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Graphs with vertex-coloring and detectable 2-edge-weighting

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Abstract

For a connected graph G of order $|V(G)| \geq 3$ and a k-edge-weighting $c : E(G) \to \{1, 2, \ldots, k\}$ of the edges of G, the *code*, $code_c(v)$, of a vertex v of G is the ordered k-tuple $(\ell_1, \ell_2, \ldots, \ell_k)$, where ℓ_i is the number of edges incident with v that are weighted i. (i) The k-edge-weighting c is *detectable* if every two adjacent vertices of G have distinct codes. The minimum positive integer k for which G has a detectable k-edge-weighting is *the detectable chromatic number* $det(G)$ of G. (ii) The k-edge-weighting c is a *vertex-coloring* if every two adjacent vertices u, v of G with codes $code_c(u) = (\ell_1, \ell_2, \ldots, \ell_k)$ and $code_c(v) = (\ell'_1, \ell'_2, \ldots, \ell'_k)$ have $1\ell_1 + 2\ell_2 + \cdots + k\ell_k \neq 1\ell'_1 + 2\ell'_2 + \cdots + k\ell'_k$. The minimum positive integer k for which G has a vertex-coloring k-edge-weighting is denoted by $\mu(G)$. In this paper, we have enlarged the known families of graphs with $det(G) = \mu(G) = 2$.

Keywords: Detectable edge-weighting; Vertex-coloring edge-weighting; Cartesian product; Tensor product

1. Introduction

For graph-theoretical terminology and notation, we in general follow [1]. In this paper, we assume that the graphs G in discussion are finite, connected, undirected and simple with order $|V(G)| \geq 3$.

Let $c : E(G) \to \{1, 2, \ldots, k\}$ be a k-edge-weighting of G, where k is a positive integer. The *color code* of a vertex v of G is the ordered k-tuple $code_c(v) = (\ell_1, \ell_2, \ldots, \ell_k)$, where ℓ_i is the number of edges incident with v that are weighted i for $i \in \{1, 2, \ldots, k\}$. Therefore, $\ell_1 + \ell_2 + \cdots + \ell_k = d_G(v)$, the degree of v in G. It follows that for $u, v \in V(G)$ if $d_G(u) \neq d_G(v)$, then $code_c(u) \neq code_c(v)$. The k-edge-weighting c of G is called *detectable* if every two adjacent vertices of G have distinct color codes. The *detectable chromatic number* $det(G)$ of G is the minimum positive integer k for which G has a detectable k-edge-weighting.

Any k-edge-weighting $c : E(G) \to \{1, 2, \ldots, k\}$ induces a vertex-weighting $f_c : V(G) \to \mathbb{N}$ defined by $f_c(v) = \sum_{e \text{ is incident with } v} c(e)$. An edge-weighting c is a *vertex-coloring* if $f_c(u) \neq f_c(v)$ for any edge uv. Denote by $\mu(G)$ the minimum k for which G has a vertex-coloring k-edge-weighting.

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If a graph has an edge as a component, then it neither has a detectable edge-weighting nor has a vertex-coloring edge-weighting. So in this paper, we only consider graphs without a K_2 component and such graphs are called *nice graphs*. As the graph G in discussion is connected and as $|V(G)| \geq 3$, G is nice.

Karoński et al. [2] initiated the study of vertex-coloring k-edge-weighting and they posed the following conjecture:

Conjecture 1.1 (*1-2-3-Conjecture*). *Every nice graph admits a vertex-coloring* 3*-edge-weighting.*

Consider a vertex-coloring k-edge-weighting c of G. For $uv \in E(G)$, let ℓ_i, ℓ_i', respectively, be the number of edges incident with u, v that are weighted i in c. Then $1\ell_1 + 2\ell_2 + \cdots + k\ell_k \neq 1\ell_1' + 2\ell_2' + \cdots + k\ell_k'$ and hence $(\ell_1, \ell_2, \ldots, \ell_k) \neq (\ell_1', \ell_2', \ldots, \ell_k')$. So c is a detectable k-edge-weighting. Consequently, $det(G) \leq \mu(G)$.

Proposition 1.1. $det(G) \leq \mu(G)$.

Proposition 1.2. *For every nice graph G, following three conditions are equivalent:*
(i) $det(G) = 1$,
(ii) $\mu(G) = 1$,
(iii) *G has no adjacent vertices with the same degree.*

Proposition 1.3. *If* $\mu(G) = 2$*, then* $det(G) = 2$.

If c is a detectable 2-edge-weighting of a k-regular graph G with $k \geq 3$, then c is a vertex-coloring 2-edge-weighting. This follows from the fact that $\ell_1 + \ell_2 = k = \ell_1' + \ell_2'$ and $(\ell_1, \ell_2) \neq (\ell_1', \ell_2')$ imply $1\ell_1 + 2\ell_2 \neq 1\ell_1' + 2\ell_2'$.

Proposition 1.4. *Let G be a k-regular graph with* $k \geq 3$*. If* $det(G) = 2$*, then* $\mu(G) = 2$.

In [2], Karoński et al. proved that: (i) $det(G) \leq 183$, and (ii) if $d_G(v) \geq 10^{99}$ for every $v \in V(G)$, then $det(G) \leq 30$.

In [3], Addario-Berry et al. proved that: (i) $det(G) \leq 4$, (ii) if $d_G(v) \geq 1000$ for every $v \in V(G)$, then $det(G) \leq 3$, and (iii) if $\chi(G) \leq 3$, then $det(G) \leq 3$.

In [4], among other results, Escuadro et al. proved that: (i) $det(K_{n_1,n_2,\ldots,n_k}) = 1$ if $n_1 < n_2 < \cdots < n_k$, $det(K_{n_1,n_2,\ldots,n_k}) = 3$ if $n_1 = n_2 = \cdots = n_k = 1$ and $det(K_{n_1,n_2,\ldots,n_k}) = 2$ otherwise, where $K_{n_1,n_2,\ldots,n_k}$ is the complete k-partite graph with partite sizes $n_1, n_2, \ldots, n_k$ ($k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_k$), (ii) $det(C_3 \,\square\, K_2) = 3$, $det(C_5 \,\square\, K_2) = 3$ and if $n \geq 7$ is an odd integer, then $det(C_n \,\square\, K_2) = 2$, where $\square$ denotes the Cartesian product, and (iii) if G is a unicyclic graph that is not a cycle, then $det(G) \leq 2$.

See Fig. 5 of [4]; detectable 3-edge-weighting of $C_3 \,\square\, K_2$ and that of $C_5 \,\square\, K_2$, in the figure, are vertex-coloring 3-edge-weightings. Hence, $\mu(C_3 \,\square\, K_2) = 3$ and $\mu(C_5 \,\square\, K_2) = 3$. If $n \geq 7$ is an odd integer, then it follows from $det(C_n \,\square\, K_2) = 2$ and Proposition 1.4 that $\mu(C_n \,\square\, K_2) = 2$.

Theorem 1.1. $det(C_3 \,\square\, K_2) = \mu(C_3 \,\square\, K_2) = 3$, $det(C_5 \,\square\, K_2) = \mu(C_5 \,\square\, K_2) = 3$ *and if* $n \geq 7$ *is an odd integer, then* $det(C_n \,\square\, K_2) = \mu(C_n \,\square\, K_2) = 2$.

From [5,6], and [4], we have:

Theorem 1.2. *For the path* P_n *on n vertices,* $det(P_3) = \mu(P_3) = 1$ *and* $det(P_n) = \mu(P_n) = 2$ *if* $n \geq 4$.

Theorem 1.3. *For the cycle* C_n *on n vertices,* $det(C_n) = \mu(C_n) = 2$ *if* $n \equiv 0 \,(mod\, 4)$ *and* $det(C_n) = \mu(C_n) = 3$ *if* $n \equiv 1,\ 2$ *or* $3 \,(mod\, 4)$.

Theorem 1.4. *For the complete graph* K_n *on* $n \geq 3$ *vertices,* $det(K_n) = \mu(K_n) = 3$.

Theorem 1.5. *For* $r + s \geq 3$, $det(K_{r,s}) = \mu(K_{r,s}) = 1$ *if* $r \neq s$ *and* $det(K_{r,s}) = \mu(K_{r,s}) = 2$ *if* $r = s$*, where* $K_{r,s}$ *is the complete bipartite graph with partite sizes r and s.*

The *theta graph* $\theta(\ell_1, \ell_2, \ldots, \ell_r)$ is the graph obtained from r disjoint paths $P_1(u_1, v_1)$, $P_2(u_2, v_2), \ldots, P_r(u_r, v_r)$ of lengths $\ell_1, \ell_2, \ldots, \ell_r$, respectively, by identifying their end-vertices $u := u_1 = u_2 = \cdots = u_r$ and $v := v_1 = v_2 = \cdots = v_r$, where $P_i(u_i, v_i)$ is a path of length ℓ_i with origin u_i and terminus v_i. Note that $\theta(\ell_1) = P_{\ell_1+1}$ and $\theta(\ell_1, \ell_2) = C_{\ell_1+\ell_2}$.

Theorem 1.6. *Let $G = \theta(\ell_1, \ell_2, \ldots, \ell_r)$ with $r \geq 3$, $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_r$, and $\ell_1 = 1$ implies $\ell_2 > 1$. Then $det(G) = \mu(G) = 1$ when $\ell_i = 2$ for all i; $det(G) = \mu(G) = 3$ when $\ell_1 = 1$ and $\ell_i \equiv 1\,(mod\,4)$ for all $i \neq 1$; and $det(G) = \mu(G) = 2$ otherwise.*

Proof of Theorem 1.6 follows from: the proof of Proposition 6 in [5], $det(G) \leq \mu(G)$, and the following: For $\ell_1 = 1$ and $\ell_i \equiv 1\,(mod\,4)$ for all $i \neq 1$, we claim that $det(G) \geq 3$. Suppose, to the contrary that G admits a detectable 2-edge-weighting c. Then, in each path the kth edge must have different weight from the $(k+2)$th edge, and has the same weight with the $(k+4)$th edge. Consequently, the first edge has the same weight with the last edge in each path of the theta graph. Then, $code_c(u) = code_c(v)$, however, this is impossible as u and v are adjacent.

Theorem 1.7. *Let G be a nice connected bipartite graph with bipartition (A, B) and G has at least one pair of adjacent vertices with the same degree. If one of the following conditions holds:*
(i) *$|A|$ or $|B|$ is even,*
(ii) *$\delta(G) = 1$,*
(iii) *$\left\lfloor \frac{d(u)}{2} \right\rfloor + 1 \neq d(v)$ for any edge $uv \in E(G)$,*
then $det(G) = \mu(G) = 2$.

Consequently,
(i) if G is a tree, then $det(G) = \mu(G) = 2$;
(ii) if G is r-regular with $r \geq 3$, then $det(G) = \mu(G) = 2$; and
(iii) if $\delta(G) \geq 4$ and $\Delta(G) + 3 \leq 2\delta(G)$, then $det(G) = \mu(G) = 2$.

The converse of Theorem 1.7 is in general not true. Consider the cycle C_{4n+2} of length $4n + 2$ ($n \geq 1$). For $G = C_{4n+2}$, both $|A|$ and $|B|$ are odd, $\delta(G) \neq 1$, $\left\lfloor \frac{d(u)}{2} \right\rfloor + 1 = d(v)$ for any edge $uv \in E(G)$, and $det(G) = \mu(G) = 3$. Next, consider the complete bipartite graph $K_{2n+1,4n+1}$ ($n \geq 1$). For $G = K_{2n+1,4n+1}$, both $|A|$ and $|B|$ are odd, $\delta(G) \neq 1$, $\left\lfloor \frac{d(u)}{2} \right\rfloor + 1 = d(v)$ for any edge $uv \in E(G)$ with $d(u) \geq d(v)$ and $det(G) = \mu(G) = 2$.

Theorem 1.8. *Let G be a nice graph and assume that G has at least one pair of adjacent vertices with the same degree. If $\delta(G) \geq 8\chi(G)$, then $det(G) = \mu(G) = 2$.*

Theorem 1.9. *Let G be nice, bipartite, and G has at least one pair of adjacent vertices with the same degree. If one of the following conditions holds:*
(i) *there exists a vertex v such that $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ and $G - v - N(v)$ is connected,*
(ii) *there exists a vertex v of degree $\delta(G)$ such that $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ and $G - v$ is connected,*
(iii) *G is 3-connected,*
(iv) *$\delta(G) \geq 3$ and there exists a vertex v of degree $\delta(G)$ such that $G - v - N(v)$ is connected,*
then $det(G) = \mu(G) = 2$.

In this paper, we have enlarged the known class of graphs with $det(G) = \mu(G) = 2$.

Let G_1 and G_2 be graphs. The *Cartesian product* $G_1 \Box G_2$ of G_1 and G_2 is the graph with $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \Box G_2$ if, and only if, either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $u_1u_2 \in E(G_1)$ and $v_1 = v_2$. The *tensor product* $G_1 \times G_2$ of G_1 and G_2 is the graph with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if, and only if, $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$.

2. Bipartite graphs

In this section, we find detectable 2-edge-weighting for some bipartite graphs.

Theorem 2.1. *Let G be a bipartite graph with bipartition (X, Y). If Y has a partition into two nonempty subsets Y_1 and Y_2, and if every vertex of X has at least one neighbor in Y_1 and one neighbor in Y_2, then $det(G) \leq 2$.*

Proof. Assign weight 1 to the edges with one end in Y_1 and 2 to the edges with one end in Y_2. Then, $code(y_1) = (d_G(y_1), 0)$ for every $y_1 \in Y_1$, and $code(y_2) = (0, d_G(y_2))$ for every $y_2 \in Y_2$. Now, let $x \in X$. If $code(x) = (\ell_1, \ell_2)$, then by hypothesis $\ell_1 \geq 1$ and $\ell_2 \geq 1$. Hence G has a detectable 2-edge-weighting. ■

Note that the partition in previous theorem is impossible for cycles C_{4n+2}, $n \geq 1$, and it is known that $det(C_{4n+2}) = 3$. Consider for $n \geq 1$, the graph G_{4n+2} obtained from C_{4n+2} by adding a pendant edge at only one vertex of C_{4n+2}. Let $G_{4n+2} := x_1 y_1 x_2 y_2 x_3 y_3 \ldots x_{2n+1} y_{2n+1} x_1 \oplus x_1 y$. Observe that the partition in previous theorem is impossible for G_{4n+2} and $det(G_{4n+2}) = 2$. $det(G_{4n+2}) = 2$ follows from the fact that G_{4n+2} is bipartite with $\delta(G_{4n+2}) = 1$.

3. Cartesian product of two graphs

Recently, in [7], we and Havet have shown that if G is bipartite and the minimum degree of G is at least 3, then $\det(G) \leq 2$.

In this section, we find some Cartesian products $G_1 \Box G_2$ of graphs G_1 and G_2 with $det(G_1 \Box G_2) = 2$ and some Cartesian products $H_1 \Box H_2$ of graphs H_1 and H_2 with $det(H_1 \Box H_2) = \mu(H_1 \Box H_2) = 2$.

Denote by $\mathscr{G}_3$, the set of tripartite graphs G with tripartition (X, Y, Z) such that for any $x \in X$, $y \in Y$ and $z \in Z$, $d_{G[X \cup Y]}(x) = r = d_{G[X \cup Y]}(y)$, $d_{G[X \cup Z]}(x) = s = d_{G[X \cup Z]}(z)$ and $d_{G[Y \cup Z]}(y) = t = d_{G[Y \cup Z]}(z)$; $r \geq 1$, $s \geq 1$, $t \geq 1$; i.e., the subgraphs induced by $X \cup Y$, $X \cup Z$ and $Y \cup Z$ are, respectively, r, s and t-regular.

Theorem 3.1. *If $G_1, G_2 \in \mathscr{G}_3$, then $det(G_1 \Box G_2) \leq 2$.*

Proof. Let (X', Y', Z') be the tripartition of G_1 such that for $x' \in X'$, $y' \in Y'$ and $z' \in Z'$, $d_{G_1[X' \cup Y']}(x') = r' = d_{G_1[X' \cup Y']}(y')$, $d_{G_1[X' \cup Z']}(x') = s' = d_{G_1[X' \cup Z']}(z')$ and $d_{G_1[Y' \cup Z']}(y') = t' = d_{G_1[Y' \cup Z']}(z')$; and let (X'', Y'', Z'') be the tripartition of G_2 such that for $x'' \in X''$, $y'' \in Y''$ and $z'' \in Z''$, $d_{G_2[X'' \cup Y'']}(x'') = r'' = d_{G_2[X'' \cup Y'']}(y'')$, $d_{G_2[X'' \cup Z'']}(x'') = s'' = d_{G_2[X'' \cup Z'']}(z'')$ and $d_{G_2[Y'' \cup Z'']}(y'') = t'' = d_{G_2[Y'' \cup Z'']}(z'')$. Define c as follows:

Assign weight 1 to edges having both ends in $X' \times V(G_2)$, to edges having both ends in $V(G_1) \times X''$, to edges having one end in $Z' \times X''$ and other end in $Z' \times Y''$, and to edges having one end in $X' \times Z''$ and other end in $Y' \times Z''$; assign weight 2 to edges having both ends in $Y' \times V(G_2)$, to edges having both ends in $V(G_1) \times Y''$, and to edges having one end in $Z' \times Z''$ and other end in $(Z' \times X'') \cup (Z' \times Y'') \cup (X' \times Z'') \cup (Y' \times Z'')$.

Let $x' \in X'$, $y' \in Y'$, $z' \in Z'$, $x'' \in X''$, $y'' \in Y''$, and $z'' \in Z''$.

Color code is given by:

$code_c((x', x'')) = (r' + s' + r'' + s'', 0)$,
$code_c((x', y'')) = (r'' + t'', r' + s')$,
$code_c((x', z'')) = (r' + s'' + t'', s')$,
$code_c((y', x'')) = (r' + t', r'' + s'')$,
$code_c((y', y'')) = (0, r' + t' + r'' + t'')$,
$code_c((y', z'')) = (r', t' + s'' + t'')$,
$code_c((z', x'')) = (s' + t' + r'', s'')$,
$code_c((z', y'')) = (r'', s' + t' + t'')$, and
$code_c((z', z'')) = (0, s' + t' + s'' + t'')$.

Hence c is a detectable 2-edge-weighting of $G_1 \Box G_2$. ■

Theorem 3.2. *If G is a k-regular bipartite graph, $k \geq 2$, and if $H \in \mathscr{G}_3$, then $det(G \Box H) \leq 2$.*

Proof. Let (A, B) be the bipartition of G, and let (X, Y, Z) be the tripartition of H such that for $x \in X$, $y \in Y$ and $z \in Z$, $d_{H[X \cup Y]}(x) = r = d_{H[X \cup Y]}(y)$, $d_{H[X \cup Z]}(x) = s = d_{H[X \cup Z]}(z)$ and $d_{H[Y \cup Z]}(y) = t = d_{H[Y \cup Z]}(z)$.

Define c as follows: Assign weight 1 to the edges having both ends in $A \times V(H)$, and edges having one end in $B \times Y$ and other end in $(A \times Y) \cup (B \times X)$; assign weight 2 to the edges having one end in $A \times X$ and other end in $B \times X$, and edges having one end in $B \times Z$ and other end in $(B \times X) \cup (B \times Y)$. Finally, we have to assign weights to the edges having one end in $A \times Z$ and other end in $B \times Z$.

For $a \in A$, $b \in B$, $x \in X$, $y \in Y$ and $z \in Z$, $code_c$ is given by:

$code_c((a, x)) = (r + s, k)$,
$code_c((a, y)) = (r + t + k, 0)$,
$code_c((b, x)) = (r, s + k)$, and
$code_c((b, y)) = (r + k, t)$.

Case 1. $|\{r, s, t\}| \geq 2$. Assume without loss of generality that $r \neq t$.

Assign weight 2 to the edges having one end in $A \times Z$ and other end in $B \times Z$. Now, $code_c((a, z)) = (s + t, k)$ and $code_c((b, z)) = (0, k + t + s)$.

Case 2. $r = s = t$.

$code_c((a, x)) = (2r, k)$, $code_c((a, y)) = (2r + k, 0)$, $code_c((b, x)) = (r, r + k)$, and $code_c((b, y)) = (r + k, r)$.

Subcase 2.1. $r \geq 2$.

Find a 1-factor F in the k-regular bipartite graph $(G \,\square\, H)[(A \times Z) \cup (B \times Z)]$. Assign weight 1 to the edges of F and the remaining edges having one end in $A \times Z$ and other end in $B \times Z$ are assigned weight 2. Now, $code_c((a, z)) = (2r + 1, k - 1)$ and $code_c((b, z)) = (1, 2r + k - 1)$.

Subcase 2.2. $r = 1$.

$code_c((a, x)) = (2, k)$, $code_c((a, y)) = (k + 2, 0)$, $code_c((b, x)) = (1, k + 1)$, and $code_c((b, y)) = (k + 1, 1)$.

If $k \geq 3$, find two edge-disjoint 1-factors F_1 and F_2 in the k-regular bipartite graph $(G \,\square\, H)[(A \times Z) \cup (B \times Z)]$. Assign weight 1 to the edges of $F_1 \cup F_2$ and the remaining edges having one end in $A \times Z$ and other end in $B \times Z$ are assigned weight 2. Now, $code_c((a, z)) = (4, k - 2)$ and $code_c((b, z)) = (2, k)$.

Finally, assume that $k = 2$. Interchange the weight for the edges having one end in $B \times X$ and other end in $B \times Y$ by 2. Find two edge-disjoint 1-factors F_1 and F_2 in the k-regular bipartite graph $(G \,\square\, H)[(A \times Z) \cup (B \times Z)]$. Assign weight 1 to the edges of F_1 and the edges of F_2 by 2. Now, $code_c((a, x)) = (2, 2)$, $code_c((a, y)) = (4, 0)$, $code_c((a, z)) = (3, 1)$, $code_c((b, x)) = (0, 4)$, $code_c((b, y)) = (2, 2)$, and $code_c((b, z)) = (1, 3)$.

In any case, c is a detectable 2-edge-weighting of $G \,\square\, H$. ■

For convenience, let $V(P_r) = V(C_r) = \{0, 1, 2, \ldots, r - 1\}$, $E(P_r) = \{\{i, i + 1\} : i \in \{0, 1, 2, \ldots, r - 2\}\}$ and $E(C_r) = E(P_r) \cup \{\{r - 1, 0\}\}$.

For any $n \geq 0$, $C_{6n+3} \in \mathscr{G}_3$; hence by previous theorem for any k-regular bipartite graph G with $k \geq 2$, we have $det(G \,\square\, C_{6n+3}) \leq 2$.

Theorem 3.3. *If G is a k-regular bipartite graph, $k \geq 2$, and if $n \geq 1$, then $det(G \,\square\, C_{2n+1}) = \mu(G \,\square\, C_{2n+1}) = 2$.*

Proof. Let (X, Y) be the bipartition of G. Define c as follows:

Case 1. $n \geq 2$.

Assign weight 1 to the edges having one end in $X \times \{0, 2, 4, \ldots, 2n\}$ and the other end in $Y \times \{0, 2, 4, \ldots, 2n\}$, edges having both ends in $X \times \{0, 1, 2, \ldots, 2n - 1\}$, and edges having both ends in $Y \times \{2n - 2, 2n - 1, 2n\}$; and assign weight 2 to the edges having one end in $X \times \{1, 3, 5, \ldots, 2n - 1\}$ and the other end in $Y \times \{1, 3, 5, \ldots, 2n - 1\}$, edges having both ends in $X \times \{2n - 1, 2n, 0\}$, and edges having both ends in $Y \times \{2n, 0, 1, 2, \ldots, 2n - 2\}$. $code_c$ is given by: for $x \in X$ and $y \in Y$,

$code_c((x, i)) = (2, k)$ if $i \in \{1, 3, 5, \ldots, 2n - 3\}$;
$code_c((x, i)) = (k + 2, 0)$ if $i \in \{2, 4, 6, \ldots, 2n - 2\}$;
$code_c((x, 0)) = (k + 1, 1)$;
$code_c((x, 2n - 1)) = (1, k + 1)$;
$code_c((x, 2n)) = (k, 2)$;
$code_c((y, i)) = (0, k + 2)$ if $i \in \{1, 3, 5, \ldots, 2n - 3\}$;
$code_c((y, i)) = (k, 2)$ if $i \in \{0, 2, 4, \ldots, 2n - 4\}$;
$code_c((y, 2n - 2)) = (k + 1, 1) = code_c((y, 2n))$; and
$code_c((y, 2n - 1)) = (2, k)$.

Case 2. $n = 1$.

Subcase 2.1. $k \geq 3$.

Assign weight 1 to the edges having one end in $X \times \{1\}$ and the other end in $Y \times \{1\}$, edges having both ends in $X \times \{0, 1, 2\}$, and edges having both ends in $Y \times \{0, 1\}$; and assign weight 2 to the edges having one end in $X \times \{0\}$ and the other end in $Y \times \{0\}$, edges having both ends in $Y \times \{1, 2\}$, and edges having both ends in $Y \times \{2, 0\}$. Find two edge-disjoint 1-factors F_1 and F_2 in the k-regular bipartite subgraph induced by the partite sets $X \times \{2\}$ and $Y \times \{2\}$. Assign weight 1 to the edges of $F_1 \cup F_2$ and the remaining edges having one end in $X \times \{2\}$ and other end in $Y \times \{2\}$ are by 2. $code_c$ is given by: for $x \in X$ and $y \in Y$,

$code_c((x, 0)) = (2, k)$; $code_c((x, 1)) = (k + 2, 0)$; $code_c((x, 2)) = (4, k - 2)$;

$code_c((y, 0)) = (1, k + 1)$; $code_c((y, 1)) = (k + 1, 1)$; $code_c((y, 2)) = (2, k)$.

Subcase 2.2. $k = 2$.

Assign weight 1 to the edges having one end in $X \times \{1\}$ and the other end in $Y \times \{1\}$, and edges having both ends in $X \times \{0, 1, 2\}$; and assign weight 2 to the edges having one end in $X \times \{0\}$ and the other end in $Y \times \{0\}$, and edges having both ends in $Y \times \{0, 1, 2\}$. Find two edge-disjoint 1-factors F_1 and F_2 in the 2-regular bipartite subgraph induced by the partite sets $X \times \{2\}$ and $Y \times \{2\}$. Assign weight 1 to the edges of F_1 and 2 to the edges of F_2. Now,

$code_c((x, 0)) = (2, 2)$; $code_c((x, 1)) = (4, 0)$; $code_c((x, 2)) = (3, 1)$;

$code_c((y, 0)) = (0, 4)$; $code_c((y, 1)) = (2, 2)$; $code_c((y, 2)) = (1, 3)$.

In any case, the 2-edge-weighting c of $G \,\square\, C_{2n+1}$ is detectable and hence $det(G \,\square\, C_{2n+1}) = 2$. By Proposition 1.4, $\mu(G \,\square\, C_{2n+1}) = 2$. ∎

Theorem 3.4. *If* $m, n \geq 3$, *then* $det(C_m \,\square\, C_n) = \mu(C_m \,\square\, C_n) = 2$.

Proof. If both m and n are even, then $C_m \,\square\, C_n$ is a 4-regular bipartite graph and hence the result follows from the result quoted in the beginning of this section, and Propositions 1.2 and 1.4. If m and n are of opposite parity, say, m is odd and n is even, then the result follows from Theorem 3.3. Hence, assume that both m and n are odd.

Define c as follows:

Assign weight 1 to the edges having both ends in $\{0, 2, 4, \ldots, m - 3\} \times V(C_n)$, and edges having both ends in $V(C_m) \times \{0, 2, 4, \ldots, n - 3\}$; assign weight 2 to the edges having both ends in $\{1, 3, 5, \ldots, m - 2\} \times V(C_n)$, and edges having both ends in $V(C_m) \times \{1, 3, 5, \ldots, n - 2\}$;

$c((m - 1, j)(m - 1, j + 1)) = 1$ if $j \in \{1, 3, 5, \ldots, n - 2\}$;

$c((m - 1, j)(m - 1, j + 1)) = 2$ if $j \in \{0, 2, 4, \ldots, n - 3\}$;

$c((m - 1, n - 1)(m - 1, 0)) = 1$;

$c((i, n - 1)(i + 1, n - 1)) = 1$ if $i \in \{1, 3, 5, \ldots, m - 2\}$;

$c((i, n - 1)(i + 1, n - 1)) = 2$ if $i \in \{0, 2, 4, \ldots, m - 3\}$; and

$c((m - 1, n - 1)(0, n - 1)) = 1$.

$code_c$ is given by:

$code_c((i, j)) = (4, 0)$ if $i \in \{0, 2, 4, \ldots, m - 3\}$ and $j \in \{0, 2, 4, \ldots, n - 3\}$;

$code_c((i, j)) = (0, 4)$ if $i \in \{1, 3, 5, \ldots, m - 2\}$ and $j \in \{1, 3, 5, \ldots, n - 2\}$;

$code_c((m - 1, j)) = (3, 1)$ if $j \in \{0, 2, 4, \ldots, n - 3\}$;

$code_c((m - 1, j)) = (1, 3)$ if $j \in \{1, 3, 5, \ldots, n - 2\}$;

$code_c((i, n - 1)) = (3, 1)$ if $i \in \{0, 2, 4, \ldots, m - 3\}$;

$code_c((i, n - 1)) = (1, 3)$ if $i \in \{1, 3, 5, \ldots, m - 2\}$;

$code_c((m - 1, n - 1)) = (4, 0)$; and

$code_c((i, j)) = (2, 2)$ otherwise.

This 2-edge-weighting c is detectable and hence $det(C_m \,\square\, C_n) = 2$. By Proposition 1.4, $\mu(C_m \,\square\, C_n) = 2$. ∎

Recently, in [8], Davoodi and Omoomi have shown that if G and H are two connected bipartite graphs and $G \square H \neq K_2$, then $\mu(G \square H) \leq 2$.

Theorem 3.5. *If* $m, n \geq 3$, *then* $det(C_m \,\square\, P_n) = \mu(C_m \,\square\, P_n) = 2$.

Proof. If m is even, then the result follows from the above result of Davoodi and Omoomi, and Propositions 1.2 and 1.3. Hence, assume that m is odd. We consider two cases.

Case 1. n is odd.

Define c as follows: Assign weight 1 to the edges having both ends in $\{1, 3, 5, \ldots, m - 2\} \times V(P_n)$, and edges having both ends in $V(C_m) \times \{2, 4, 6, \ldots, n - 3\}$; assign weight 2 to the edges having both ends in $\{0, 2, 4, \ldots, m - 3\} \times V(P_n)$, and edges having both ends in $V(C_m) \times \{1, 3, 5, \ldots, n - 2\}$;

$c((m - 1, j)(m - 1, j + 1)) = 1$ if $j \in \{0, 2, 4, \ldots, n - 3\}$;

$c((m - 1, j)(m - 1, j + 1)) = 2$ if $j \in \{1, 3, 5, \ldots, n - 2\}$;

$c((i, 0)(i + 1, 0)) = 1$ if $i \in \{0, 1, 2, \ldots, m - 2\}$;

$c((m - 1, 0)(0, 0)) = 2$;

$c((i, n - 1)(i + 1, n - 1)) = 1$ if $i \in \{0, 2, 4, \ldots, m - 3\}$;

$c((i, n-1)(i+1, n-1)) = 2$ if $i \in \{1, 3, 5, \ldots, m-2\}$; and
$c((m-1, n-1)(0, n-1)) = 2$.
f_c is given by:
$f_c((i, j)) = 8$ if $i \in \{0, 2, 4, \ldots, m-3\}$ and $j \in \{1, 3, 5, \ldots, n-2\}$;
$f_c((i, j)) = 4$ if $i \in \{1, 3, 5, \ldots, m-2\}$ and $j \in \{2, 4, 6, \ldots, n-3\}$;
$f_c((0, 0)) = 5$;
$f_c((i, 0)) = 3$ if $i \in \{1, 3, 5, \ldots, m-2\}$;
$f_c((i, 0)) = 4$ if $i \in \{2, 4, 6, \ldots, m-1\}$;
$f_c((i, n-1)) = 5$ if $i \in \{0, 2, 4, \ldots, m-3\}$;
$f_c((i, n-1)) = 4$ if $i \in \{1, 3, 5, \ldots, m-2\}$;
$f_c((m-1, n-1)) = 6$;
$f_c((m-1, j)) = 7$ if $j \in \{1, 3, 5, \ldots, n-2\}$;
$f_c((m-1, j)) = 5$ if $j \in \{2, 4, 6, \ldots, n-3\}$;
$f_c((i, j)) = 6$ otherwise.

Case 2. n is even.

Define c as follows: Assign weight 1 to the edges having both ends in $\{0, 2, 4, \ldots, m-3\} \times V(P_n)$, and edges having both ends in $V(C_m) \times \{2, 4, 6, \ldots, n-2\}$; assign weight 2 to the edges having both ends in $\{1, 3, 5, \ldots, m-2\} \times V(P_n)$, and edges having both ends in $V(C_m) \times \{1, 3, 5, \ldots, n-3\}$;
$c((m-1, j)(m-1, j+1)) = 1$ if $j \in \{0, 2, 4, \ldots, n-2\}$;
$c((m-1, j)(m-1, j+1)) = 2$ if $j \in \{1, 3, 5, \ldots, n-3\}$;
$c((i, 0)(i+1, 0)) = 1$ if $i \in \{0, 1, 2, \ldots, m-3\}$;
$c((m-2, 0)(m-1, 0)) = 2$;
$c((m-1, 0)(0, 0)) = 1$;
$c((i, n-1)(i+1, n-1)) = 1$ if $i \in \{0, 1, 2, \ldots, m-3\}$;
$c((m-2, n-1)(m-1, n-1)) = 2$; and
$c((m-1, n-1)(0, n-1)) = 1$.
f_c is given by:
$f_c((i, j)) = 8$ if $i \in \{1, 3, 5, \ldots, m-2\}$ and $j \in \{1, 3, 5, \ldots, n-3\}$;
$f_c((i, j)) = 4$ if $i \in \{0, 2, 4, \ldots, m-3\}$ and $j \in \{2, 4, 6, \ldots, n-2\}$;
$f_c((i, 0)) = 3$ if $i \in \{0, 2, 4, \ldots, m-3\}$;
$f_c((i, 0)) = 4$ if $i \in \{1, 3, 5, \ldots, m-4\}$;
$f_c((m-2, 0)) = 5$;
$f_c((m-1, 0)) = 4$;
$f_c((i, n-1)) = 3$ if $i \in \{0, 2, 4, \ldots, m-3\}$;
$f_c((i, n-1)) = 4$ if $i \in \{1, 3, 5, \ldots, m-4\}$;
$f_c((m-2, n-1)) = 5$;
$f_c((m-1, n-1)) = 4$;
$f_c((m-1, j)) = 7$ if $j \in \{1, 3, 5, \ldots, n-3\}$;
$f_c((m-1, j)) = 5$ if $j \in \{2, 4, 6, \ldots, n-2\}$;
$f_c((i, j)) = 6$ otherwise.

In any case, the 2-edge-weighting c is a vertex-coloring and hence $\mu(C_m \square P_n) = 2$. By Proposition 1.3, $det(C_m \square P_n) = 2$. ■

Denote by $\mathscr{G}_b^{(2)}$, the set of graphs $G = (V, E)$ for which there exists a partition (X, Y) of V such that
(i) if $x', x'' \in X$ and $x'x'' \in E$, then $|d_G(x') - d_G(x'')| \geq 2$; and
(ii) if $y', y'' \in Y$ and $y'y'' \in E$, then $|d_G(y') - d_G(y'')| \geq 2$.
Clearly, (i) if G is bipartite, then $G \in \mathscr{G}_b^{(2)}$; and (ii) if $G \in \mathscr{G}_b^{(2)}$ is regular, then G is bipartite.

Theorem 3.6. *If $G \in \mathscr{G}_b^{(2)}$, then $det(G \square K_2) = \mu(G \square K_2) \leq 2$.*

Proof. Let $V(G) = V$, $E(G) = E$, $\Delta(G) = \Delta$, the maximum degree of G, and $V(K_2) = \{0, 1\}$. By the definition of $\mathscr{G}_b^{(2)}$, there exists a partition (X, Y) of V such that: if $x', x'' \in X$ and $x'x'' \in E$, then $|d_G(x') - d_G(x'')| \geq 2$; and if $y', y'' \in Y$ and $y'y'' \in E$, then $|d_G(y') - d_G(y'')| \geq 2$. For $1 \leq i \leq \Delta$, set $X_i = \{x \in X : d_G(x) = i\}$ and $Y_i = \{y \in Y : d_G(y) = i\}$.

Now we give a 2-edge-weighting c for $G \square K_2$. Assign:
weight 1 to the edges with ends in $V \times \{0\}$;
weight 2 to the edges with ends in $V \times \{1\}$;
for odd i, weight 1 to the edges with one end in $X_i \times \{0\}$ and other end in $X_i \times \{1\}$;
for even i, weight 2 to the edges with one end in $X_i \times \{0\}$ and other end in $X_i \times \{1\}$;
for odd i, weight 2 to the edges with one end in $Y_i \times \{0\}$ and other end in $Y_i \times \{1\}$;
for even i, weight 1 to the edges with one end in $Y_i \times \{0\}$ and other end in $Y_i \times \{1\}$.
Next, we compute f_c for adjacent vertices of $G \square K_2$.

• Let $x \in X$. Then $x \in X_i$ for some i with $1 \leq i \leq \Delta$. Hence,

$$f_c((x,0)) = \begin{cases} i+1 & \text{if } i \text{ is odd,} \\ i+2 & \text{if } i \text{ is even;} \end{cases} \quad \text{and} \quad f_c((x,1)) = \begin{cases} 2i+1 & \text{if } i \text{ is odd,} \\ 2i+2 & \text{if } i \text{ is even.} \end{cases}$$

Consequently, $f_c((x,0)) \neq f_c((x,1))$.

• Let $y \in Y$. Then $y \in Y_i$ for some i with $1 \leq i \leq \Delta$. Hence,

$$f_c((y,0)) = \begin{cases} i+2 & \text{if } i \text{ is odd,} \\ i+1 & \text{if } i \text{ is even;} \end{cases} \quad \text{and} \quad f_c((y,1)) = \begin{cases} 2i+2 & \text{if } i \text{ is odd,} \\ 2i+1 & \text{if } i \text{ is even.} \end{cases}$$

Consequently, $f_c((y,0)) \neq f_c((y,1))$.

• Let $x', x'' \in X$ and $x'x'' \in E$. Then $|d_G(x') - d_G(x'')| \geq 2$. Without loss of generality, assume that $x' \in X_i$, $x'' \in X_j$ with $1 \leq i < j \leq \Delta$. As $|d_G(x') - d_G(x'')| \geq 2$, $j - i \geq 2$. Hence,

$$f_c((x',0)) = \begin{cases} i+1 & \text{if } i \text{ is odd,} \\ i+2 & \text{if } i \text{ is even;} \end{cases} \qquad f_c((x',1)) = \begin{cases} 2i+1 & \text{if } i \text{ is odd,} \\ 2i+2 & \text{if } i \text{ is even;} \end{cases}$$
$$f_c((x'',0)) = \begin{cases} j+1 & \text{if } j \text{ is odd,} \\ j+2 & \text{if } j \text{ is even;} \end{cases} \quad \text{and} \quad f_c((x'',1)) = \begin{cases} 2j+1 & \text{if } j \text{ is odd,} \\ 2j+2 & \text{if } j \text{ is even.} \end{cases}$$

As $j \geq i + 2$, $f_c((x',0)) \neq f_c((x'',0))$ and $f_c((x',1)) \neq f_c((x'',1))$.

• Let $y', y'' \in Y$ and $y'y'' \in E$. Then $|d_G(y') - d_G(y'')| \geq 2$. Without loss of generality, assume that $y' \in Y_i$, $y'' \in Y_j$ with $1 \leq i < j \leq \Delta$. As $|d_G(y') - d_G(y'')| \geq 2$, $j - i \geq 2$. Hence,

$$f_c((y',0)) = \begin{cases} i+2 & \text{if } i \text{ is odd,} \\ i+1 & \text{if } i \text{ is even;} \end{cases} \qquad f_c((y',1)) = \begin{cases} 2i+2 & \text{if } i \text{ is odd,} \\ 2i+1 & \text{if } i \text{ is even;} \end{cases}$$
$$f_c((y'',0)) = \begin{cases} j+2 & \text{if } j \text{ is odd,} \\ j+1 & \text{if } j \text{ is even;} \end{cases} \quad \text{and} \quad f_c((y'',1)) = \begin{cases} 2j+2 & \text{if } j \text{ is odd,} \\ 2j+1 & \text{if } j \text{ is even.} \end{cases}$$

As $j \geq i + 2$, $f_c((y',0)) \neq f_c((y'',0))$ and $f_c((y',1)) \neq f_c((y'',1))$.

• Let $x \in X$, $y \in Y$ and $xy \in E$. Then, $x \in X_i$, $y \in Y_j$ with $1 \leq i, j \leq \Delta$.

$$f_c((x,0)) = \begin{cases} i+1 & \text{if } i \text{ is odd,} \\ i+2 & \text{if } i \text{ is even;} \end{cases} \qquad f_c((y,0)) = \begin{cases} j+2 & \text{if } j \text{ is odd,} \\ j+1 & \text{if } j \text{ is even;} \end{cases}$$
$$f_c((x,1)) = \begin{cases} 2i+1 & \text{if } i \text{ is odd,} \\ 2i+2 & \text{if } i \text{ is even;} \end{cases} \quad \text{and} \quad f_c((y,1)) = \begin{cases} 2j+2 & \text{if } j \text{ is odd,} \\ 2j+1 & \text{if } j \text{ is even.} \end{cases}$$

Since $f_c((x,0))$ is even and $f_c((y,0))$ is odd, we have $f_c((x,0)) \neq f_c((y,0))$. Since $f_c((x,1)) \equiv 2$ or $3 \ (mod\ 4)$ and $f_c((y,1)) \equiv 0$ or $1 \ (mod\ 4)$, we have $f_c((x,1)) \neq f_c((y,1))$.

This completes the proof of $\mu(G \square K_2) \leq 2$ and $det(G \square K_2) = \mu(G \square K_2)$ follows from this inequality and Propositions 1.2 and 1.3. ■

Theorem 3.7. *For positive integers n_1, n_2, n_3, with $(n_1, n_2, n_3) \neq (1, 1, 1)$, $det(K_{n_1,n_2,n_3} \square K_2) = \mu(K_{n_1,n_2,n_3} \square K_2) = 2$.*

Proof. Let $V(K_2) = \{0, 1\}$ and $V = V(K_{n_1,n_2,n_3}) = V_1 \cup V_2 \cup V_3$, where, for $i \in \{1, 2, 3\}$, V_i is an independent set of cardinality n_i. Without loss of generality, assume that $n_1 \leq n_2 \leq n_3$. If $n_3 - n_1 \geq 2$, then $K_{n_1,n_2,n_3} \in \mathscr{G}_b^{(2)}$, to see this take the set V_2 for one part and $V_1 \cup V_3$ for other part. In this case, theorem follows from Theorem 3.6. Hence, assume that $n_3 - n_1 \leq 1$. We consider three cases and in each case we give a 2-edge-weighting c for $K_{n_1,n_2,n_3} \square K_2$.

Case 1. $n_1 + 1 = n_2 = n_3$.

Let $n = n_1 + 1 = n_2 = n_3$. Assign:
weight 1 to the edges with ends in $V \times \{0\}$;
weight 2 to the edges with ends in $V \times \{1\}$;
weight 1 to the edges with one end in $V_2 \times \{0\}$ and other end in $V_2 \times \{1\}$;
weight 2 to the edges with one end in $(V_1 \cup V_3) \times \{0\}$ and other end in $(V_1 \cup V_3) \times \{1\}$.

Next, we compute f_c. For $v_1 \in V_1, v_2 \in V_2, v_3 \in V_3$, $f_c((v_1, 0)) = 2n+2$, $f_c((v_2, 0)) = 2n$, $f_c((v_3, 0)) = 2n+1$, $f_c((v_1, 1)) = 4n + 2$, $f_c((v_2, 1)) = 4n - 1$, $f_c((v_3, 1)) = 4n$. As $n \neq 1$, the 2-edge-weighting c is a vertex-coloring.

Case 2. $n_1 = n_2 = n_3 - 1$.

Let $n = n_1 = n_2 = n_3 - 1$. Assign:
weight 1 to the edges with ends in $V \times \{0\}$;
weight 2 to the edges with ends in $V \times \{1\}$;
weight 1 to the edges with one end in $(V_1 \cup V_3) \times \{0\}$ and other end in $(V_1 \cup V_3) \times \{1\}$;
weight 2 to the edges with one end in $V_2 \times \{0\}$ and other end in $V_2 \times \{1\}$.

Next, we compute f_c. For $v_1 \in V_1$, $v_2 \in V_2$, $v_3 \in V_3$, $f_c((v_1, 0)) = 2n + 2$, $f_c((v_2, 0)) = 2n + 3$, $f_c((v_3, 0)) = 2n + 1$, $f_c((v_1, 1)) = 4n + 3$, $f_c((v_2, 1)) = 4n + 4$, $f_c((v_3, 1)) = 4n + 1$. For any n, the 2-edge-weighting c is a vertex-coloring. Note that for $n = 1$, $f_c((v_2, 0)) = 5 = f_c((v_3, 1))$ and the set $(V_2 \times \{0\}) \cup (V_3 \times \{1\})$ is an independent set in $K_{n_1,n_2,n_3} \square K_2$.

Case 3. $n_1 = n_2 = n_3$.

Let $n = n_1 = n_2 = n_3 \geq 2$. Choose two edge-disjoint 1-factors F_1, F_2 in the subgraph induced by the edges with one end in $V_2 \times \{0\}$ and other end in $V_3 \times \{0\}$ and choose a 1-factor F in the subgraph induced by the edges with one end in $V_2 \times \{1\}$ and other end in $V_3 \times \{1\}$. Assign:
weight 2 to the edges of $F_1 \cup F_2$;
weight 1 to the edges with ends in $V \times \{0\}$ but not belonging to $F_1 \cup F_2$;
weight 1 to the edges of F;
weight 2 to the edges with ends in $V \times \{1\}$ but not belonging to F;
weight 1 to the edges with one end in $V_2 \times \{0\}$ and other end in $V_2 \times \{1\}$;
weight 2 to the edges with one end in $(V_1 \cup V_3) \times \{0\}$ and other end in $(V_1 \cup V_3) \times \{1\}$.

Next, we compute f_c. For $v_1 \in V_1$, $v_2 \in V_2$, $v_3 \in V_3$, $f_c((v_1, 0)) = 2n + 2$, $f_c((v_2, 0)) = 2n + 3$, $f_c((v_3, 0)) = 2n + 4$, $f_c((v_1, 1)) = 4n + 2$, $f_c((v_2, 1)) = 4n$, $f_c((v_3, 1)) = 4n + 1$. For any n, the 2-edge-weighting c is a vertex-coloring. Note that for $n = 2$, $f_c((v_3, 0)) = 8 = f_c((v_2, 1))$ and the set $(V_3 \times \{0\}) \cup (V_2 \times \{1\})$ is an independent set in $K_{2,2,2} \square K_2$. ■

4. Tensor product of two graphs

In this section, we find some tensor product $G_1 \times G_2$ of graphs G_1 and G_2 with $det(G_1 \times G_2) = \mu(G_1 \times G_2) = 2$.

For $i \in \{0, 1, \ldots, m - 1\}$, let $R_i = \{(i, j) \mid j \in \{0, 1, \ldots, n - 1\}\}$; and for $j \in \{0, 1, \ldots, n - 1\}$, let $C_j = \{(i, j) \mid i \in \{0, 1, \ldots, m - 1\}\}$.

Consider $C_m \times G$, where G is any graph with $V(G) = \{0, 1, \ldots, n - 1\}$. For $i \in \{0, 1, \ldots, m - 2\}$, we denote by E_i the set of edges having one end in R_i and other end in R_{i+1}; and denote by E_{m-1} the set of edges having one end in R_{m-1} and other end in R_0.

Theorem 4.1. *Let G be a k-regular graph, $k \geq 2$, containing a 2-factor F. Then $det(C_m \times G) = \mu(C_m \times G) = 2$.*

Proof. If $m \equiv 0 \ (mod\ 2)$, then as $C_m \times G$ is bipartite and $2k$-regular, the result follows from the result quoted in the beginning of Section 3, and Propositions 1.2 and 1.4. For $m \equiv 1 \ (mod\ 2)$, we consider two cases.

Case 1. $m \equiv 1\ (mod\ 4)$.

Define c as follows:

If $i \in \{0, 4, 8, \ldots, m-5, m-1\} \cup \{3, 7, 11, \ldots, m-6\}$, then assign weight 1 to the edges of E_i. If $i \in \{1, 5, 9, \ldots, m-4\} \cup \{2, 6, 10, \ldots, m-3\}$, then assign weight 2 to the edges of E_i.

Finally, consider E_{m-2}. For each cycle $j_0 j_1 j_2 \ldots j_k j_0$ in F, assign weight 2 to the edges $(m-2, j_0)(m-1, j_1)$, $(m-2, j_1)(m-1, j_2)$, $(m-2, j_2)(m-1, j_3)$, …, $(m-2, j_{k-1})(m-1, j_k)$ and $(m-2, j_k)(m-1, j_0)$. Assign weight 1 to the remaining edges of E_{m-2}. In other words, the edges of a 1-factor of the subgraph induced by E_{m-2} are assigned weight 2 and the edges of the remaining $(k-1)$-factor of the subgraph are assigned weight 1.

$code_c$ is given by: for $j \in V(G)$,

$code_c((i, j)) = (k, k)$ if $i \in \{1, 3, 5, \ldots, m-4\}$,

$code_c((i, j)) = (2k, 0)$ if $i \in \{0, 4, 8, \ldots, m-5\}$,

$code_c((i, j)) = (0, 2k)$ if $i \in \{2, 6, 10, \ldots, m-3\}$,

$code_c((m-2, j)) = (k-1, k+1)$, and

$code_c((m-1, j)) = (2k-1, 1)$.

Case 2. $m \equiv 3\ (mod\ 4)$.

First, assume that $m \neq 3$.

Define c as follows:

If $i \in \{0, 4, 8, \ldots, m-7\} \cup \{3, 7, 11, \ldots, m-8\} \cup \{m-1\}$, then assign weight 1 to the edges of E_i. If $i \in \{1, 5, 9, \ldots, m-6, m-2\} \cup \{2, 6, 10, \ldots, m-5\}$, then assign weight 2 to the edges of E_i.

Now, consider E_{m-4}. For each cycle $j_0 j_1 j_2 \ldots j_k j_0$ in F, assign weight 1 to the edges $(m-4, j_0)(m-3, j_1)$, $(m-4, j_1)(m-3, j_2)$, $(m-4, j_2)(m-3, j_3)$, …, $(m-4, j_{k-1})(m-3, j_k)$ and $(m-4, j_k)(m-3, j_0)$. Assign weight 2 to the remaining edges of E_{m-4}.

Finally, consider E_{m-3}. For each cycle $j_0 j_1 j_2 \ldots j_k j_0$ in F, assign weight 1 to the edges $(m-3, j_0)(m-2, j_1)$, $(m-3, j_1)(m-2, j_2)$, $(m-3, j_2)(m-2, j_3)$, …, $(m-3, j_{k-1})(m-2, j_k)$ and $(m-3, j_k)(m-2, j_0)$. Assign weight 2 to the remaining edges of E_{m-3}.

$code_c$ is given by: for $j \in V(G)$,

$code_c((i, j)) = (k, k)$ if $i \in \{1, 3, 5, \ldots, m-6\} \cup \{m-1\}$,

$code_c((i, j)) = (2k, 0)$ if $i \in \{0, 4, 8, \ldots, m-7\}$,

$code_c((i, j)) = (0, 2k)$ if $i \in \{2, 6, 10, \ldots, m-5\}$,

$code_c((m-4, j)) = code_c((m-2, j)) = (1, 2k-1)$, and

$code_c((m-3, j)) = (2, 2k-2)$.

Finally, assume that $m = 3$.

Define c as follows:

Assign weight 2 to all the edges of E_1, and assign weight 1 to all the edges of E_2.

Now consider E_0. For each cycle $j_0 j_1 j_2 \ldots j_k j_0$ in F, assign weight 2 to the edges $(0, j_0)(1, j_1)$, $(0, j_1)(1, j_2)$, $(0, j_2)(1, j_3)$, …, $(0, j_{k-1})(1, j_k)$ and $(0, j_k)(1, j_0)$. Assign weight 1 to the remaining edges of E_0.

$code_c$ is given by: for $j \in V(G)$,

$code_c((0, j)) = (2k-1, 1)$,

$code_c((1, j)) = (k-1, k+1)$, and

$code_c((2, j)) = (k, k)$.

In any case, the 2-edge-weighting c of $C_m \times G$ is detectable. Hence, $det(C_m \times G) = 2$. By Proposition 1.4, $\mu(C_m \times G) = 2$. ■

Corollary 4.1. *For $m, n \geq 3$, $det(C_m \times C_n) = \mu(C_m \times C_n) = 2$.*

5. Conclusion

In conclusion, we ask: *does there exist a graph G with $det(G) \neq \mu(G)$?*

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