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On cyclically embeddable $(n, n-1)$-graphs

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Abstract

An *embedding* of a simple graph G into its complement $\bar{G}$ is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. In this note we consider the embeddable $(n, n-1)$-graphs. We prove that with few exceptions the corresponding permutation may be chosen as cyclic one. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs of order $n = |V(G)|$ and size $|E(G)|$. All graphs will be assumed to have neither loops nor multiple edges. If a graph G has order n and size m, we say that G is an (n, m)-graph.

Assume now that G_1 and G_2 are two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph is the union of k ($\geqslant 2$) disjoint copies of a graph H, then we write $G = kH$.

An *embedding* of G (in its complement $\bar{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. In others words, an embedding is an (edge-disjoint) *placement* (or *packing*) of two copies of G (of order n) into a complete graph K_n. If, additionally, an embedding of G is a cyclic permutation we say that G is *cyclically embeddable* (CE for short).

In the paper, we continue the study of families of CE graphs of [6]. It will be helpful to formulate some results proved in [6] as a theorem.

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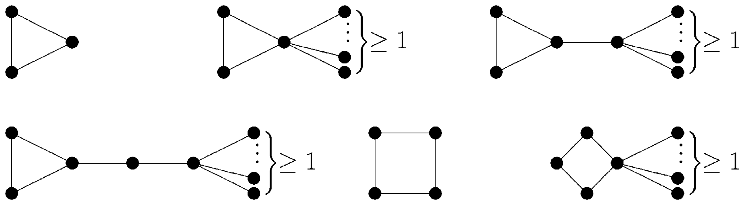

Fig. 1. Non-embeddable unicyclic graphs.

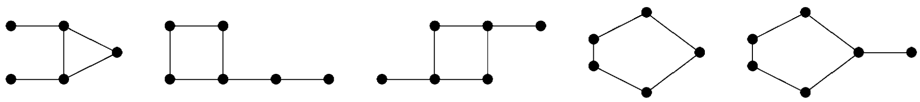

Fig. 2. Five embeddable unicyclic graphs which are not CE.

Theorem 1. *The following graphs are cyclically embeddable*:

1. $(n, n-2)$-*graphs*,
2. *non-star trees*,
3. *cycles* C_i *for* $i \geqslant 6$,
4. *unicyclic graphs* (*connected* (n, n)-*graphs*) *except for graphs that are not embeddable at all* (*see Fig.* 1), *and five graphs given in Fig.* 2.

The aim of this note is to consider the family of $(n, n-1)$-graphs. The following theorem, originally proved in [2], completely characterizes those graphs with n vertices and $n-1$ edges that are embeddable.

Theorem 2. *Let* $G=(V,E)$ *be a graph of order n. If* $|E(G)| \leqslant n-1$ *then either G is embeddable or G is isomorphic to one of the following graphs*: $K_{1,n-1}, K_{1,n-4} \cup K_3$ *with* $n \geqslant 8$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$, $K_1 \cup C_4$. (*see Fig.* 3).

Note that the graphs $K_{1,2} \cup K_3$ and $K_{1,3} \cup K_3$ are embeddable but cannot be embedded without fixed vertices. It is interesting to note that all other $(n, n-1)$-graphs that are contained in their complements can be embedded without fixed vertices. More precisely, we have the following theorem mentioned first in [4].

Theorem 3. *Let* $G=(V,E)$ *be a graph of order n with* $|E(G)| \leqslant n-1$ *and such that*

(a) *G is not an exceptional graph of Theorem* 2,
(b) $G \neq K_{1,2} \cup K_3$ *and* $G \neq K_{1,3} \cup K_3$.

Then there exists a fixed-point-free embedding of G.

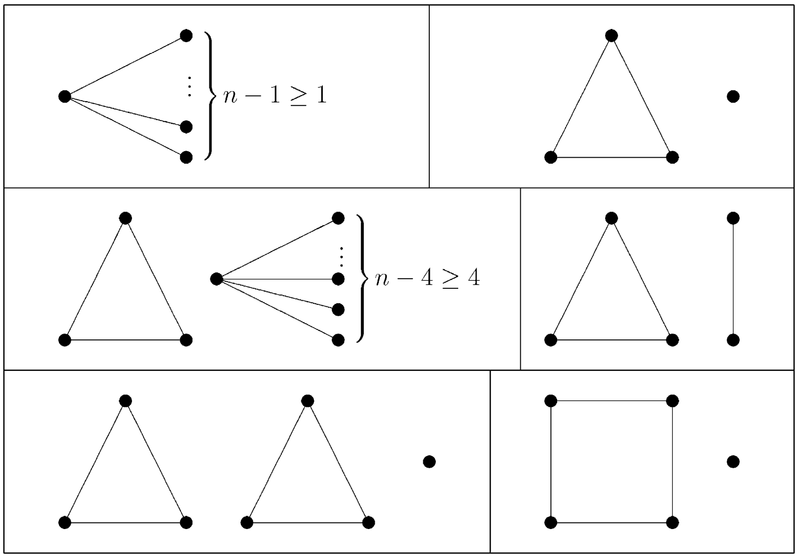

Fig. 3. Non-embeddable $(n, n-1)$ graphs.

Somewhat surprisedly, with only one exceptional graph more we have considerably stronger result. Namely, we shall prove that:

Theorem 4. *Let $G=(V,E)$ be a graph of order n with $|E(G)| \leqslant n-1$ and such that*

(a) *G is not an exceptional graph of Theorem* 2,
(b) *$G \neq K_{1,2} \cup K_3$ and $G \neq K_{1,3} \cup K_3$,*
(c) *$G \neq K_1 \cup C_5$.*

Then there exists a cyclic embedding of G.

The general references for these and other packing problems are [1,7,5] (see also [8]).

We shall need some additional definitions in order to formulate the results. Let G and H be two rooted graphs at u and x, respectively. The graph of order $|V(G)|+|V(H)|-1$ obtained from G and H by identifying u with x will be called the *touch* of G and H and will be denoted by $G \cdot H$. A similar operation consisting in the identification of a couple of vertices of G, say (u_1,u_2) with a couple of vertices of H, say (x_1,x_2) will be called the 2-*touch* of G and H and will be denoted by $G:H$. The graph $G:H$ is of order $|V(G)|+|V(H)|-2$. By definition, the edge say u_1u_2 belongs to $E(G:H)$ if $u_1u_2 \in E(G)$ or $x_1x_2 \in E(H)$.

Let σ be a cyclic permutation defined on of $V(G)$. For $u \in V(G)$, we denote often the vertex $\sigma(u)$ by u^+ and $\sigma^{-1}(u)$ by u^-. The edge uu^+ is said to be *of length one* (with respect to σ).

2. Some lemmas

The proofs of the following lemmas are relatively easy. They can be found in [6].

Lemma 5. *Let G be a graph obtained from the graph H by removing a pendent vertex. If G is CE then H is CE.*

Lemma 6. *Let H be a graph with at least one isolated vertex v and let $G=H-\{v,x\}$ be a graph obtained from the graph H by removing v and another vertex x. If G has an isolated vertex and is CE then H is CE.*

Lemma 7. *Let G and H be two CE graphs. Then $G \cup H$ is CE.*

Lemma 8. *Let G and H be two CE graphs rooted at u and x, respectively. Then the graph $G \cdot H$ is CE.*

Remark. A similar result holds also if "cyclically embeddable" is replaced by "embeddable" (see [3]).

Lemma 9. *Let G and H be two CE graphs such that the vertices v,u of G and x,y of H are consecutive with respect to the cyclic embeddings of G and H, respectively. Suppose that*:

(C) *the edges uu^+ and xx^- as well as the edges yy^+ and vv^- are not simultaneously present.*

Then the graph $G:H$ obtained by identifying u with x and v with y is CE.

3. Proof of Theorem 4

Let $G=(V,E)$ be an $(n,n-1)$-graph. We shall consider some cases in dependence of the number of components of G.

Case 1. G has exactly one component.

Then G is a tree, and since by assumption G is not a star, the existence of a cyclic embedding of G follows from Theorem 1.

Case 2. G has exactly two components.

Then G is of the form $G=U \cup T$ where T is a tree and U is a unicyclic graph.

Subcase 2a: $T=K_1$. Denote the unique vertex of T by u. If U is a cycle C_i, then by assumptions $i \geqslant 6$. Since, by Theorem 1, each C_i for $i \geqslant 6$ has a cyclic embedding, by Lemma 5 it is also true for the graphs of the form $C_i \cup K_1$. So, we can assume that U is not a cycle. Remove the pendent vertices of U one by one as long as we obtain a graph G' having a pendent vertex x adjacent to a vertex, say y, of degree at least three. Removing y from G' we get a graph G'' of order k and of size at most $k-2$ and with at least one isolated vertex. By Theorem 1 this graph is CE. By Lemma 6

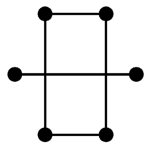

Fig. 4. Cyclic embeddings of $K_2 \cup C_4$.

also the graph G', obtained from G'' by adding y and u is CE. Applying Lemma 5 we are done.

Subcase 2b: $T = K_2$. Denote the vertices T by u and v. After removing v we get the previous subcase. Denote the obtained graph by G'. If G' is CE then, by Lemma 5, G is also CE. Hence, we have to examine two possibilities, $K_2 \cup C_4$ and $K_2 \cup C_5$. The graph $K_2 \cup C_4$ is drawn in Fig. 4 in such a way that the corresponding cyclic embedding is easy to guess as a "rotation". It is easy to see that the graph $K_2 \cup C_5$ can be considered as a subgraph of a unicyclic CE graph.

Subcase 2c: $T = K_{1,k}$, $k > 1$. By assumptions, $U \neq C_3$. In other cases, it is easy to see that G can be considered as a subgraph of a unicyclic CE graph.

Subcase 2d: T is not a star. Then always G can be considered as a subgraph of a unicyclic CE graph.

Case 3. G has exactly three components.

Subcase 3a: G has two tree components. Then $G = G_1 \cup T_1 \cup T_2$, where G_1 is a connected $(p, p+1)$-graph and T_1 and T_2 are trees. It suffices to consider the case, where T_1 and T_2 are isomorphic to K_1 i.e. G has two isolated vertices u and v. Denote by x a vertex of G_1 of degree at least three and put $G' = (G_1 - \{x\}) \cup \{u\}$. G' is a (sub)graph of a $(p, p-2)$-graph; hence, G' is CE by Theorem 1. On the other hand, since G' has at least one isolated vertex, this implies, by Lemma 6 that G is CE.

Subcase 3b: G has exactly one tree component T. Suppose first that $T = K_1$ and let $G = U_1 \cup U_2 \cup \{u\}$ where U_i, $i = 1, 2$, are unicyclic graphs. If one of these graphs, say U_1, were CE, then G could be considered as a touch $G = U_1 \cdot \tilde{U}$ where $\tilde{U} = U_2 \cup 2K_1$. Then G would be CE by Lemma 8, since it is easy to see that a unicyclic graph with two added isolated vertices is CE. If U_1 or U_2 is not a cycle we can proceed as in Subcase 2a.

Therefore, both unicyclic graphs are cycles. By assumptions, at least one of them is not the triangle. On the other hand, if one of them is of length greater than five G is CE by Lemma 8 because it can be viewed as a touch of this cycle and a $(p, p-2)$-graph.

So, it remains to verify the existence of the cyclic embedding of five following graphs: $C_3 \cup C_4 \cup K_1$, $C_3 \cup C_5 \cup K_1$, $C_4 \cup C_4 \cup K_1$, $C_4 \cup C_5 \cup K_1$, $C_5 \cup C_5 \cup K_1$. In fact, we are able to show the existence of CE of five graphs obtained from the above graphs by deleting isolated vertices. These embeddings are shown in Fig. 5.

It remains the case where the unicyclic graphs are triangles and T is not K_1. This case follows from the cyclic embedding of the graph $C_3 \cup C_3 \cup K_2$ which is given in Fig. 6.

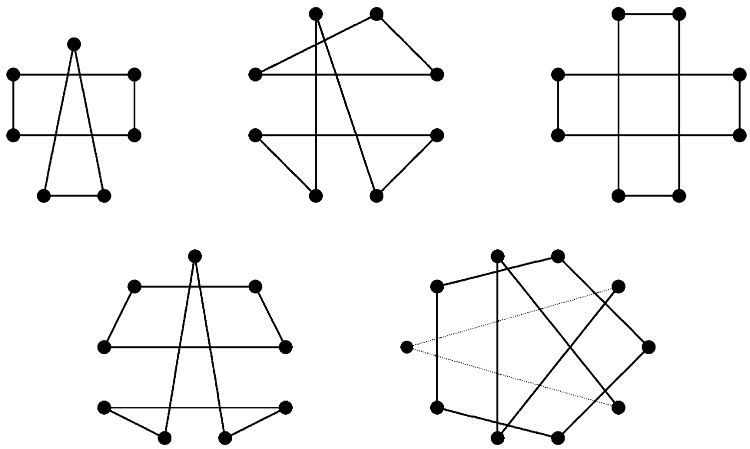

Fig. 5. Cyclic embeddings of $C_3 \cup C_4$, $C_3 \cup C_5$, $2C_4$, $C_4 \cup C_5$, and $2C_5$.

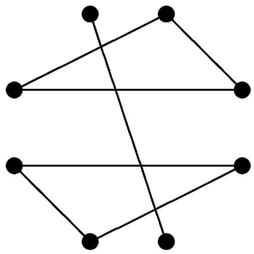

Fig. 6. Cyclic embeddings of $2C_3 \cup K_2$.

Case 4. G has more than three components.

Consider first the case that G has at least two tree components and suppose that these components are reduced to the isolated vertices. Removing one of these vertices together with a vertex of degree at least three we get a $(p, p-k)$-graph with $k \geqslant 2$. Applying Theorem 1 and Lemma 6 we obtain easily a cyclic embedding of G. Let now u be the isolated vertex of G being the unique tree component of G. This implies that the other components of G are unicyclic. If one of these components is not a cycle, then we proceed similarly as in the Case 2a.

So, it remains the case where $G = K_1 \cup \bigcup_{i=1}^{k} C_{n_i}$, $k \geqslant 3$, $n_1 \leqslant n_2 \leqslant \cdots$. We shall use the induction with respect to the number of cycles k. Observe that by previous case the graph G is CE for $k=2$ except for $C_3 \cup C_3 \cup K_1$. Moreover, the graph $K_1 \cup 3C_3$ is CE. This follows from Lemma 5 and the CE permutation for $3C_3$ defined in Fig. 7.

Let $k \geqslant 3$ and $G \neq K_1 \cup 3C_3$. Let G' be the graph obtained from G by removing C_{n_1}. By induction hypothesis G' is CE. It suffices now to remark that G can be considered as the 2-touch of G' and the graph $2K_1 \cup C_{n_1}$. Since the last graph is CE the result

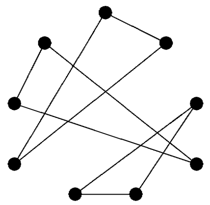

Fig. 7. Cyclic embeddings of $3C_3$.

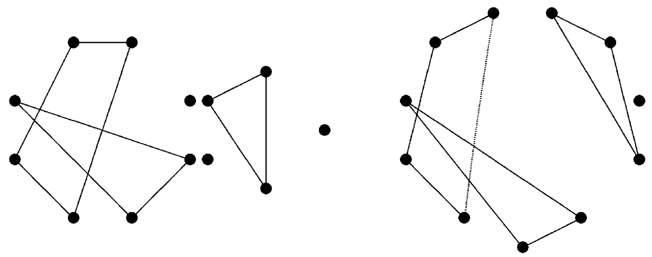

Fig. 8. 2-touch of $C_3 \cup C_4 \cup K_1$ and $C_3 \cup 2K_1$ and the resulting cyclic embedding of $2C_3 \cup C_4 \cup K_1$.

follows from Lemma 9. An example of this construction is shown in Fig. 8. This finishes the proof of our theorem. □

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