example, Conlisk and Watts[3]). Mathematically, we can express this by the following constraints (Wagner[60]):

$$n_{jt} = \begin{cases} 1, & \text{if the } j\text{th design point is chosen at time } t, \\ 0, & \text{otherwise,} \end{cases} \quad (13a)$$

$$\sum_{j=1}^{l} n_{jt} = 1. \quad (13b)$$

The system corresponding to the model (7) can be written as

$$\mathbf{Y}_t = \left[\sum_{j=1}^{l} n_{jt}H_{(SF),jt}\right]\boldsymbol{\delta}_t + \mathbf{v}_t^0. \quad (14)$$

It is apparent that, since

$$H_{(SF),t} = \left\{\sum_{j=1}^{l} n_{jt}H_{(SF),jt}\right\}$$

we can write the Kalman filter and its variance–covariance matrix as follows:

$$\hat{\boldsymbol{\delta}}_t = (\Phi_t^\delta\hat{\boldsymbol{\delta}}_{t-1}) + (\Phi_t^\delta V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^\delta})\left(\sum_{j=1}^{l} n_{jt}H'_{[SF],jt}\right)$$
$$\times\left(\sum_{j=1}^{l} n_{jt}H_{[SF],jt}[\Phi_t^\delta V(\hat{\boldsymbol{\delta}}_{t-1})\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^\delta}]H_{[SF],jt} + \Omega_{\mathbf{v}_t^0}\right)^{-1}$$
$$\times\left(\mathbf{Y}_t - \left[\sum_{j=1}^{l} n_{jt}H_{(SF),jt}\right][\Phi_t^\delta\hat{\boldsymbol{\delta}}_{t-1}]\right), \quad (15)$$

$$V(\hat{\boldsymbol{\delta}}_t) = (\Phi_t V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^\delta}) - (\Phi_t^\delta V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^\delta}) \left[\sum_{j=1}^{l} n_{jt} H'_{(SF),jt} \left(\sum_{j=1}^{l} n_{jt} H_{[SF],jt} [\Phi_t^\delta V(\hat{\boldsymbol{\delta}}_{t-1})\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^\delta}] H'_{[SF],jt} + \Omega_{\mathbf{v}_t^0} \right)^{-1} H_{(SF),jt} \right] (\Phi_t V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^\delta}) \quad (16)$$

where use has been made of the fact that $n_{jt}^2 = n_{jt}$ and $n_{jt}n_{ht}(H'_{[SF],jt} \cdot [\Phi_t^\delta V(\hat{\boldsymbol{\delta}}_{t-1})\Phi_t^{\delta'} + \Omega_{a_t^\delta}]H_{[SF],ht}) = n_{jt}n_{ht}(H'_{[SF],jt}\Omega_{\bar{v}_t^0}^{-1} H_{[SF],ht}) = 0$, since either n_{jt} or n_{ht} is zero, for all $j \neq h$.

Finally, if different design points have different costs and the experimenter has a budget constraint to worry about (the typical situation in economics), the experimenter has to specify the unit cost of an observation at the jth design point and the total budget available for all $t = 1, 2, \ldots, T$. Specifically, we shall assume that the budget constraint for our estimation control problem is

$$\left\{ \sum_{t=1}^{T} \left[\sum_{j=1}^{l} n_{jt} c_{jt} \right\} = M_T; c_{jt} > 0 \quad \text{for all} \quad j = 1, 2, \ldots, l; t = 1, 2, \ldots, T, \quad (17)$$

where c_{jt} is the cost of the jth design point at time t, and M_T is the total budget associated with the use of the design sequence $\{\mathbf{n}_T = (n_t)_{t=1}^T\}$. To keep the problem from becoming trivial, we assume $c_t^m T > M_T$ where c_t^m is the cost of the most expensive design point.

It is clear that given the initial condition $V(\boldsymbol{\delta}_0) = \hat{\Sigma}_{\boldsymbol{\delta}_0}$ and given any design sequence which satisfies the constraints (13) and (17), we can obtain by straightforward calculation the values of the matrix $V(\hat{\boldsymbol{\delta}}_t)$ for all t. Of course thc values of the elements of this matrix will depend on the design sequence which was chosen. In this context the experimenter may view the equations (12) or (16) as defining a dynamical system whose state variables are the elements of the matrix $V(\hat{\boldsymbol{\delta}}_t)$ and where the elements of the design sequence play the role of the control variables. In this manner the design sequence controls the time evolution of the elements of the error covariance matrix.

Since the variance–covariance matrix of the estimated structural parameters depends through $H_{(SF),jt}$ on the various design points, it is reasonable to require that a scalar functional of the estimation error covariance matrix is minimized. Obviously, the "smaller" the estimation error covariance is, the better the estimates.

A sensible design objective function can be built around $V(\hat{\boldsymbol{\delta}}_T)$. Suitable scalar functions were discussed in Papakyriazis[48, 51]. This paper will focus on the trace criterion. Suppose the experimenter's goal is accurate estimation of a vector $\boldsymbol{\delta}_T^0 = P\boldsymbol{\delta}_T$ of linear combinations of the elements of $\boldsymbol{\delta}_T$ where P is a known matrix of dimensions $\{p \times [\sum_{i=1}^N (N_i + K_i)]\}$. In particular, suppose the experimenter (or the sponsor of the experiment) is interested in predicting the expected values of the left-hand variable of each structural equation evaluated at each of the admissible design points: $(\mathbf{x}_1, \ldots, \mathbf{x}_l)$. Then the expected values of all structural equations is $(I \otimes x^0)H\boldsymbol{\delta}$, where $x^0 = \text{diag}(\mathbf{x}_1, \ldots, \mathbf{x}_l)$. The dimension p may be larger or smaller than $(\sum_{i=1}^N [N_i + K_i])$. The estimate of $\boldsymbol{\delta}_T^0$ and its variance–covariance matrix are: $\hat{\boldsymbol{\delta}}_T^0 = P\hat{\boldsymbol{\delta}}_T$ and $V\hat{\boldsymbol{\delta}}_T^0 = [PV(\hat{\boldsymbol{\delta}}_T)P']$, respectively. If the experimenter wishes to minimize the weighted sum of variances of the elements of $\hat{\boldsymbol{\delta}}_T^0$ the objective function may be written: $\text{tr}\{Q(V[\hat{\boldsymbol{\delta}}_T^0])\}$ where tr $(\cdot)$ is the trace, $Q = (P'WP)$, and W denotes the $(P \times P)$ diagonal matrix whose diagonal elements $(w_1, \ldots, w_p)$ indicate the policy importance to the experimenter of the elements of $\boldsymbol{\delta}_T^0 = (P\boldsymbol{\delta}_T)$.

Finally, since $V(\boldsymbol{\delta}_0) = \hat{\Sigma}_{\boldsymbol{\delta}_0}$, it is apparent from (12) that maximizing $\text{tr}\{Q(V[\hat{\boldsymbol{\delta}}_T^0])\}$ with respect to $\{H_{(SF),t}$ or $n_{jt}\}_{t=1}^T$ is equivalent to maximizing the following expression:

$$\Gamma \equiv \mathrm{tr}\left\{\sum_{t=1}^{T} K_t[(\Phi_t^{\delta} V[\hat{\delta}_{t-1}]\Phi_t^{\delta'} + \Omega_{a_t^{\delta}})H'_{(SF),t}(H_{[SF],t}[\Phi_t^{\delta} V(\hat{\delta}_{t-1})\Phi_t^{\delta'}\right.$$

$$\left. + \Omega_{a_t^{\delta}}]H'_{[SF],t} + \Omega_{v_t^0})^{-1}H_{(SF),t}(\Phi_t^{\delta} V[\hat{\delta}_{t-1}]\Phi_t^{\delta'} + \Omega_{a_t^{\delta}})]\right\}$$

$$= \left\{\sum_{t=1}^{T} K_t[(\Phi_t^{\delta} V[\hat{\delta}_{t-1}]\Phi_t^{\delta'} + \Omega_{a_t^{\delta}})\left[\sum_{j=1}^{l} n_{jt}H'_{(SF),jt}\right.\right.$$

$$\left.\left.\times\left(\sum_{j=1}^{l} n_{jt}H_{[SF],jt}\right](\Phi_t^{\delta} V[\hat{\delta}_{t-1}]\Phi_t^{\delta'} + \Omega_{a_{it}^{\delta}})\right]\right\}, \tag{18}$$

where

$$K_t = \left(\left[\prod_{j=t}^{T} \Phi_j^{\delta}\right]' Q_t \left[\prod_{j=t}^{T} \Phi_j^{\delta'}\right]'\right) \tag{19}$$

with respect to $H_{(SF),t}$ or n_{jt}.

There are now all the "essential ingredients" for the statement of the estimation control problem, which can be formulated in a manner that suggests the use of the discrete maximum principle of Pontryagin (see Halkin[58], for example). In particular, the estimation control problem can be stated as follows: For a given terminal time $T > 0$, determine the optimal $\{n_{jt}; j = 1, 2, \ldots, l\}_{t=1}^{T}$ such that the objective function given by (18), Γ, is maximized subject to: (i) (16); (ii) (13) and (17); (iii) $V(\delta_0) = \hat{\Sigma}_{\delta_0}$ (given positive definite matrix) and $V(\hat{\delta}_T)$: unrestricted. Note that this estimation control (design) problem is a standard (discrete time) optimal control problem where the elements $V(\hat{\delta}_t)_{kk'}$, $\{k, k' = 1, 2, \ldots, (\sum_{i=1}^{N} [M_i + K_i])\}$ of the covariance matrix $V(\hat{\delta}_t)$ play the role of the state variables of a dynamical system whose "equation of motion" is governed by the matrix variance–covariance difference equation (16), n_{jt}'s play the role of control variables, and the "cost functional" (objective function) depends on the values of the control and state variables n_{jt}, $V(\hat{\delta}_t)_{kk'}$, for all t.

We shall use the discrete Maximum Principle to derive a set of necessary conditions for optimality. The discrete maximum principle is essentially equivalent to the Kuhn–Tucker theorem (see, for example, Chow[59, 60]). Before we apply the maximum principle, however, it is necessary to have the "state" equation (16) in the form: $(V[\hat{\delta}_t) - V[\hat{\delta}_{t-1}]) = F(V[\hat{\delta}_{t-1}[, \{n_{jt}\}_{j=1}^{l})$ and also to transform the total budget constraint $\sum_{t=1}^{T} (\sum_{j=1}^{l} n_{jt}c_{jt}) = M_T$ into a difference equation type constraint. To have the "state" equation (16) in the appropriate form, $V(\hat{\delta}_{t-1})$ will be subtracted from both sides of (16) to give the final "state" equation. To transform the total budget constraint, on the other hand, we define a new state variable $\tilde{\psi}_{0t} = \sum_{k=1}^{t} (\sum_{j=1}^{l} n_{jt}c_{jt})$. It is apparent that $\tilde{\psi}_{0t}$ satisfies the first-order difference equation $(\tilde{\psi}_{0t} - \tilde{\psi}_{0t-1}) = \sum_{j=1}^{l} n_{jt}c_{jt}$ with $\tilde{\psi}_{00} = 0$ (initial condition) and $\tilde{\psi}_{0t} = M_T$ (terminal condition).

We now define the real-valued function H, called the Hamiltonian, as follows:

$$H = H(V[\hat{\delta}_t], \Psi_t, \tilde{\psi}_{0t}, p_{0t}, n_{jt}, t)$$

$$= (\mathrm{tr}\{K_t[(\Phi_t^{\delta} V(\hat{\delta}_{t-1}]\Phi_t^{\delta'}$$

$$+ \Omega_{a_t^{\delta}})\left[\sum_{j=1}^{l} n_{jt}H'_{(SF),jt}\left(\sum_{j=1}^{l} n_{jt}H_{[SF],jt}[\Phi_t^{\delta} V(\hat{\delta}_{t-1})\Phi_t^{\delta'} + \Omega_{a_t^{\delta}}]H'_{[SF],jt}\right.\right.$$

$$\left.+ \Omega_{v_t^0})^{-1}H_{(SF),jt}\right](\Phi_t^{\delta} V[\hat{\delta}_{t-1}]\Phi_t^{\delta'} + \Omega_{a_t^{\delta}})\} + \mathrm{tr}\{(V[\hat{\delta}_t] - V[\hat{\delta}_{t-1}])\Psi_t'\}$$

$$\left. + (p_{0t}[\tilde{\psi}_{0t} - \tilde{\psi}_{0t-1}])\right), \tag{20}$$

where $\{p_{0t};\ t = 1, 2, \ldots, T\}$ is the costate at time t corresponding to the state variable $\tilde{\psi}_{0t}$ and $\{\psi_t;\ t = 1, 2, \ldots, T\}$ is the costate matrix (whose kk'th element is the costate which corresponds to the $V[\hat{\boldsymbol{\delta}}_t]_{kk'}$ state variable) corresponding to the variance–covariance matrix $V(\hat{\boldsymbol{\delta}}_t)$.

Alternatively, utilizing the properties of the trace function, the Hamiltonian can be written as

$$\begin{aligned} H &= H(\cdot) \\ &= \Bigg\{ \sum_{j=1}^{l} n_{jt}[(p_{0t}c_{jt}) + \operatorname{tr}([H'_{(SF),jt}(H_{[SF],jt}[\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}]H'_{[SF],jt} \\ &\quad + \Omega_{\mathbf{v}_t^0})^{-1} H_{(SF),jt}]\ [(\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}})\ (K_t - \Psi_t')\ (\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta'} \\ &\quad + \Omega_{\mathbf{a}_t^{\delta}})])] + \operatorname{tr}(\{[\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}] - [V(\hat{\boldsymbol{\delta}}_{t-1})]\}\psi_t') \Bigg\} . \end{aligned} \tag{21}$$

The necessary conditions for optimality of the estimation control problem can now be stated. Assume that an optimal control sequence exists. Let $\{n_{jt}^*\}$ denote the optimal (estimation) control sequence; let $\{V(\hat{\boldsymbol{\delta}}_t)_*\}$ be the resultant variance–covariance matrix and let $\{\tilde{\psi}_{0t}^*\}$ be the resulting state variable. Then there exist costate variables $\{p_{0t}^*\}$ and $\{\psi_{kk'}^*;\ k, k' = 1, 2, \ldots, [\sum_{i=1}^{N} (N_i + K_i)]\}$ such that the following conditions hold:

(1) *Hamiltonian maximization*

The inequality

$$H(V[\hat{\boldsymbol{\delta}}_t]_*, \psi_t^*, \tilde{\psi}_{0t}^*, p_{0t}^*, n_{jt}^*, t) \geqslant H(V[\hat{\boldsymbol{\delta}}_t]_*, \psi_t^*, \tilde{\psi}_{0t}^*, p_{0t}^*, n_{jt}, t)$$

or [see eqn (21)],

$$\begin{aligned} &\Bigg(\sum_{j=1}^{l} n_{jt}^*[(p_{0t}^* c_{jt}) + \operatorname{tr}([H'_{(SF),jt}(H_{[SF],jt}[\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})_*\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}]H'_{[SF],jt} \\ &\quad + \Omega_{\mathbf{v}_t^0})^{-1} H_{(SF),jt}]\ [(\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_*\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}})\ (K_t - \psi_t^{*'})\ (\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_*\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}})])]\Bigg) \\ &\geqslant \Bigg(\sum_{j=1}^{l} n_{jt}[(p_{0t}^* c_{jt}) + \operatorname{tr}([H'_{(SF),jt}\ (H_{[SF],jt}[\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})_*\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}]H'_{[SF],jt} \\ &\quad + \Omega_{\mathbf{v}_t^0})^{-1} H_{(SF),jt}]\ [(\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_*\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}})\ (K_t - \Psi_t^{*'})\ (\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_*\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}})])]\Bigg) \end{aligned} \tag{22}$$

must hold for each $t = 1, 2, \ldots, T$ and for all $n_{jt} \in (0, 1)$, $\sum_{j=1}^{l} n_{jt} = 1$.

In view of the above constraints on n_{jt}, we have the following result:

$$n_{jt}^* = \begin{cases} 1 & \begin{aligned}[t] &\text{if } [(p_{0t}^* c_{jt} + \operatorname{tr}([H'_{(SF),jt}(H_{[SF],jt}[\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})_*\Phi_t^{\delta'} \\ &+ \Omega_{\mathbf{a}_t^{\delta}}]H'_{[SF],jt} + \Omega_{\mathbf{v}_t^0})^{-1} H_{(SF),jt}]\ [(\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_*\Phi_t^{\delta'} \\ &+ \Omega_{\mathbf{a}_t^{\delta}})\ (K_t - \Psi_t^{*'})\ (\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_*\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}})])] \\ &\geqslant [(p_{0t}^0 c_{st}) + \operatorname{tr}([H'_{(SF),st}(H_{[SF],st}[\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})_*\Phi_t^{\delta'} \\ &+ \Omega_{\mathbf{a}_t^{\delta}}]H'_{[SF],\mathrm{st}} + \Omega_{\mathbf{v}_t^0})^{-1} H_{(SF),st}]\ [(\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_*\Phi_t^{\delta'} \\ &+ \Omega_{\mathbf{v}_t^{\delta}})\ (K_t - \Psi_t^{*'})\ (\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_*\Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}})])] \\ &\quad \text{(for all } s, j = 1, 2, \ldots, l;\ s \neq j) \end{aligned} \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

(2) *Canonical equations*

$$
\begin{aligned}
(V[\hat{\boldsymbol{\delta}}_t]_* - V[\hat{\boldsymbol{\delta}}_{t-1}]_*) &= ([\partial H/\partial\psi_t]\,|_*) \\
&= ([\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1})_* \Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}] - [V(\hat{\boldsymbol{\delta}}_{t-1})_*]) \\
&\quad - ([\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})_* \Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}] \Bigg[\sum_{j=1}^{l} n_{jt}(H'_{[SF],jt} \\
&\quad \times [H_{(SF),jt}(\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_* \Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}) H'_{(SF),jt} + \Omega_{\mathbf{v}_t^{0}})^{-1} \\
&\quad \times H'_{(SF),jt}])] \Bigg] [\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})_* \Phi_t' + \Omega_{\mathbf{a}_t^{\delta}}]),
\end{aligned} \tag{24}
$$

$$
(\bar{\psi}_{0t}^* - \bar{\psi}_{0t-1}^*) = \left(\sum_{j=1}^{l} n_{jt}^* c_{jt} \right), \tag{25}
$$

$$
\begin{aligned}
(\Psi_t^* - \Psi_{t-1}^*) &= -([\partial H/\partial(V[\hat{\boldsymbol{\delta}}_{t-1}])]\,|_*) \\
&= -\Bigg\{ \Bigg(\sum_{j=1}^{l} n_{jt}^*[(H'_{[SF],jt}[H_{(SF),jt}(\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_* \Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}) \\
&\quad \times H'_{(SF),jt} + \Omega_{\mathbf{v}_t^{0}}]^{-1} H_{[SF],jt}] \Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})_* \Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}] \\
&\quad \times \Phi_t^{\delta'}[K_t - \psi_t^{*\prime}]\Phi_t^{\delta}) - (\Phi_t^{\delta'}[H_{(SF),jt}(\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_* \Phi_t^{\delta'} \\
&\quad + \Omega_{\mathbf{a}_t^{\delta}}) H'_{(SF),jt} + \Omega_{\mathbf{v}_t^{0}}]^{-1} H_{[SF],jt}[\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})_* \Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}] \\
&\quad \times [K_t - \Psi_t^{*\prime}]\, [\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})_* \Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}[H'_{[SF],jt}[H_{(SF),jt} \\
&\quad \times (\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]_* \Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}) H'_{(SF),jt} + \Omega_{\mathbf{v}_t^{0}}]^{-1} H_{[SF],jt} \\
&\quad \times \Phi_t^{\delta}) + ([K_t - \Psi_t']\, [\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1}) \Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}]\, [H'_{(SF),jt} \\
&\quad \times (H_{[SF],jt}[\Phi_t V(\hat{\boldsymbol{\delta}}_{t-1}) \Phi_t^{\delta'} + \Omega_{\mathbf{a}_t^{\delta}}] H'_{[SF],jt} + \Omega_{\mathbf{v}_t^{0}})^{-1} \\
&\quad \times H_{(SF),jt}]\, [\Phi_t^{\delta'} \Phi_t^{\delta}])] \Bigg) - (\Phi_t^{\delta'} \Psi_t^* \Phi_t^{\delta} - I) \Bigg\},
\end{aligned} \tag{26}
$$

$$
(p_{0t}^* - p_{0t-1}^*) = 0. \tag{27}
$$

(3) *Boundary conditions*

$$
V(\boldsymbol{\delta}_0) = \hat{\Sigma}_{\boldsymbol{\delta}_0} \quad \text{and} \quad \psi_{00}^* = 0 \quad \text{at the initial time } t = 0, \tag{28}
$$

$$
\psi_T^* = 0, \quad \bar{\psi}_{0T}^* = M_T \quad \text{at the terminal time } t = T. \tag{29}
$$

The necessary conditions given above can be used to determine in each particular case the optimal design sequence $(\{n_{jtij=1,2,\ldots,l}^*\}_{t=1}^{T})$. A number of computational algorithms that use the necessary conditions of the maximum principle to obtain in an iterative manner numerical solutions to the optimal control problem that have been proposed in the literature (standard gradient techniques, for example) cannot be used in our case because of the constraints on n_{jt}. Other techniques, however, could be used to obtain numerical solutions for the optimal design problem. The essence of one such technique can be outlined as follows: (i) an initial guess is made for the values of n_{jt} (say n_{jt}^0 for all $j = 1, 2, \ldots, l$; $t = 1, 2, \ldots, T$); (ii) the initial guess permits the solution for $\tilde{\psi}_{0t}$ and $V(\hat{\boldsymbol{\delta}}_t)$ forward in time (starting at the known condition $V[\boldsymbol{\delta}_0] = \hat{\Sigma}_{\delta_0}$) as well as for ψ_t backward in time (using the known condition $\psi_T = 0$); (iii) the maximization condition (23) may now be used to determine new values for n_{jt} (say n_{jt}^1); (iv) steps two and three are repeated until a suitable criterion (based on the change in the objective function with successive iterations, perhaps) is satisfied.

The obvious difficulty with using the maximum principle to solve the experimental design problem is that it is basically an open-loop technique in the sense that it uses only the initial *a priori* knowledge about the parameters to compute all future designs. Since in practice the true parameter values are unknown and the *a priori* means are therefore used, it is apparent, that one may be hesitant to design all observations on the basis of questionable prior means. A sequential approach, where we improve the design as we find out more about the parameters seems to be more appropriate. In the next subsection we will consider feedback designs; that is, designs that are generated with the knowledge of measurements. In particular, we consider open-loop feedback designs. At each time t, the future design is obtained by replacing the unknown parameters by their most recent estimates. Since the feedback approach uses all the information up to the current time to determine the future designs, it has the merit of giving improved parameter estimates.

3.3. *Open-loop feedback (OLF) estimation control*

Since $H_{(SF),t} = (I \otimes \mathbf{x}_t)H_t$, $H_t = \text{diag}(H_{t1}H_{t2} \cdots H_{tN})$ and $H_{ti} = (\Pi_i : J_i)$, $i = 1, 2, \ldots, N$, the objective function Γ is a function of the design sequence $\tilde{N}_T = \{n_{jt}; j = 1, 2, \ldots, l\}_{t=1}^T$ and the unknown parameters $\{\Pi_i\}_{i=1}^N$ or $\boldsymbol{\Pi}_t = (\boldsymbol{\Pi}'_{t1}\boldsymbol{\Pi}'_{t2} \cdots \boldsymbol{\Pi}'_{tN})'$. One way around the unknown parameters problem is to approximate the estimation control problem in real time; that is, at the beginning of the design process, the experimenter calculates a series of estimation controls, $\{n_{jt}^0; j = 1, 2, \ldots, l\}_{t=1}^T$, utilizing the prior guesses about $\boldsymbol{\delta}_0$ (i.e. $\hat{\boldsymbol{\delta}}_0$ and $\hat{\Sigma}_{\delta_0}$), and hence about $\boldsymbol{\Pi}_0$ (i.e. $\hat{\boldsymbol{\Pi}}_0$ and $\hat{\Sigma}_{\Pi_0}$). Only the first of these estimation (design) controls $\{n_{j1}; j = 1, 2, \ldots, l\}$, will actually be implemented, of course; it is necessary to make tentative plans, however, as to what will be done later in the experiment in order to make an optimal choice of what is to be done in the first period. As the observations of the endogenous variables of the first period, $\mathbf{Y}_1$, become available, the experimenter estimates $\hat{\boldsymbol{\delta}}_1$, $V(\hat{\boldsymbol{\delta}}_1)$, and hence $\hat{\boldsymbol{\Pi}}_1$, and recalculates the optimum values for the remaining estimation controls, $\{n_{jt}^1; j = 1, 2, \ldots, l\}_{t=2}^T$. Thus, as the experimental process progresses, more and more observations become available and the initial parameters may be replaced by better estimates, which in turn are used to update the design of the remainder of the experiment.

Formally, the open-loop feedback (OSF) estimation control procedure can be described as follows: Denote the current period by t. Let us assume that the estimation control sequence $\{n_{jk}^*; j = 1, 2, \ldots, l\}_k^t$ has been applied and the corresponding observation sequence $\{\mathbf{Y}_k\}_{k=1}^t$ is obtained. We wish to choose the future estimation control sequence $\{n_{jt+1}^*, \ldots, n_{jT}^*; j = 1, 2, \ldots, l\}$ based on the information up to time t. Let us denote the maximization at this point as follows:

$$\Lambda_{T-t}^* = \underset{\{n_{jk};\, j = 1, 2, \ldots, l\}_{k=(t+1)}^T}{\text{Maximize}}$$
$$\{\Gamma = \Gamma([\{n_{jk}^*;\ j = 1, 2, \ldots, l\}_{k=1}^t;\ \ j = 1, 2, \ldots, l\}_{k=(t+1)}^T;\ \ \boldsymbol{\Pi}] \mid \boldsymbol{\Pi} = \hat{\boldsymbol{\Pi}}_t)\}$$

subject to

(1) $$\{(V[\hat{\boldsymbol{\delta}}_t] - V[\hat{\boldsymbol{\delta}}_{t-1}]) \mid [n_{j1} = n_{j1}^*, \ldots, n_{jt} = n_{jt}^*;\ j = 1, 2, \ldots, l;\ \boldsymbol{\Pi} = \hat{\boldsymbol{\Pi}}_t]\}$$
$$= \{[(\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta\prime} + \Omega_{\mathbf{a}_t^{\delta}} - V[\hat{\boldsymbol{\delta}}_{t-1}]) - (\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta\prime} + \Omega_{\mathbf{a}_t^{\delta}})$$
$$\times \left[\sum_{j=1}^{l} n_{jt} H'_{(SF),jt} \left(\sum_{j=1}^{l} n_{jt} H_{[SF],jt}[\Phi_t^{\delta} V(\hat{\boldsymbol{\delta}}_{t-1})\Phi_t^{\delta\prime} + \Omega_{\mathbf{a}_t^{\delta}}]\right.\right.$$
$$\times H'_{[SF],jt} + \Omega_{\mathbf{v}_t^0})^{-1} H_{(SF),jt}]\Phi_t^{\delta} V[\hat{\boldsymbol{\delta}}_{t-1}]\Phi_t^{\delta\prime} + \Omega_{\mathbf{a}_t^{\delta}})] \mid [n_{j1} = n_{j1}^*, \ldots, n_{jt}$$
$$= n_{jt}^*;\ j = 1, 2, \ldots, l; \boldsymbol{\Pi} = \hat{\boldsymbol{\Pi}}_t], \tag{30}$$

(2) $$\{(\tilde{\psi}_{0t} - \tilde{\psi}_{0t-1} \mid_{[n_{j1} = n_{j1}^* \ldots, n_{jt} = n_{jt}^*; j = 1, 2, \ldots, l]}\}$$
$$= \left\{\left(\sum_{j=1}^{l} n_{jt} c_{jt}\right) \middle| \, [n_{j1} = n_{j1}^*, \ldots, n_{jt} = n_{jt}^*; j = 1, 2, \ldots, l]\right\},$$

(3) $$\left\{\sum_{j=1}^{l} n_{jk} = 1;\quad n_{jk} \in (0, 1); k = (t+1), \ldots, T\right\},$$

(4) $$V(\hat{\boldsymbol{\delta}}_0) = \hat{\Sigma}_{\boldsymbol{\delta}_0}, \qquad V(\hat{\boldsymbol{\delta}}_T)\text{: unrestricted}, \qquad \tilde{\psi}_{00} = 0 \quad \text{and} \quad \tilde{\psi}_{0T} = M_T,$$

where $\hat{\boldsymbol{\Pi}}_t$ is the estimate of $\boldsymbol{\Pi}$ based on the observations $\{\mathbf{Y}_k\}_{k=1}^{t}$ and Γ is defined in (18).

The maximization carried out by (30) yields the following sequence of parameter estimation controls

$$\{n_{j[k \mid t]};\quad j = 1, 2, \ldots, l;\quad t = 1, 2, \ldots, T;\quad k = (t+1), \ldots, T\}. \tag{31}$$

The optimal OLF estimation control sequence is obtained from (31) by choosing those design points for which $k = t$; that is,

$$\{n_{jt}^* = n_{j(t \mid t)};\quad j = 1, 2, \ldots, l;\quad t = 1, 2, \ldots, T\}. \tag{32}$$

3.4. *One-step-ahead (OSA) estimation control*

The main problem with the implementation of the OLF estimation control scheme stems from the fact that, for each period t, we have to solve a deterministic constrained maximization problem where the sequential equations of the error covariance matrix act as the state equations. Hence, determining the OLF estimation control sequence reduces to solving, for each period t, a nonlinear optimization problem and has all the difficulties associated with this type of problem. In this subsection an approximation of the deterministic constrained maximization problem will be given which is easy to implement. We call this computationally efficient design one-step-ahead (OSA) estimation control.

The procedure for one-step-ahead (recursive) estimation control can be described as follows: First, a preliminary experiment is conducted with t observations; t is the minimum number of observations which is necessary for a single-valued estimate of the unknown structural and the variance–covariance of the $\boldsymbol{\delta}$ estimates. Since the optimal estimation control scheme cannot be used to generate these preliminary observations, a random selection of the t observations may be used. The preliminary observations can now be used to obtain $\{\hat{\boldsymbol{\delta}}_t, \hat{\Omega}_{\mathbf{v}_t^0}, \hat{\boldsymbol{\Pi}}_t, V(\hat{\boldsymbol{\delta}}_t)\}$ by the method of three-stage least-squares (3SLS), or any other full-information (system) estimation method. Second, the optimal values of $\{n_{j(t+1)}; j = 1, 2, \ldots, l\}$ are determined from the solution of the following problem:

$$\underset{\{n_{j(t+1)};j=1,2,\ldots,l\}}{\text{Maximize:}} \left(\left[\sum_{j=1}^{l} n_{j(t+1)}(c_{j[t+1]})^{-1} \operatorname{tr}\{(H'_{[SF],j[t+1]}[H_{(SF),j(t+1)} \right. \right.$$
$$\times (\Phi^{\delta}_{t+1} V[\hat{\boldsymbol{\delta}}_t]\Phi^{\delta\prime}_{t+1} + \Omega_{\mathbf{a}^{\delta}_{t+1}})H'_{(SF),j(t+1)} + \hat{\Omega}_{\mathbf{v}^{\text{o}}_t}]^{-1} H_{[SF],j[t+1]})$$
$$\left. \left. \times (\Phi_{t+1} V[\hat{\boldsymbol{\delta}}_t]\Phi^{\delta\prime}_{t+1} + \Omega_{\mathbf{a}^{\delta}_{t+1}})Q(\Phi^{\delta}_{t+1} V[\hat{\boldsymbol{\delta}}_t]\Phi^{\delta\prime}_{t+1} + \Omega_{\mathbf{a}^{\delta}_{t+1}})\} \right|_{\mathbf{\Pi}=\hat{\mathbf{\Pi}}_t} \right] \right) \tag{33}$$

subject to

$$\sum_{j=1}^{l} n_{j(t+1)} = 1; \quad n_{j(t+1)} \in (0, 1); \quad j = 1, 2, \ldots, l, \tag{34}$$

where $c_{j(t+1)}(j = 1,2,\ldots,l)$ is the cost of the jth design point. It is clear that if different design points have different costs and the experimenter has a budget constraint to worry about, it may be "optimal" (cheaper) to gain a certain amount of information by performing several cheap though inefficient experiments, rather than a single efficient though expensive one. In this case, one solution is to divide the expected gain from one more observation by the cost of that observation, and maximize the expected information per unit cost ("benefit-cost" ratio). When the optimum $\{n^*_{j(t+1)}\}^l_{j=1}$ have been chosen, the $(t + 1)$-st observation (i.e. $\mathbf{Y}_{[t+1]}$) can be processed to obtain: $\{\hat{\boldsymbol{\delta}}_{t+1}, \hat{\mathbf{\Pi}}_{t+1}, V(\hat{\boldsymbol{\delta}}_{t+1}), \hat{\Omega}_{\mathbf{v}^{\text{o}}_{t+1}}\}$.

For the purpose of sequentially estimating the structural form parameters, we can use a two-step system (3SLS alike) Kalman filtering method which can be described as follows: (1) Estimates of the structural form parameters $\boldsymbol{\delta}$, $\hat{\boldsymbol{\delta}}_t$, and their variance–covariance matrix, $V(\hat{\boldsymbol{\delta}}_t)$, are computed based on the initial observations, $\{\mathbf{Y}_t, \mathbf{x}_t\}$, using the 3SLS or any other full information method. During this computation a consistent estimate of the covariance $\Omega_{\mathbf{v}^{\text{o}}_t}$, $\hat{\Omega}_{\mathbf{v}^{\text{o}}_t}$, is obtained. Estimates of the reduced form parameters $\mathbf{\Pi}$, $\hat{\mathbf{\Pi}}$, are also obtained from the structural estimates $\hat{\boldsymbol{\delta}}_t$. In addition, $\hat{\mathbf{Y}}_{(t+1)} = H_{(RF),t+1}\hat{\mathbf{\Pi}}$ is computed. (2) The Kalman filter (Kalman[56]) is applied to (7) and (10) to obtain $\hat{\boldsymbol{\delta}}_{t+1}$ and $V(\hat{\boldsymbol{\delta}}_{t+1})$, where $\{\mathbf{y}_{(t+1)i}\}^N_{i=1}$ in $H_{(SF),t+1}$ are replaced by $\{\hat{\mathbf{y}}_{(t+1)}\}^N_{i=1}$ and the covariance matrix $\Omega_{\mathbf{v}^{\text{o}}}$ is replaced by $\hat{\Omega}_{\mathbf{v}^{\text{o}}_t}$. Also, a new estimate of $\Omega_{\mathbf{v}^{\text{o}}}$, based on $\hat{\boldsymbol{\delta}}_{t+1}$, is obtained. Finally, estimates of the updated reduced form parameters $\mathbf{\Pi}$, $\hat{\mathbf{\Pi}}_{(t+1)}$, are obtained from the structural form estimates $\hat{\boldsymbol{\delta}}_{t+1}$; furthermore $\hat{\mathbf{Y}}_{(t+2)} = H_{(RF),t+2}\hat{\mathbf{\Pi}}_{(t+1)}$ is computed. (3) Go to step (2) and the whole procedure can be repeated. When the structural form parameters are constant (i.e. $\boldsymbol{\delta}_t = \boldsymbol{\delta}_{t-1}$), the above sequential procedure closely resembles the 3SLS method. The two estimation methods, however, are not identical: first, the estimates of $\Omega_{\mathbf{v}^{\text{o}}}$ are going to be different under the two methods and, second, in the case of the 3SLS method, $\hat{\mathbf{Y}}^t = \{[H_{(SF),1}\hat{\mathbf{\Pi}}_t] \cdots (H_{(SF),t}\hat{\mathbf{\Pi}}_t]$, while in the sequential procedure suggested above, $\hat{\mathbf{Y}}^t = \{(H_{(SF),1}\hat{\mathbf{\Pi}}_1] \cdots [H_{(SF),t}\hat{\mathbf{\Pi}}_t]\}$. An iterative procedure which uses the full set of data available each time to estimate $\mathbf{Y}$ and $\Omega_{\mathbf{v}^{\text{o}}}$ could be combined with the sequential procedure to improve on parameter estimates. Since in the post-experiment estimation phase, we are free to disown the sequential estimates on which we based the design, the use of noniterative sequential estimates of the structural form parameters is in a way "safer" in estimation control (design) than in estimation. This procedure can be repeated until the budget constraint is met or until a desired degree of accuracy is obtained.

4. CONCLUSIONS

The problem of choosing optimal estimation control strategies in the context of a simultaneous equations system with time-varying parameters for the special case (typical in public policy experimentation) in which attention is limited to l admissible design points, has been considered in this paper. Since the nonlinear restrictions on reduced form coefficients implied by the structural form cause the design objective function to depend on unknown parameters, an adaptive procedure, where we improve the design as we find out more about the parameters immediately suggests itself.

Two sequential approaches to the simultaneous equations estimation control problem are looked at in the paper: (1) the "open-loop-feedback design" (OLFD) approach in which, at each time t, a dynamic optimization technique such as the Maximum Principle of Pontryagin is utilized to obtain the optimal design by replacing the unknown parameters by their most recent estimates, and (2) the "recursive (one-step-ahead) estimation control" approach in which the weighted sum of variances of the estimated structural form parameters in every period is minimized. Since the feedback approach uses all the information up to the current time to determine the future design and at the same time takes into account future designs, it has the merit of giving improved estimation control sequences. However, determining the optimal OLFD sequence reduces to solving, for each period t, a nonlinear dynamic optimization problem and has all the difficulties associated with this type of problem. The one-step-ahead (OSA) estimation control procedure, on the other hand, is relatively easy to implement and thus computationally efficient.

The estimation control sequential procedures suggested in this paper could be extended in several directions. For example, one might wish to consider the case of dynamic simultaneous equations systems where the predetermined variables in a particular structural equation include not only current exogenous but also lagged endogenous and/or exogenous variables. Also, the parameters $\{\Phi_t^{\delta}$ and $\Omega_{\mathbf{a}_t^{\delta}}\}_{t=1}^{T}$ in the "state equation" (10) might be unknown and hence have to be estimated along with the structural form ("measurement equation") parameters. Finally, using t-period estimated parameters for the true ones is a "seat-of-the-pants" device that might be replaced by a Bayesian or other more elaborate device which allows the application of the stochastic maximum principle (Kushner and Schweppe[61]) to solve the intertemporal estimation control problem. These extensions, however, carry heavy informational and computational costs and are beyond the scope of this paper.

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