Flag-transitive $C_3$-geometries

Antonio Pasini
Department of Mathematics, University of Siena, Via del Capitano 15, 53100 Siena, Italy

Received 28 June 1990

Abstract

We obtain conditions on the structure and the parameters of an anomalous finite thick flag-transitive $C_3$-geometry.

1. Introduction

Let $\Gamma$ denote a residually connected finite $C_3$-geometry with thick lines, admitting parameters $x, y, z$:

<table>
<thead>
<tr>
<th>points</th>
<th>lines</th>
<th>planes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$x$</td>
<td>$y$</td>
</tr>
</tbody>
</table>

and let $z$ be the Ott–Liebler number of $\Gamma$ (see [12]). This means that $\Gamma$ consists of a set $S_0$ of points, a set $S_1$ of lines and a set $S_2$ of planes together with an incidence relation $*$ such that:

(1) For each plane $u$, the points and lines incident with $u$ constitute a projective plane of finite order $x > 1$.
(2) For each point $a$, the lines and planes incident with $a$ constitute a generalized quadrangle of finite orders $x, y$.
(3) For every line $r$, every point incident with $r$ and every plane incident with $r$ are incident.
(4) The graph defined by the incidence relation $*$ is connected.

For every $i=0, 1, 2$, $\sigma_i$ will be the shadow operator relative to $S_i$. Given a point–plane flag $(a, u)$ in $\Gamma$, let $x$ be the number of planes $v$ incident with $a$, collinear with $u$, distinct from $u$ and such that the line incident with $u$ and $v$ does not pass through $a$. This number $x$ will be called the Ott–Liebler number of $\Gamma$.
We shall shortly write $A$ instead of $\text{Aut}(\Gamma)$ to denote the automorphism group of $\Gamma$. It is easily seen that $A$ acts faithfully both on $S_2$ and on $S_1$, because $\ast$ induces a partial plane on $S_1 \cup S_2$. But $A$ need not act faithfully on $S_0$. The kernel of the action of $A$ on $S_0$ will be denoted by $K$ and we set $\overline{A} = A/K$.

The geometry $\Gamma$ is flat if all of its points are incident with all of its planes. If $\Gamma$ is neither a building nor flat, then we say that it is anomalous. This definition is motivated by the fact that no such anomalous example is presently known (apart from nonthick ones). Anyway, just one example is presently known of a nonbuilding finite $C_3$-geometry with thick lines, namely, the so-called $\mathcal{A}_7$-geometry (or 7-geometry). It is flat with parameters $x = y = 2$ and its automorphism group is the alternating group $\mathfrak{A}_7$, in its natural action of degree 7. The reader is referred to [1, 16] for further details.

The following theorem gives some necessary conditions for $\Gamma$ to be both anomalous and flag-transitive.

**Theorem 1.1.** Let $\Gamma$ be anomalous with a flag-transitive automorphism group $A$. Then the following hold:

(A) The number $x$ is even, $1 + x + x^2$ is prime and $x + 1 \equiv 0 (\mod 3)$. We have $x^2 - x - y > x$. $(x + y)(x + 1)$ divides $(1 + xy)(xy - \alpha/x)$ and $(x^2 + y)(x + 1)$ divides $(1 + x^2 y)(x^2 y - \alpha/x)$. Let $d = (x^2, y)$ be the greatest common divisor of $x^2$ and $y$. Then $x > d^2, y \geq (x - 1)d^2 + d, xd$ divides $\alpha$ and $x + 1$ divides $xy/d - \alpha/xd$.

(B) The stabilizer $A_u$ in $A$ of a plane $u$ of $\Gamma$ acts on the residue $\Gamma_u$ of $u$ as a Frobenius group of order $(1 + x)(1 + x + x^2)$, regular on the set of flags of $\Gamma_u$, with Frobenius kernel cyclic of order $1 + x + x^2$ regular on the set of points (lines) of $\Gamma_u$, and the Frobenius complements are stabilizers of antiflags of $\Gamma_u$, cyclic of order $x + 1$.

(C) Either $y$ is odd or $A$ acts imprimitively on the set $S_0$ of points of $\Gamma$.

We might give some more information in (C) (see the remarks at the end of this paper), but it would not yet be sufficient to obtain very severe restrictions.

We observe that, by (A) of Theorem 1.1, flag-transitive finite thick anomalous $C_3$-geometries cannot admit ‘known’ parameters in the sense of [12].

**Remark.** We note that the conditions given in (A) do not seem to fit with the Bruck–Ryser condition on orders of finite projective planes (that condition must hold on $x$, of course) and with the divisibility condition $x^2(x^2 - 1) \equiv 0 (\mod x + y)$ ([15, 1.2.2]). Dr. U. Ciocca (UCES, Siena) has tested them by a computer and it turned out that they never hold together when $x \leq 1000$.

The next theorem immediately follows from Theorem 1.1 and [11].

**Theorem 1.2.** Let the automorphism group of $\Gamma$ be flag-transitive. Then one of the following holds:

(i) the geometry $\Gamma$ is a building;
(ii) $\Gamma$ is the $\mathcal{A}_7$-geometry;
(iii) the geometry $\Gamma$ is anomalous as in Theorem 1.1.
Let us mention the following consequences of Theorem 1.2 before coming to the proof of Theorem 1.1.

**Corollary 1.3.** Let $x \geq y$ and let $A$ be flag-transitive. Then $\Gamma$ is either a building or the $\mathcal{A}_7$-geometry.

**Corollary 1.4.** A finite thick geometry of type $C_n$ ($n \geq 4$) or $F_4$ is a building if its automorphism group is flag-transitive.

Corollary 1.3 is a straightforward consequence of Theorem 1.2. The reader is referred to [14] for the proof of Corollary 1.4. It depends also on the classification of flag-transitive subgroups of finite Chevalley groups by Seitz [18].

2. **Proof of Theorem 1.1**

The proof is an application of the classification of finite flag-transitive projective planes by Kantor [7]. It depends on a subsidiary result stated in [13, Theorem 2], on results on finite primitive groups obtained in [7 (Theorem C), 8, 9] and, of course, on representation theory (see [10]).

Given a plane $u$ of $\Gamma$, let $A_u$ be the stabilizer of $u$ in $A$, let $\bar{A}_u$ be the action of $A_u$ on $\Gamma_u$ and let $K_u$ be the kernel of that action, so that $\bar{A}_u = A_u/K_u$ and $K_u \leq A_u \cap K$.

By [7, Theorem A], either $\bar{A}_u$ is desarguesian and $\bar{A}_u \cong PSL(3, \mathbb{F})$, or $x$ is even, $1 + x + x^2$ is prime and (B) of Theorem 1.1 of this paper holds.

In the first case, the number of lines through two distinct collinear points $a$ and $b$ does not depend on the choice of the collinear pair $(a, b)$. Then $\Gamma$ is either a building or flat [13, Theorem 2]. This conflicts with the assumption that $\Gamma$ is anomalous.

Then the latter case occurs.

We have $x + 1 \equiv 0 \pmod{3}$ by [4, 4.4.4.c] (indeed, the orders of Hall multipliers divide $x + 1$ in our case).

Let us prove that

1. $1 + x + x^2$ does not divide $1 + xy$.

   Indeed, assume the contrary. We get that $1 + x + x^2$ divides $y - x - 1$. Then $x + 1 \leq y$.

   If $x + 1 = y$, then $2x + 1$ divides $x(x + 1)^2(x + 2)$, by a well-known restriction on parameters of generalized quadrangles [15, 1.2.2]. So, $2x + 1$ divides $x + 2$. This conflicts with the fact that $x > 1$. Then $x + 1 < y$. So, we get that $x^2 < y$. This conflicts with another restriction on parameters of generalized quadrangles [15, 1.2.3]. Then (1) is proved. We have also:

2. $1 + x + x^2$ does not divide any of $x + y$ and $1 + y$.

   Indeed, if otherwise, we get the contradiction $x^2 < y$ again. It is known that $1 + x$ divides $(1 + x^2 y)n$, where $n$ is the greatest common divisor of $1 + x + x^2$ and $(1 + xy)(1 + y)$ (see [12, Section 4]). But $n = 1$ by (1) and (2). Then:

3. $1 + x$ divides $1 + x^2 y$. 

Now we exploit formulas for multiplicities of irreducible representations of the Hecke algebra of $\Gamma$.

Every such representation is associated with a double partition of the set $\{0, 1, 2\}$ of types of $\Gamma$, where 0 is the type of points, 1 the type of lines and 2 of planes (see [6, 10]). There are 10 essentially distinct such double partitions. The multiplicities of the associated representations can be computed by techniques developed in [6] (see also [10, 21]). Doing that is a tiresome but easy job. We obtain the list given in Table 1.

By the formulas for $1^2/1$ and $1^2/2$ we easily see that, if $d$ is the greatest common divisor of $x^2$ and $y$, then $xd$ divides $x$. Then the divisibility condition

$$\frac{xy}{d} - \frac{x}{xd} \equiv 0 \pmod{x + 1}$$

easily follows from (3).

Let us prove that

(4) $1 + x + x^2$ does not divide $x^2 + y$.

Indeed, if $1 + x + x^2$ divides $x^2 + y$, then we either $1 + x + x^2 = x^2 + y$ or $1 + x + x^2 \leq (x^2 + y)/2$. But the earlier case violates 1.2.2 of [15] and the latter case conflicts with [15, 1.2.3].

(5) $1 + x + x^2$ does not divide $1 + x^2 y$ (then it does not divide $x + 1$, by (3)).

Indeed, $1 + x + x^2$ divides $x + y$ if it divides $1 + x^2 y$. So, (5) follows from (2).

Exploiting (4) and (5) in formulas for $2/1$ and $1^2/1$ (Table 1), we easily get the remaining divisibility conditions listed in (A) of Theorem 1.1. It is worth observing that these are actually all divisibility conditions that can be obtained from Table 1.

Let us set

$$U = \frac{xy}{d} - \frac{x}{xd}$$

and

$$V = \frac{d}{x + 1}.$$

We have $UVxd + U + V = xy/d$. Then $U + V \equiv 0 \pmod{x}$. That is, there is a positive integer $W$ such that $U + V = Wx$. Then we have $UVd + W = y/d$. We have $x \neq 0$ because $\Gamma$ is not a building (see [12]). Then $U \neq 0$. Moreover, $x < x^2 y$ because $\Gamma$ is not flat (see [12]). Then $V \neq 0$. So, we have $UV \geq Wx - 1$ because $U + V = Wx$. Then

(6) $y \geq (Wx - 1)d^2 + Wd \geq (x - 1)d^2 + d \geq x$.

We have also $x^2 > y > x$ because $\Gamma$ is neither a building nor flat (see [12, Section 4]). Then $x^2 - x \geq y > x$ by [15, 1.2.5]. Now, by (6) and the inequality $x^2 - x \geq y$, we get $x^2 - x \geq (x - 1)d^2 + d$ and the inequality $x > d^2$ easily follows. Thus, (A) is proved.

Let us come to (C). We need several preliminary lemmas.

Henceforth, $n_i$ will be the number of elements in $S_i$ ($i = 0, 1, 2$); we set $p = 1 + x + x^2$ and $L$ will be the socle of the action $\bar{A} = A/K$ of $A$ over $S_0$.

\[1\] I got the knowledge of this list from Liebler [19] first.
Table 1

<table>
<thead>
<tr>
<th>Double partition</th>
<th>Shortened symbol</th>
<th>Multiplicity of the associated representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0, 1, 2}; 0</td>
<td>3/0</td>
<td>1</td>
</tr>
<tr>
<td>{0, 1}; {2}; 0</td>
<td>2, 1/0</td>
<td>(\frac{(1+x^3) y(1+x)(x^3+a)}{x(x+y)(1+a)})</td>
</tr>
<tr>
<td>{0}; {1}; {2}; 0</td>
<td>1^3/0</td>
<td>(\frac{(1+xy)(1+x^2 y)(x^9+a)}{(x^2+y)(x+y)(1+a)})</td>
</tr>
<tr>
<td>{0}; {2}</td>
<td>2/1</td>
<td>(\frac{(1+x^2 y)(1+xy)(x^7 y-a)}{x(x+y)(1+a)})</td>
</tr>
<tr>
<td>{0}; {1}; {2}</td>
<td>1^2/1</td>
<td>(\frac{(1+x^2 y)(1+x+x^2)(x^4 y-a)}{x(x^2+y)(1+a)})</td>
</tr>
<tr>
<td>{0}; {1}; {2}</td>
<td>1/1^2</td>
<td>(\frac{(1+xy)(1+x+x^2)(x^4 y^2+a)}{x(x+y)(1+a)})</td>
</tr>
<tr>
<td>{0}; {1, 2}</td>
<td>1/2</td>
<td>(\frac{(1+x^2 y)(1+x+x^2)(x^2 y^2+a)}{x(x^2+y)(1+a)})</td>
</tr>
<tr>
<td>{0}; {0}, {1}, {2}</td>
<td>0/1^3</td>
<td>(\frac{x^8 y^3-a}{1+a})</td>
</tr>
<tr>
<td>{0}; {0, 1}, {2}</td>
<td>0/2, 1</td>
<td>(\frac{(1+x)(1+x^2 y)(x^3 y^3-a)}{x(x+y)(1+a)})</td>
</tr>
<tr>
<td>{0}; {0, 1, 2}</td>
<td>0/3</td>
<td>(\frac{(1+xy)(1+x^2 y)(y^3-a)}{(x+y)(x^2+y)(1+a)})</td>
</tr>
</tbody>
</table>

The shortened symbols listed in the 2nd column will be taken also as names of the representations. The representation 3/0 is the index representation. 2/1 is the so-called reflection representation. The formula given above for its multiplicity has been found independently also by Scharlau [20].

**Lemma 2.1.** Let \(g \in A\) have order a power of \(p\) and let \(g \neq 1\). Then \(g\) does not fix any point of \(\Gamma\).

**Proof.** Assume that \(g(a)=a\) for some \(a \in S_0\), by way of contradiction. Then \(g\) fixes some plane \(u\) in \(\Gamma_a\) by (1) and (2). By (B) of Theorem 1.1 \(g\) induces the identity over \(\Gamma_u\). Then it fixes all lines incident with \(u\) and all planes sharing a line with \(u\), because \(p>y+1\) [see 15, 1.2.3]. Moreover, it fixes all points of any line fixed by it. So, \(g\) induces the identity over the residue \(\Gamma_v\) of \(\Gamma_u\) for every plane \(\nu\) sharing a line with \(u\). Iterating this argument, we get that \(g\) fixes everything. We have the contradiction. \(\square\)

**Lemma 2.2.** Let \(g\) be as above. Then \(g\) has order \(p\) and its orbits over \(S_0\) have size \(p\).
A. Pasini

Proof. Indeed, let \( o(g) \) be the order of \( g \). Each of the orbits of \( g \) over \( S_0 \) has size \( o(g) \) by Lemma 2.1. Then \( o(g) \) divides \( n_0 \). Then \( o(g) = p \) because \( n_0 = p(x^2y + 1)/(x + 1) \) (see [12]) and \( p \) does not divide \( 1 + x^2y \), by (5). □

Lemma 2.3. The \( p \)-Sylow subgroups of \( A \) are cyclic of order \( p \).

We omit the proof. It is similar to that of Lemma 2.2.

From now on, \( A \) is assumed to act primitively on \( S_0 \).

Lemma 2.4. The socle \( L \) of \( \bar{A} \) is simple of Lie type and acts transitively on \( S_0 \).

Proof. The transitivity of \( L \) on \( S_0 \) is a trivial (and well-known) consequence of the primitivity of \( \bar{A} \).

We have \( n_0 = p(x^2y + 1)/(x + 1) \) (see [12]). Then \( n_0 \) is odd because \( x \) is even. Moreover, it is not a prime power because \( x + 1 \) is a proper divisor of \( x^2y + 1 \) (by (3) and because \( \Gamma \) is not flat) and \( p \) does not divide \( x^2y + 1 \) (by (5)). \( n_0 \) is not a proper power by the same reasons. Then \( L \) is a nonabelian simple group (see [2]).

\( p \) divides the order of \( L \) because it divides \( n_0 \) and \( L \) is transitive on \( S_0 \). Then \( L \) cannot be sporadic either. Indeed, \( p > 10^6 \) because \( x > 1000 \) (see the remark after the statement of Theorem 1.1), and no sporadic simple group has order divisible by such large primes (see [3]).

Let us assume that \( L \) is the alternating group \( \mathcal{A}_d \) for some \( d \). Then, if \( L_a \) is the stabilizer in \( L \) of a point \( a \) of \( \Gamma \), one of the following holds (see [7, Theorem C] or [8]):

(i) \( L_a \) is the stabilizer of a \( k \)-subset of the relevant \( d \)-set \( Y \) of \( \mathcal{A}_d \). We can always assume that \( k \leq d/2 \).

(ii) We have \( d = hk \) \((h, k > 1)\) and \( L_a \) is the stabilizer of a partition of the relevant \( d \)-set \( Y \) of \( \mathcal{A}_d \) into \( h \) classes of size \( k \).

(iii) \( d = 7 \) and \( L_a = PSL(3, 2) \).

The last case is clearly impossible because \( p \) divides the order \( d!/2 \) of \( \mathcal{A}_d \) and \( p > 10^6 \). In the second case we have

\[
\prod_{i=2}^{h} \frac{\binom{ik}{k}}{i!} = n_0 = p(x^2y + 1)/(x + 1).
\]

But we have \((x^2y + 1)/(x + 1) < (x - 1)^2 < p\), by (A). Exploiting these inequalities and the fact that \( p > 10^6 \), a contradiction is obtained easily. So, we are led to case (i). We have \( t_f = n_0 \) and an argument similar to that used above in case (ii) forces \( k = 1 \) or \( 2 \).

We have \( d \geq p \) because \( L \) contains elements of order \( p \). If \( d - p \geq k \), then an element of \( L_a \) of order \( p \) is easily found, contradicting Lemma 2.1. Then \( d - p < k \). So, we have \( d = p \) if \( k = 1 \) and \( d = p \) or \( p + 1 \) if \( k = 2 \). So, \( n_0 = p \) if \( k = 1 \) and this forces \( \Gamma \) to be flat. We have a contradiction. Then \( k = 2 \) and \( d = p \) or \( p + 1 \). Let \( d = p \). Then we have
Flag-transitive $C_2$-geometries

$p(p-1)/2 = n_0 = p(x^2y + 1)/(x + 1).$ So, we get $x + 1 = 2(x^2y + 1)/(x + 1)x,$ which is clearly impossible, as $x$ is relatively prime with respect to $x^2y + 1.$ Then $d - p + 1$ and we have $x + 1 = 2(x^2y + 1)/(p + 1) = 2(x^2 + x + 2).$ Then $x^2 + x + 2$ divides $x^2y + 1.$ By easy computations, we get that $x^2 + x + 2$ divides $2(4y + x - 1).$ Then $2(4y + x - 1) = z(x^2 + x + 2)$ for some positive integer $z.$ We have $z \leq 7$ because $y \leq x^2 - x$ (by (A)).

It is easily seen that the greatest common divisor of $x$ and $y$ divides $2(z + 1).$ Then, by [15, 1.2.2], we get the following:

\[ z(x^2 + x + 2) + 6x + 2 = 8(x + y) \] divides $32(z + 1)^2(x^2 - 1).$

From the above, by easy computations, we see that

\[ z(x^2 + x + 2) + 6x + 2 \] divides $32(z^2(x + 3) + 8z^2(x + 1) + z(13x + 7) + 6x + 2).$

This contradicts the fact that $x > 1000$ and $z \leq 7.$ So, $L \not\in \mathcal{A}.$

The following lemma is a trivial consequence of Lemma 2.4.

**Lemma 2.5.** The group $A$ contains involutions.

Henceforth, $i$ will always be an involution in $A.$ $i$ fixes at least one point of $\Gamma,$ because $n_0$ is odd. Henceforth, if $a$ is a point fixed by $i,$ then $\mathcal{C}_a$ is the configuration fixed by $i$ in $\Gamma_a.$

**Lemma 2.6.** One of the following holds for all points $a$ fixed by $i$:

(i) $\mathcal{C}_a$ is a nonempty set of pairwise noncoplanar lines. Then $y$ is odd. No plane of $\Gamma$ is fixed by $i$ in this case.

(ii) $\mathcal{C}_a$ is a grid in $\Gamma_a$ with parameters as below:

<table>
<thead>
<tr>
<th>lines</th>
<th>planes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>1</td>
</tr>
</tbody>
</table>

$y$ is odd.

(iii) $\mathcal{C}_a$ consists of a nonempty set of planes passing through the same line $r$ of $\Gamma_a$ and of all lines incident with ($a$ and with) any of those planes. The number of planes in $\mathcal{C}_a$ is odd. $y$ is even.

**Proof.** The line $r$ in case (iii) is uniquely determined iff $\mathcal{C}_a$ contains at least two planes. If that is the case, then $r$ will be called the center of $\mathcal{C}_a.$

Let us come to the proof of Lemma 2.6. Let $a$ be a point fixed by $i.$ First of all we observe that, as the number $(xy + 1)(x + 1)$ of lines incident with $a$ is odd, $i$ fixes at least one line in $\Gamma_a.$ Moreover, if $i$ fixes a plane $u$ in $\Gamma_a,$ then it fixes everything in $\Gamma_a,$ by (B).
By this information and Theorem 2.4.1 of [15], we see that only the following possibilities might occur on \( \mathcal{E}_a \) besides those described in (i), (ii) and (iii) above:
(iv) \( \mathcal{E}_a = \Gamma_a \).
(v) \( \mathcal{E}_a \) is a subquadrangle of \( \Gamma_a \) with parameters as below:

\[
\begin{array}{c|c}
\text{lines} & \text{planes} \\
\hline
\odot & \odot \\
X & Z \\
\hline
\end{array}
\]

where \( 2 \leq z < y \).

Let us prove that none of the previous two cases can occur.

Assume that (iv) occurs. Then \( i \) fixes another point \( b \) of \( \Gamma \) collinear with \( a \), every line through \( a \) and \( b \) and every plane incident with such a line. Then only (iv) of (iii) can occur on \( \mathcal{E}_b \). The latter case occurs only if \( y \) is even, there is exactly one line through \( a \) and \( b \) and \( \mathcal{E}_b \) contains exactly \( y+1 \) planes. If it is not possible to find any pair of points \( (a, b) \) both fixed by \( i \) and such that (iv) holds on \( \mathcal{E}_a \) whereas (iii) holds on \( \mathcal{E}_b \), then it is easily seen that \( i \) fixes all points and (iv) holds on \( \mathcal{E}_a \) for every point \( a \) of \( \Gamma \). So, \( i \) fixes everything in \( \Gamma \) and we have a contradiction. So, we can assume that (iii) holds on \( \mathcal{E}_b \). Let \( r \) be the line through \( a \) and \( b \) (that is, the center of \( \mathcal{E}_b \)). Let \( w \) be any plane incident with \( r \) and let \( c \) be any point in \( w \) non incident with \( r \). Of course, \( c \) is fixed by \( i \) because \( i \) fixes \( w \). Either (iv) or (iii) holds on \( \mathcal{E}_c \), because \( c \) is collinear with \( a \). But (iv) cannot hold on \( \mathcal{E}_c \), otherwise, \( r \) and any line through \( b \) and \( c \) would give different centers of \( \mathcal{E}_b \); this is a contradiction. Then (iii) holds on \( \mathcal{E}_c \). Now, interchanging the roles of \( b \) and \( c \), we get that (iii) holds on \( \mathcal{E}_c \) for every point \( d \) on \( r \) different from \( a \) and \( c \). Then \( a \) is homogeneous, in the meaning of [13], and \( \Gamma \) is either a building or flat by [13, Theorem 2]. We have a contradiction. Then (iv) never occurs.

Let (v) occur on \( \mathcal{E}_c \). Let us set

\[
\begin{align*}
m_1 &= -(1 + z)(1 + xz) \\
m_2 &= -(1 + x)(1 + xz).
\end{align*}
\]

Let \( t \) be the number of planes \( u \) in \( \Gamma_a \) such that \( u \neq i(u) \) but \( u \) and \( i(u) \) are incident with the same line in \( \Gamma_a \). It is easily seen that \( t \) is the number of planes of \( \Gamma_a \) that are tangent with the subquadrangle \( \mathcal{E}_a \) (see [15, Chapter 2]). Then \( t = (1 + z)(1 + xz)(y - z) \) (see [15, Proof of 2.2.1]). Moreover, it is easily seen that \( m_1(y + 1) = m_2(x + 1) + t \). Then \( x = z \). Hence, \( y = x^2 \) by [15, 2.2.2(iii)] and this contradicts (A). So, (v) cannot occur either.

Of course, \( y \) must be odd if (i) or (ii) occur on \( \mathcal{E}_a \), and even if (iii) occurs. We have still to prove that, if \( \mathcal{E}_a \) is as in (i) for some point \( a \) fixed by \( i \), then (ii) never occurs, that is, no plane of \( L \) is fixed by \( i \). Indeed, let \( \mathcal{E}_a \) be as in (i) and let \( u \) be a plane fixed by \( i \). Of course, \( u \notin \sigma_3(a) \). There are exactly \( x + 1 \) incident line–plane pairs \( (r, v) \) such that \( a \ast v \) and \( r \ast u \) (see [12]) and \( i \) must fix some of them because \( x + 1 \) is odd (by (A)). Then \( i(v) = v \), contradicting our assumption on \( \mathcal{E}_a \). □
From now on we assume that $y$ is even. Let $\mathcal{C}_i$ be the configuration fixed by $i$ in $\Gamma$.

**Lemma 2.7.** One of the following holds:

(i) The configuration $\mathcal{C}_i$ consists of exactly one plane $u$ and of its residue $\Gamma_u$.

(ii) The configuration $\mathcal{C}_i$ consists of planes $u_1, \ldots, u_m$ ($1 < m \leq y+1$, $m$ odd), all incident with the same line $r$ (called axis of $\mathcal{C}_i$), and of their residues. We have $\sigma_0(u_i) \cap \sigma_0(u_j) = \sigma_0(r)$ ($i \neq j$).

**Proof.** $\mathcal{C}_i$ contains at least one point. Then it contains at least one plane $u$ together with its residue $\Gamma_u$, by Lemma 2.6. Let $a$ be a point in $\mathcal{C}_i - \sigma_0(u)$. Then there is at least one incident line–plane pair $(r, v)$ fixed by $i$ and such that $a \ast v$ and $r \ast u$, because there are exactly $x+1$ line–plane pairs $(r, v)$ where $a \ast v$ and $r \ast u$ (see [12]) and $x+1$ is odd (by (A)). The line $r$ is the center of $\mathcal{C}_b$, for every point $b \in \sigma_0(r)$. If $\mathcal{C}_a$ contains some plane $w$ other than $r$, then $\mathcal{C}_a$ has a center $s$. If $b$ is the meeting point of $s$ and $r$ in $\Gamma_v$, then $\mathcal{C}_b$ has two distinct centers, namely $r$ and $s$, contradicting Lemma 2.6. Then $\mathcal{C}_a$ consists only of $v$ and all lines incident with $v$ and $a$. Then the pair $(r, v)$ as above is uniquely determined by $a$. We call it $(r(a), v(a))$. Let $a'$ be another point in $\mathcal{C}_i - \sigma_0(u)$. Then $r(a) = r(a')$; otherwise, $r(a)$ and $r(a')$ would be different centers of the configuration $\mathcal{C}_b$, where $b$ is the meeting point of $r(a)$ and $r(a')$ in $\Gamma_v$. So, we have proved that, if $\mathcal{C}_i$ contains some point nonincident with $u$, then all planes of $\mathcal{C}_i$ pass through the same line $r$ (the axis of $\mathcal{C}_i$).

Let $v$ be another plane in $\mathcal{C}_i$. If $\sigma_0(u) \neq \sigma_0(v)$, then $v \ast r$ by the previous argument. If $a \in \sigma_0(u) \cap \sigma_0(v) - \sigma_0(r)$, then $\mathcal{C}_a$ has a center $r' \neq r$. So, we have two distinct lines $r$ and $r'$ both incident with both $u$ and $v$. It is well known that this cannot happen. Then $\sigma_0(r) = \sigma_0(u) \cap \sigma_0(v)$. So, (ii) holds in this case.

We have still to consider the case when $\sigma_0(u)$ coincides with the set of points in $\mathcal{C}_i$. In this case, if $v$ is a plane in $\mathcal{C}_i$ different from $u$, we have $\sigma_0(u) = \sigma_0(v)$. If $b$ is a point in $\sigma_0(u)$, let $r$ be the center of $\mathcal{C}_b$, let $c \in \sigma_0(u) - \sigma_0(r)$ and let $r'$ be the center of $\mathcal{C}_c$. Then $r$ and $r'$ are distinct lines both incident with $u$ and $v$. We have a contradiction again. So, (i) holds in this case. $\square$

**Lemma 2.8.** The kernel $K$ of the action of $A$ on $S_0$ has odd order.

**Proof.** Indeed, if $i$ is an involution in $K$, then (i) of Lemma 2.7 cannot hold on $\mathcal{C}_i$; otherwise, $\Gamma$ is flat. Then (ii) holds. So, we have $mx^2 + x + 1 = n_0$. Then $1 + x + x^2$ divides $1 + x + mx^2$. Hence, it divides $m - 1$. Then $1 + x + x^2 \leq y$ and this conflicts with [15, 1.2.3]. $\square$

The Sylow 2-subgroups of $A$ can be viewed also as Sylow 2-subgroups of $\tilde{A}$, by Lemma 1.11. Henceforth, $S$ will always be a Sylow 2-subgroup of $A$ and $\mathcal{C}_S$ will be the configuration fixed by $S$ in $\Gamma$.

**Lemma 2.9.** The configuration $\mathcal{C}_S$ consists of one plane $u$ and of its residue $\Gamma_u$. 


Proof. Indeed, $S$ fixes at least one plane because $n_2$ is odd. Moreover, $\mathcal{C}_S$ is a subconfiguration of $\mathcal{C}_i$, for every involution $i \in S$. Then an analogue of Lemma 2.7 holds on $\mathcal{C}_S$. We have to show that the situation described at (ii) of Lemma 2.7 cannot occur on $\mathcal{C}_S$.

Assume that it occurs, by way of contradiction. Then the axis of $\mathcal{C}_S$ is fixed by all elements of $A$ normalizing $S$. Now, let $u$ be a plane fixed by $S$ and let $A_u$ be the stabilizer of $u$ in $A$. Let $K_u$ be the kernel of the action of $A_u$ on $\Gamma_u$. All Sylow 2-subgroups of $A_u$ are contained in $K_u$. So, if $N$ and $N^*$ are the normalizers of $S$ in $A_u$ and $K_u$, respectively, we have $[A_u:N^*]=[K_u:N^*]$. But $p$ does not divide the order of $K_u$, by Lemma 2.1. So, it divides the order of $N$, because it divides $|A_u|$. Then there is an element $g$ of $A_u$, normalizing $S$ and acting cyclically on the set of lines of $\Gamma_u$. Thus, $g$ cannot fix the axis of $\mathcal{C}_S$ and we have a contradiction. □

Henceforth, $u(S)$ will be the plane fixed by $S$ (it is uniquely determined by Lemma 2.9) and $A_S$ will be the stabilizer of $u(S)$ in $A$. By the uniqueness of $u(S)$, we easily get the following lemma.

Lemma 2.10. We have $N_A(S) \leq A_S$.

Lemma 2.11. Every involution fixes exactly one plane.

Proof. Indeed, let $i$ be an involution in $A$. Let $S$ be a Sylow 2-subgroup containing $i$. Let us set $u=u(S)$ and assume that $i$ fixes another plane $v$ besides $u$, by way of contradiction. Let $2^m$ be the order of $S$ and $2^k$ be the order of the stabilizer $S_v$ of $v$ in $S$. We have $1 < k < m$ because $1 \neq S_v \neq S$. The configuration fixed by $S_v$ has one axis; so, if $g$ is an element of order $p$ in $N_A(S)$, the conjugates $S^g (j=1, \ldots, p)$ of $S$ are pairwise distincts and any two of them intersect on the identity element $1$. Then $p(2^k-1) \leq 2^m-1$. Of course, $2^m-1 \leq y$. If $2^m-1 < y$, then we can always assume to have chosen $i$ and $v$ so that the orbit of $v$ under the action of $S$ is as small as possible. Then $2^m-1 < y$ and a contradiction is easily obtained exploiting the inequalities $y \leq x^2-x$ and $(2^m-1)/(2^k-1) \geq 1+x+x^2$.

Then $y=2^m-k$. So, the orbit of $v$ under the action of $S$ consists of all planes different from $u$ and incident with the axis $r$ of $\mathcal{C}_i$. Moreover, acting by $g$ and $S$ we can map $v$ onto any other plane $v$ sharing a line with $u$. So, $S_v$ has order $2^k$ for every such plane $v$. Moreover, substituting $i$ with any nonidentical subgroup $S'$ of $S$, we can easily see that either $u$ is the only plane fixed by $S'$ or $S' \leq S_u$ for some plane $v$ sharing a line with $u$. Let $v, w$ be two distinct such planes and let us assume that there is some point $a \in \sigma_0(v) \cap \sigma_0(w) \cap \sigma_0(u)$. Then $S_v \cap S_w = 1$ by Lemma 2.7. Let $S'=\langle S_v, S_w \rangle$ be the subgroup of $S$ generated by $S_v$ and $S_w$. Then $S' \neq S$ because it fixes $a \neq \sigma_0(u)$. Then $S' \leq S_v$ for some plane $v'$ sharing a line with $u$. Hence, $S_v$ and $S_w$ are proper subgroups of $S_v$, contradicting the fact that all subgroups of $S$ of this kind have order $2^k$. Then
\(\sigma_0(v) \cap \sigma_0(w) - \sigma_0(u) = \emptyset\). It is easily seen that this forces \(\alpha = 0\). Then \(\Gamma\) is a building (see [12]).

We have a contradiction. \(\square\)

**Lemma 2.12.** The socle \(L\) of \(\bar{A}\) is one of the following groups:

\[
\text{SL}(2,2^n) \ (n \geq 2), \quad \text{PSU}(3,2^n) \ (n \geq 2) \quad \text{or} \quad 2B_2(2^{m+1}) \ (m \geq 1).
\]

**Proof.** Let \(i\) be an involution of \(L\) (see Lemmas 2.4 and 2.8). Let \(S\) be a Sylow 2-subgroup of \(L\). We can identify \(S\) with a 2-subgroup \(S^*\) of \(A\), by Lemma 1.11. Of course, \(S^*\) need not be a Sylow 2-subgroup of \(A\). Anyway, \(S^*\) fixes exactly one plane \(u(S^*)\), by Lemma 2.11. The plane \(u(S^*)\) and the 2-group \(S^*\) need not be uniquely determined by \(S\). Anyway, the set of points \(\sigma_0(u(S^*))\) is uniquely determined by \(S\). Let us set \(U(S) = \sigma_0(u(S^*))\) and let \(M_S\) be the stabilizer in \(L\) of \(U(S)\). We have \(M_S \neq L\) because \(L\) is transitive on \(S_0\) and \(\Gamma\) is not flat. Of course, \(S \leq M_S\), because \(S\) fixes \(u(S)\) elementwise.

Assume that \(i \in M_S\) and \(g \in L\) be such that \(i^g\) centralizes \(i\) and belongs to \(M_S\). Then \(i\) and \(i^g\) belong to the same Sylow 2-subgroup of \(M_S\). Moreover, the Sylow 2-subgroups of \(M_S\) are Sylow 2-subgroups of \(L\) because \(S \leq M_S\) and \(M_S = M_{S'}\) for every Sylow 2-subgroup \(S'\) of \(M_S\) because \(S\) and \(S'\) are conjugated in \(M_S\). So, we can assume to have chosen \(S\) in \(M_S\) so that \(i\) and \(i^g\) belong to \(S\).

As we have observed above, \(S\) can be identified with a 2-subgroup \(S^*\) of \(A\). So, \(i\) and \(i^g\) can be identified with involutions in \(S^*\). Of course, we can identify \(g\) with any of its representatives in \(A\). As \(i \in S^*\), \(u(S^*)\) is the only plane fixed by \(i\), by Lemma 2.11. Similarly, \(u(S^*)\) is the only plane fixed by \(i^g\). Then \(u(S^*)\) and \(g(u(S^*))\) have the same 0-shadow. Coming back to \(L\), we have \(U(S) = g(U(S))\); so, \(g \in M_S\).

Then, by Aschbacher’s strong embedding criterion (see [5, Theorem 4.31(a)]), \(L\) possesses a strongly embedded subgroup. Now the conclusion follows by a theorem of Bender [5, Theorem 4.24]. \(\square\)

### 2.1. End of the proof of Theorem 1.1

Now we get the final contradiction. This will show that \(y\) must be odd if \(A\) acts primitively over \(S_0\).

In each of the cases listed in Lemma 2.12, the stabilizer in \(L\) of a point of \(\Gamma\) is the normalizer \(N_L(S)\) in \(L\) of a Sylow 2-subgroup \(S\) of \(L\), by [7, Theorem C] or [8]. Then \(n_0 = [L:N_L(S)]\) and \(p\) divides \([L:N_L(S)]\).

Given a Sylow 2-subgroup \(S\) of \(L\), let \(S^*, U(S)\) and \(M_S\) be defined as in the proof of Lemma 2.12. \(L\) acts primitively on the set of its Sylow 2-subgroups. Indeed, they are parabolic subgroups now. Then \(M_S = N_L(S)\).
Of course, $L$ possesses elements of order $p$. Let $g$ be any such element. We can identify $g$ with an element of order $p$ in $A$, by Lemmas 2.1 and 2.3. Viewed as an element of $A$, $g$ fixes at least one plane. Indeed,

\[ n_2 = \frac{(x^2y + 1)}{x + 1} - (xy + 1)(y + 1) \]

(see [12]) and $p$ does not divide any of $x^2y + 1$, $xy + 1$ and $y + 1$ by (1), (2) and (5). Let $u$ be a plane fixed by $g$. Let $S$ be a Sylow 2-subgroup of $L$ such that $u(S^*) = u$. Then $g \in M_S$. Then $p$ divides the order $|N_L(S)|$ of $N_L(S)$. So, $p^2$ divides $|L|$ because $p$ divides both $|N_L(S)|$ and $[L:N_L(S)]$. But this contradicts Lemma 2.3. We are done.

**Remark 1.** As for (C) of Theorem 1.1, let $A$ act primitively on $S_0$ and let $L$ be the socle of its action $\bar{A} = A/K$ over $S_0$. Then $L$ is a simple group of Lie type (see Lemma 2.4). Now, observing that the degree $|S_0|$ of $\bar{A}$ is odd and divisible by $1 + x + x^2$ and $1 + x + x^2$ is prime and not so small compared with $|S_0|$, we can get some information on $\bar{A}$ exploiting either the classification of primitive groups of odd degree [7, Theorem C, 8] or the classification of primitive groups with a large prime factor [9]. For instance, exploiting [9] we get that either $y - 3 \geq x$ (so $x^2 - 2x \geq x$ and $y \geq xd + 3 \geq d^3 + d + 3$) or the possibilities for $L$ and the stabilizer $L_u$ in $L$ of a point $a$ of $\Gamma$ are those listed in Table 2.

Note that the last case of Table 2 cannot occur if $A$ acts primitively also on the set of lines of $\Gamma$. Indeed, the number of lines is odd in any case. So, by [8] or

<table>
<thead>
<tr>
<th>$L$</th>
<th>Comment</th>
<th>$L_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$PSL(d, q)$</td>
<td>$d \geq 2$</td>
<td>Parabolic of type $A_{d-2}$ or $A_1 \times A_{d-3}$.</td>
</tr>
<tr>
<td>$PSL(d, q)$</td>
<td>$d \geq 3$</td>
<td>Stabilizer of a nonincident point–hyperplane pair $(X, Y)$, $X$ and $Y$ interchanged by a graph automorphism of $L$ contained in $\bar{A}$</td>
</tr>
<tr>
<td>$PSL(d, q)$</td>
<td>$q$ even, $d \geq 3$</td>
<td>Stabilizer of a point–hyperplane flag $(X, Y)$, $X$ and $Y$ interchanged by a graph automorphism of $L$ contained in $\bar{A}$</td>
</tr>
<tr>
<td>$PSL(2, q)$</td>
<td>$q$ odd, $q$ square</td>
<td>$PGL(2, q^{1/2})$</td>
</tr>
<tr>
<td>$PSp(2m, q)$</td>
<td>$q$ even, $m \geq 1$</td>
<td>Stabilizer of a 1-space</td>
</tr>
<tr>
<td>$PSp(4, q)$</td>
<td>$q$ even</td>
<td>Stabilizer of a totally isotropic 2-space</td>
</tr>
<tr>
<td>$PSU(d, q)$</td>
<td>$q$ even, $d$ odd prime</td>
<td>Stabilizer of a singular 1-space</td>
</tr>
<tr>
<td>$PO^+(2m, q)$</td>
<td>$q$ even, $m \geq 1$</td>
<td>Stabilizer of a singular 1-space</td>
</tr>
<tr>
<td>$^2B_2(2m-1)$</td>
<td></td>
<td>Parabolic</td>
</tr>
</tbody>
</table>

(Note that $L_u$ is parabolic in all cases, except in the third and fourth cases).
[7, Theorem C], lines and points should be stabilized by the same kind of subgroups of $B(2^{2m+1})$ and this conflicts with the fact that the number of points properly divides the number of lines.

Of course, similar arguments would allow one to improve Theorem 1.1. But that job would be long and tiresome, considering that we do not know so much about the action in $\Gamma_w$ of the stabilizer of $w$ in $A$ when $w$ is a point. We have some more information when $w$ is a line and we know much more if $w$ is a plane (by (B) of Theorem 1.1). But if $y$ is odd, then the number of planes is even; so, we should exploit information on lines rather than on planes and compare with what we get about points, if we had in mind to exploit the results of [8] or of [7, Theorem C] on primitive groups of odd degree.

Moreover, in order to avoid too many ugly statements of the form 'either $A$ acts imprimitively on ... or we have ...' a systematic inquiry into the imprimitive cases should be done in advance.

Perhaps, the following facts are worth mentioning here. If $A$ acts imprimitively on $S_0$, then every plane picks up at most one point from each of the imprimitivity classes of $A$ over $S_0$. So, each of these imprimitivity classes contains at most $x^2 + 2$ points, the number of those classes is a multiple of $1 + x + x^2$ and every element of $A$ of order $1 + x + x^2$ cyclically permute them.

Moreover, Lemmas from 2.6 to 2.11 still hold in the imprimitive case, provided that the existence of involutions is explicitly assumed when $y$ is even. We have already observed that Lemmas 2.1–2.3 hold in the imprimitive case as well.

Remark 2. The final part of the proof of Theorem 1.1 and the proof of Lemma 2.12 could be rearranged so as to prove that, if $y$ is odd but $A$ acts primitively and faithfully on $S_0$, then no Sylow 2-subgroup of $A$ fixes some plane. This information might be useful in future investigations.

Remark 3. The flag-transitivity of $A$ is rarely fully exploited in this paper. The following weaker condition is sufficient to obtain most of what is stated in Theorem 1.1:

(\circledast) For every plane $u$, the action $\bar{A}_u$ on $\Gamma_u$ of the stabilizer of $u$ in $A$ is flag-transitive. Moreover, $\text{PSL}(3, x)$ is contained in each of these actions if it is contained in some of them.

Condition (\circledast) is sufficient to obtain (A) and (B) of Theorem 1.1, for instance.

Thus, improving the results of this paper amounts to full exploitation of the flag-transitivity of $A$. It is likely that most of our problems arise from the fact that too little is presently known on flag-transitive generalized quadrangles. So, we have been forced to exploit only that part of the flag-transitivity of $A$ that appears in the point–line zone of $\Gamma$ (namely, (\circledast)).
References