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Adomian decomposition: a tool for solving a system of fractional differential equations

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Abstract

Adomian decomposition method has been employed to obtain solutions of a system of fractional differential equations. Convergence of the method has been discussed with some illustrative examples. In particular, for the initial value problem:

$$[D^{\alpha_1} y_1, \ldots, D^{\alpha_n} y_n]^t = A(y_1, \ldots, y_n)^t, \quad y_i(0) = c_i, \quad i = 1, \ldots, n,$$

where $A = [a_{ij}]$ is a real square matrix, the solution turns out to be $\bar{y}(x) = \mathcal{E}_{(\alpha_1,\ldots,\alpha_n),1}(x^{\alpha_1} A_1, \ldots, x^{\alpha_n} A_n)\bar{y}(0)$, where $\mathcal{E}_{(\alpha_1,\ldots,\alpha_n),1}$ denotes multivariate Mittag-Leffler function defined for matrix arguments and A_i is the matrix having ith row as $[a_{i1} \ldots a_{in}]$, and all other entries are zero. Fractional oscillation and Bagley–Torvik equations are solved as illustrative examples.

Keywords: Caputo fractional derivative; System of fractional differential equations; Adomian decomposition; Bagley–Torvik equation; Fractional oscillation equation; Mittag-Leffler function

1. Introduction

In recent years considerable interest in fractional differential equations (FDE) has been stimulated due to their numerous applications in the areas of physics and engineering

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[11,15]. Damping laws, diffusion processes [5] and fractals [15] are better formulated with the use of fractional derivatives/integrals [10–12]. Recently, Atanackovic and Stankovic [3] have analyzed lateral motion of an elastic column fixed at one end and loaded at the other, in terms of a system of FDE. Thus system of FDE is an important aspect which finds many applications. Daftardar-Gejji and Babakhani [6] have earlier presented analysis of a system of FDE. They have studied existence, uniqueness and stability of solutions of a system of FDE. In particular they have proved that, for the initial value problem:

$$D^\alpha \bar{y} = A\bar{y}, \quad \bar{y}(0) = \bar{y}_0, \tag{1}$$

where $A = [a_{ij}]$ is a real square matrix, the unique solution is

$$\bar{y}(x) = E_\alpha(x^\alpha A)\bar{y}_0,$$

where E_α is Mittag-Leffler function with matrix arguments. As a pursuit of this in the present paper we obtain analytical solution of the more general system of FDE:

$$D^{\alpha_i} y_i(x) = \sum_{j=1}^{n} \big(\phi_{ij}(x) + \gamma_{ij} D^{\alpha_{ij}}\big) y_j + g_i(x), \quad 1 \leqslant i, j \leqslant n, \tag{2}$$

where D^{α_i} denotes Caputo fractional derivative of order α_i. Amongst a variety of definitions for fractional order derivatives, Caputo fractional derivative has been used [9] as it is suitable for describing various phenomena, since the initial values of the function and its integer order derivatives have to be specified. Numerical methods [7,8], which are commonly used, encounter difficulties in terms of the size of the computational work needed and usually the rounding-off error causes loss of accuracy. A new iterative method proposed by Adomian [2] has proven rather successful in dealing with both linear as well as non-linear problems. This computational method yields analytical solutions and has certain advantages over standard numerical methods. It is free from rounding off errors as it does not involve discretization, and does not require large computer memory or power. Babolian et al. [4] have applied this method to a system of ordinary differential equations. Shawagfeh [13] has employed this method for solving non-linear FDE. In the present paper we explore Adomian decomposition method to obtain solutions of the above mentioned system of FDE. We discuss convergence problem and present illustrations encompassing Bagley and Torvik [14] and fractional oscillation equations [8].

In particular we consider the following system which is a generalization of Eq. (1):

$$[D^{\alpha_1} y_1, \ldots, D^{\alpha_n} y_n]^t = A(y_1, \ldots, y_n)^t, \quad y_i(0) = c_i, \quad i = 1, \ldots, n,$$

where $A = [a_{ij}]$ is a real square matrix, the solution turns out to be

$$\bar{y}(x) = \mathcal{E}_{(\alpha_1,\ldots,\alpha_n),1}(x^{\alpha_1} A_1, \ldots, x^{\alpha_n} A_n)\bar{y}(0),$$

where $\mathcal{E}_{(\alpha_1,\ldots,\alpha_n),1}$ denotes multivariate Mittag-Leffler function defined for matrix arguments and A_i is the matrix having ith row as $[a_{i1} \ldots a_{in}]$, and all other entries are zero. This result generalizes the result obtained by Daftardar-Gejji and Babakhani [6].

The present paper has been organized as follows. In Section 2 we give basic definitions. System of FDE and Adomian decomposition have been dealt with in Section 3, whereas convergence of the decomposition method has been discussed in Section 4. Applications have been presented in Section 5. This is followed by the conclusions, which are summarized in Section 6.

2. Basic definitions

Definition 2.1. A real function $f(x)$, $x>0$, is said to be in the space C_α, $\alpha \in \Re$ if there exists a real number p $(>\alpha)$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leqslant \alpha$.

Definition 2.2. A function $f(x)$, $x>0$, is said to be in the space C_α^m, $m\in N\cup\{0\}$, if $f^{(m)}\in C_\alpha$.

Definition 2.3. The left sided Riemann–Liouville fractional integral of order $\mu \geqslant 0$, [9–12] of a function $f\in C_\alpha$, $\alpha \geqslant -1$, is defined as

$$I^\mu f(x)=\frac{1}{\Gamma(\mu)}\int_0^x \frac{f(t)}{(x-t)^{1-\mu}}\,dt,\quad \mu>0,\ x>0,$$
$$I^0 f(x)=f(x). \tag{3}$$

Definition 2.4. Let $f\in C_{-1}^m$, $m\in N$. Then the (left sided) Caputo fractional derivative of f is defined as [9,11]

$$D^\mu f(x)=\begin{cases}[I^{m-\mu}f^{(m)}(x)], & m-1<\mu\leqslant m,\\ \frac{d^m}{dt^m}f(t), & \mu=m.\end{cases} \tag{4}$$

Note that [9,11]

$$I^\mu I^\nu f=I^{\mu+\nu}f,\quad \mu,\nu\geqslant 0,\ f\in C_\alpha,\ \alpha\geqslant -1,$$
$$I^\mu x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)}x^{\gamma+\mu},\quad \mu>0,\ \gamma>-1,\ x>0,$$
$$I^\mu D^\mu f(t)=f(t)-\sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{t^k}{k!},\quad m-1<\mu\leqslant m. \tag{5}$$

3. System of fractional differential equations and Adomian decomposition

In the present paper we consider the following system of linear fractional differential equations:

$$D^{\alpha_i}y_i(x)=\sum_{j=1}^n\bigl(\phi_{ij}(x)+\gamma_{ij}D^{\alpha_{ij}}\bigr)y_j+g_i(x),\quad y_i^{(k)}(0)=c_k^i, \tag{6}$$

where $0\leqslant k\leqslant[\alpha_i]$, if α_i is not an integer, and $0\leqslant k\leqslant \alpha_i-1$ if α_i is an integer. Here $0\leqslant \alpha_{ij}<\alpha_i$, for $1\leqslant i,j\leqslant n$, γ_{ij}'s are constants and $\phi_{ij}(x), g_i(x)\in C[0,T]$.

Applying I^{α_i} to both the sides of Eq. (6), we get

$$
\begin{aligned}
y_i(x) = {} & \sum_{k=0}^{[\alpha_i]} c_k^i \frac{x^k}{k!} + I^{\alpha_i} g_i(x) - \sum_{j=1}^{n} \gamma_{ij} \sum_{k=0}^{[\alpha_{ij}]} c_k^j \frac{x^{\alpha_i - \alpha_{ij} + k}}{\Gamma(\alpha_i - \alpha_{ij} + k + 1)} \\
& + \sum_{j=1}^{n} \bigl(I^{\alpha_i} \phi_{ij}(x) + \gamma_{ij} I^{\alpha_i - \alpha_{ij}} \bigr) y_j, \quad \text{where } 1 \leqslant i \leqslant n. \qquad (7)
\end{aligned}
$$

We employ Adomian decomposition method to solve the system of Eq. (7). The Adomian decomposition method [2] consists of representing y_i in the decomposition form given by

$$
y_i(x) = \sum_{m=0}^{\infty} y_{im}(x), \qquad (8)
$$

where the components y_{im}, $m \geqslant 0$, can be determined in a recursive manner. Substituting Eq. (8) into both sides of Eq. (7), we get

$$
\begin{aligned}
\sum_{m=0}^{\infty} y_{im}(x) = {} & \sum_{k=0}^{[\alpha_i]} c_k^i \frac{x^k}{k!} + I^{\alpha_i} g_i(x) - \sum_{j=1}^{n} \gamma_{ij} \sum_{k=0}^{[\alpha_{ij}]} c_k^j \frac{x^{\alpha_i - \alpha_{ij} + k}}{\Gamma(\alpha_i - \alpha_{ij} + k + 1)} \\
& + \sum_{m=0}^{\infty} \left(\sum_{j=1}^{n} \bigl(I^{\alpha_i} \phi_{ij}(x) + \gamma_{ij} I^{\alpha_i - \alpha_{ij}} \bigr) y_{jm}(x) \right), \quad 1 \leqslant i \leqslant n. \qquad (9)
\end{aligned}
$$

The decomposition method defines the components $y_{im}(x)$, $m \geqslant 0$, by the following recursion relation:

$$
\begin{aligned}
y_{i0}(x) &= \sum_{k=0}^{[\alpha_i]} c_k^i \frac{x^k}{k!} + I^{\alpha_i} g_i(x) - \sum_{j=1}^{n} \gamma_{ij} \sum_{k=0}^{[\alpha_{ij}]} c_k^j \frac{x^{\alpha_i - \alpha_{ij} + k}}{\Gamma(\alpha_i - \alpha_{ij} + k + 1)}, \\
y_{i,m+1}(x) &= \sum_{j=1}^{n} \bigl(I^{\alpha_i} \phi_{ij}(x) + \gamma_{ij} I^{\alpha_i - \alpha_{ij}} \bigr) y_{jm}(x), \quad 1 \leqslant i \leqslant n,\ m = 0, 1, \ldots. \qquad (10)
\end{aligned}
$$

We approximate the solution $y_i(x)$ by the truncated series

$$
f_{ik}(x) = \sum_{m=0}^{k-1} y_{im}(x) \quad \text{and} \quad \lim_{k \to \infty} f_{ik}(x) = y_i(x).
$$

Adomian decomposition method is very simple in its principles, though the difficulties consist in proving the convergence of the Adomian series [1]. In the following section we prove the convergence of the series $\sum_{m=0}^{\infty} y_{im}(x)$.

4. Convergence

In this section we show convergence of the decomposition series. In view of Eq. (10),

$$
\bigl| y_{i1}(x) \bigr| \leqslant \sum_{j=1}^{n} \bigl| I^{\alpha_i} \phi_{ij}(x) y_{j0}(x) \bigr| + \bigl| \gamma_{ij} I^{\alpha_i - \alpha_{ij}} y_{j0}(x) \bigr|
$$

$$\leqslant nML\frac{x^{\alpha_i}}{\Gamma(\alpha_i+1)}+\gamma L\sum_{j=1}^{n}\frac{x^{\alpha_i-\alpha_{ij}}}{\Gamma(\alpha_i-\alpha_{ij}+1)},$$

where

$$M=\max_{x\in[0,T]}\{\phi_{ij}(x)\},\quad i,j=1,\dots,n,$$
$$\gamma=\max_{1\leqslant i,j\leqslant n}\{\gamma_{ij}\},\qquad L=\max_{x\in[0,T]}y_{i0}(x).$$

Let $\beta_{ij}=\alpha_i-\alpha_{ij}$, $1\leqslant j\leqslant n$, $\beta_{i,n+1}=\alpha_i$, $\delta>\max\{nML,\gamma L\}$ and $z_{ij}=\delta x^{\beta_{ij}}$, $1\leqslant i,j\leqslant n$.

$$\left|y_{i1}(x)\right|\leqslant\delta\sum_{j=1}^{n+1}\frac{x^{\beta_{ij}}}{\Gamma(1+\beta_{ij})}=\sum_{\substack{l_1+\cdots+l_{n+1}=1\\ l_i\geqslant 0}}(k;l_1,\dots,l_{n+1})\left[\frac{\prod_{j=1}^{n+1}z_{ij}^{l_i}}{\Gamma(\beta+\sum_{j=1}^{n+1}\beta_{ij}l_j)}\right],$$

$$\left|y_{i2}(x)\right|\leqslant\delta^2\sum_{j=1}^{n+1}I^{\beta_{ij}}\left[\sum_{k=1}^{n+1}\frac{x^{\beta_{ik}}}{\Gamma(1+\beta_{ik})}\right]=\delta^2\sum_{j,k=1}^{n+1}\frac{x^{\beta_{ij}+\beta_{ik}}}{\Gamma(1+\beta_{ij}+\beta_{ik})}$$
$$=\sum_{\substack{l_1+\cdots+l_{n+1}=2\\ l_i\geqslant 0}}(k;l_1,\dots,l_{n+1})\left[\frac{\prod_{j=1}^{n+1}z_{ij}^{l_i}}{\Gamma(1+\sum_{j=1}^{n+1}\beta_{ij}l_j)}\right],$$

$$\vdots$$

$$\left|y_{im}(x)\right|\leqslant\delta^m\sum_{j_1=1}^{n+1}\sum_{j_2=1}^{n+1}\cdots\sum_{j_m=1}^{n+1}\frac{\prod_{k=1}^{m}x^{\beta_{ij_k}}}{\Gamma(1+\sum_{k=1}^{m}\beta_{ij_k})}$$
$$=\sum_{\substack{l_1+\cdots+l_{n+1}=m\\ l_i\geqslant 0}}(k;l_1,\dots,l_{n+1})\left[\frac{\prod_{j=1}^{n+1}z_{ij}^{l_i}}{\Gamma(1+\sum_{j=1}^{n+1}\beta_{ij}l_j)}\right],$$

$$\vdots$$

Hence $\sum_{m=0}^{\infty}y_{im}\leqslant\mathcal{E}^i_{(\beta_{i1},\dots,\beta_{i,n+1}),\beta}(z_{i1},z_{i2},\dots,z_{i,n+1})$, where $\beta=1$ and $z_{ij}=\delta x^{\beta_{ij}}$, $j=1,\dots,n+1$. The multivariate Mittag-Leffler function $\mathcal{E}^i$ [9,11] is defined below:

$$\mathcal{E}^i_{(\beta_{i1},\dots,\beta_{i,n+1}),\beta}(z_{i1},z_{i2},\dots,z_{i,n+1})$$
$$=\sum_{k=0}^{\infty}\sum_{\substack{l_1+\cdots+l_{n+1}=k\\ l_i\geqslant 0}}(k;l_1,\dots,l_{n+1})\left[\frac{\prod_{j=1}^{n+1}z_{ij}^{l_i}}{\Gamma(\beta+\sum_{j=1}^{n+1}\beta_{ij}l_j)}\right].$$

Thus $|y_{im}(x)|$ is bounded by the mth term of a multivariate Mittag-Leffler series, which is convergent. Hence by comparison test $\sum_{m=0}^{\infty}y_{im}(x)$ is convergent.

5. Applications

In this section we generalize the Mittag-Leffler function for matrix arguments $A_1, \dots, A_n$ as follows:

$$\begin{aligned}
&\mathcal{E}_{(\alpha_1,\dots,\alpha_n),\beta}(A_1,\dots,A_n)\\
&\quad= I + \sum_{i_1=1}^{n} \frac{A_{i_1}}{\Gamma(\beta+\alpha_{i_1})} + \sum_{i_1=1,\, i_2=1}^{n} \frac{A_{i_1}A_{i_2}}{\Gamma(\beta+\alpha_{i_1}+\alpha_{i_2})} + \cdots\\
&\qquad+ \sum_{i_1=1,\, i_2=1,\dots,\, i_m=1}^{n} \frac{A_{i_1}A_{i_2}\dots A_{i_m}}{\Gamma(\beta+\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_m})} + \cdots\\
&\quad= I + \sum_{m=1}^{\infty}\left[\sum_{i_1=1,\, i_2=1,\dots,\, i_m=1}^{n} \frac{A_{i_1}A_{i_2}\dots A_{i_m}}{\Gamma(\beta+\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_m})}\right].
\end{aligned} \tag{11}$$

5.1. Illustrative examples

To demonstrate the effectiveness of the method we consider here some fractional differential equations.

(I) Consider the initial value problem

$$D^{\bar{\alpha}}\bar{y} = A\bar{y}, \quad \bar{y}(0) = (c_1,\dots,c_n)^t, \quad 0<\alpha_i<1,\ i=1,\dots,n,$$

where $\bar{y} = (y_1,\dots,y_n)^t$, $D^{\bar{\alpha}}\bar{y} = (D^{\alpha_1}y_1,\dots,D^{\alpha_n}y_n)^t$ and $A = [a_{ij}]_{n\times n}$ is a square matrix of constants. This system is equivalent to the following system of integral equations:

$$y_i(x) = c_i + I^{\alpha_i}\sum_{j=1}^{n} a_{ij}y_j(x), \quad i=1,2,\dots,n.$$

Note that this is a special case of Eq. (6) and is obtained by putting $\gamma_{ij} = 0 = g_i$. In view of Eq. (10),

$$\begin{aligned}
&y_{i0} = c_i, \qquad y_{i1} = \sum_{j=1}^{n} a_{ij}\frac{x^{\alpha_i}}{\Gamma(1+\alpha_i)},\\
&y_{i2} = \sum_{j,k=1}^{n} a_{ij}a_{jk}c_k\frac{x^{\alpha_i+\alpha_j}}{\Gamma(1+\alpha_i+\alpha_j)},\\
&y_{i3} = \sum_{j,k,s=1}^{n} a_{is}a_{ij}a_{jk}c_k\frac{x^{\alpha_i+\alpha_s+\alpha_j}}{\Gamma(1+\alpha_i+\alpha_j)}, \quad \dots,\\
&y_{im} = \sum_{l_1,l_2,\dots,l_m=1}^{n} a_{l_1l_2}a_{l_1l_2}\dots a_{l_{m-1}l_m}c_{l_m}\left[\frac{x^{\sum_{j=1}^{m}\alpha_{l_j}}}{\Gamma(1+\sum_{j=1}^{m}\alpha_{l_j})}\right].
\end{aligned}$$

It is observed that $\bar{y}(x) = [\mathcal{E}_{(\alpha_1,\dots,\alpha_n),1}(x^{\alpha_1}A_1,\dots,x^{\alpha_n}A_n)]\bar{y}(0)$, where A_i has the only non-zero row as ith row with entries $[a_{i1}\dots a_{in}]$, and all other entries are zero. Here

$\mathcal{E}_{(\alpha_1,\ldots,\alpha_n),1}(x^{\alpha_1}A_1,\ldots,x^{\alpha_n}A_n)$ denotes multivariate Mittag-Leffler function for matrix arguments.

(II) Consider the fractional oscillation equation [8]

$$D^{1+\alpha}y(t)+by(t)=f(t),\quad y(0)=y_0,\quad y'(0)=y'_0,$$

where $\alpha\in(0,1)$ and f satisfies Lipschitz condition. This equation is equivalent to the following system of equations:

$$D^1y_1=y_2,\quad y_1(0)=y_0,$$
$$D^\alpha y_2=-by_1+f(t),\quad y_2(0)=y'_0.$$

Here we solve the following particular case using Adomian decomposition:

$$\begin{aligned}&D^1y_1=y_2,\quad y_1(0)=1,\\&D^\alpha y_2=y_1,\quad y_2(0)=0,\quad \alpha\in(0,1).\end{aligned}\tag{12}$$

To derive the solution, we use the following Adomian scheme:

$$y_1=1+I^1y_2,\qquad y_2=0-I^\alpha y_1,$$
$$y_{10}=1,\quad \text{and}\quad y_{20}=0,$$
$$y_{1,m+1}=Iy_{2m},\quad y_{2,m+1}=-I^\alpha y_{1m},\quad m=0,1,\ldots.$$

In the first iteration we have

$$y_{11}=Iy_{20}=0\quad\text{and}\quad y_{21}=-I^\alpha y_{10}=-\frac{\Gamma(1)}{\Gamma(\alpha+1)}x^\alpha.$$

The subsequent terms are

$$y_{12}=Iy_{21}=-\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\quad\text{and}\quad y_{22}=-I^\alpha y_{11}=0,$$
$$y_{13}=Iy_{22}=0\quad\text{and}\quad y_{23}=-I^\alpha y_{12}=-\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}.$$

In general we get

$$\begin{aligned}y_{1j}&=\begin{cases}(-1)^k\frac{x^{k(\alpha+1)}}{\Gamma(k(\alpha+1)+1)}, & j=2k,\\ 0, & j=2k+1,\end{cases}\\ y_{2j}&=\begin{cases}0, & j=2k,\\ (-1)^{(k+1)}\frac{x^{k(\alpha+1)+\alpha}}{\Gamma[(\alpha+1)(k+1)]}, & j=2k+1.\end{cases}\end{aligned}\tag{13}$$

Using Eq. (8) we can write y_1 and y_2 in the following form:

$$\begin{aligned}y_1&=\sum_{j=0}^{\infty}y_{1j}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{k(\alpha+1)}}{\Gamma(k(\alpha+1)+1)},\\ y_2&=\sum_{j=0}^{\infty}y_{2j}=\sum_{k=0}^{\infty}(-1)^{(k+1)}\frac{x^{k(\alpha+1)+\alpha}}{\Gamma[(\alpha+1)(k+1)]}.\end{aligned}\tag{14}$$

In Figs. 1–4 we plot y_1 for various values of α.

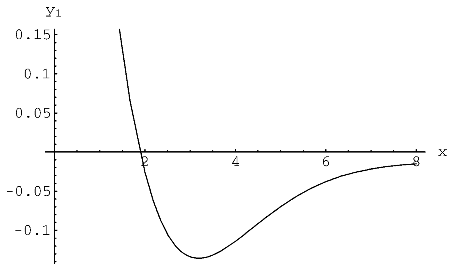

Fig. 1. $\alpha = 0.3$.

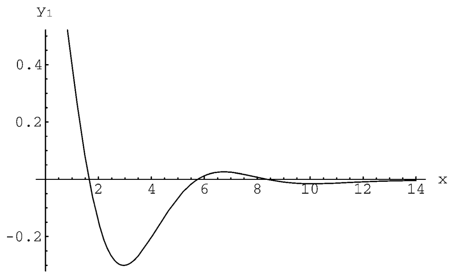

Fig. 2. $\alpha = 0.5$.

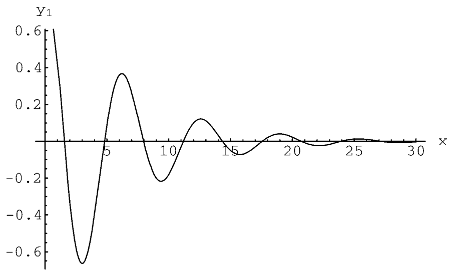

Fig. 3. $\alpha = 0.8$.

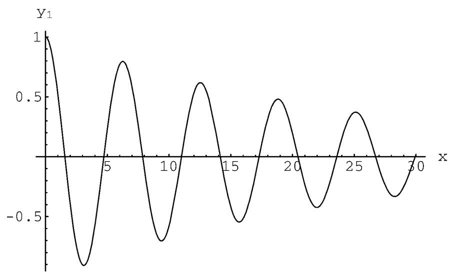

Fig. 4. $\alpha = 0.95$.

Remark. Fractional oscillation equation has been solved by numerical methods by Edwards et al. [8]. They have plotted solutions of Eq. (12) for $\alpha = 0.3$, $\alpha = 0.5$, $\alpha = 0.8$, and $\alpha = 0.95$. It should be remarked that the graphs drawn here using Adomian method are in excellent agreement with those drawn using numerical methods [8].

(III) We discuss Bagley–Torvik equation [8,14] that arises, for instance, in modelling the motion of a rigid plate immersed in Newtonian fluid,

$$D^2 y + b_2 D^{\alpha_1} y + b_1 y = f, \quad y(0) = y_0,\ y'(0) = y_0',$$

where $\alpha_1 = \alpha + 1$, $\alpha \in (0, 1)$. This equation can be viewed as the following system of equations:

$$\begin{aligned} &D^\beta y_1 = y_2, \quad y_1(0) = y_0, \\ &D^\beta + b_2 D^\alpha y_2 = f - b_1 y_1, \quad y_2(0) = y'(0), \end{aligned}$$

where $\beta = 1$. Using the Adomian decomposition method, we obtain the solution for the case $b_1 = b_2 = 1$, $f = 0$, as follows:

$$\begin{aligned} y_1 &= \sum_{k=0}^{[\beta]} c_k^1 \frac{x^k}{k!} - I^\beta y_2, \\ y_2 &= \sum_{k=0}^{[\beta]} c_k^2 \frac{x^k}{k!} - I^{\beta-\alpha} y_2 + \sum_{k=0}^{[\alpha]} c_k^2 \frac{x^{\beta-\alpha+k}}{\Gamma(\beta-\alpha+k+1)} - I^\beta y_1, \end{aligned}$$

$$\begin{cases} y_{20} = 0, \\ y_{2,m+1} = -I^{\beta-\alpha} y_{2m} - I^{\beta} y_{1m}, \end{cases} \quad \begin{cases} y_{10} = 1, \\ y_{1,m+1} = I^{\beta} y_{2m}, \end{cases} \quad m = 0, 1, 2, \ldots. \tag{15}$$

In the first iteration we get

$$\begin{cases} y_{11} = I^{\beta} y_{20} = 0, \\ y_{21} = -I^{\beta-\alpha} y_{20} - I^{\beta} y_{10} = -\frac{x^{\alpha}}{\Gamma(\alpha+1)}, \end{cases}$$

and the following terms are:

$$\begin{cases} y_{12} = I^{\beta} y_{21} = -\frac{x^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, \\ y_{22} = \frac{x^{2\beta-\alpha}}{\Gamma(2\beta-\alpha+1)}, \end{cases} \qquad \begin{cases} y_{13} = \frac{x^{3\beta-\alpha}}{\Gamma(3\beta-\alpha+1)}, \\ y_{23} = -\frac{x^{3\beta-2\alpha}}{\Gamma(3\beta-2\alpha+1)} + \frac{x^{3\beta}}{\Gamma(3\beta+1)}, \end{cases}$$

$$\begin{cases} y_{14} = \frac{x^{4\beta-2\alpha}}{\Gamma(4\beta-2\alpha+1)} + \frac{x^{4\beta}}{\Gamma(4\beta+1)}, \\ y_{24} = \frac{x^{4\beta-3\alpha}}{\Gamma(4\beta-3\alpha+1)} - 2\frac{x^{4\beta-\alpha}}{(4\beta-\alpha+1)}, \end{cases}$$

$$y_{1n} = \sum_{j=1}^{[n/2]} (-1)^{n+j} a_{nj} \frac{x^{n(\beta-\alpha)+2j\alpha}}{\Gamma(n(\beta-\alpha)+2j\alpha+1)}, \tag{16}$$

where

$$a_{nj} = \begin{cases} a_{n-1,j} + a_{n-2,j-1}, & 1 \leqslant j \leqslant \frac{n}{2}, \\ 1, & n = j = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{17}$$

Following Eq. (8) we can write y_1 and y_2 in the following form:

$$y_1 = \sum_{n=0}^{\infty} y_{1n} = 1 + \sum_{n=0}^{\infty} \sum_{j=1}^{[n/2]} (-1)^{n+j} a_{nj} \frac{x^{n(\beta-\alpha)+2j\alpha}}{\Gamma(n(\beta-\alpha)+2j\alpha+1)},$$

$$y_2 = \sum_{n=0}^{\infty} y_{2n} = \sum_{n=2}^{\infty} \sum_{j=1}^{[n/2]} (-1)^{n+j+1} b_{nj} \frac{x^{n(\beta-\alpha)+(2j-1)\alpha}}{\Gamma(n(\beta-\alpha)+(2j-1)\alpha+1)} + \sum_{k=0}^{[n/2]} (-1)^k \frac{x^{2k}}{\Gamma(2k+1)},$$

where b_{nj} is defined similar to a_{nj}.

In Figs. 5, 6 we draw y_1 for $\alpha = 0.5$, $\beta = 1$ and $\alpha = 0.25$, $\beta = 1$, respectively.

Remark. The graphs drawn in the Figs. 5 and 6 are in excellent agreement with those drawn in [8] using numerical methods.

6. Conclusions

Adomian decomposition method is a powerful tool which enables to find analytical solutions in case of linear as well as non-linear equations. The method has been successfully applied to a system of FDE. It is interesting to note that for the initial value

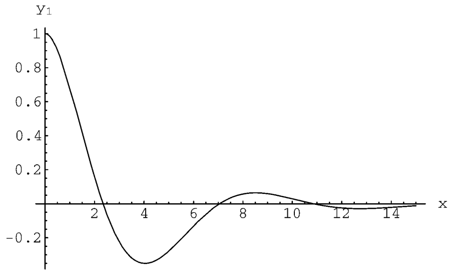

Fig. 5. $\alpha = 0.25$.

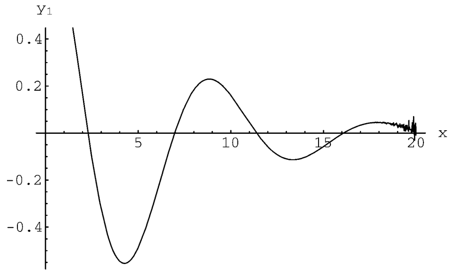

Fig. 6. $\alpha = 0.5$.

problem $[D^{\alpha_1} y_1, \ldots, D^{\alpha_n} y_n]^t = A(y_1, \ldots, y_n)^t$, $y_i(0) = c_i$, $i = 1, \ldots, n$, where A is a real square matrix, the solution turns out to be $\bar{y}(x) = \mathcal{E}_{(\alpha_1,\ldots,\alpha_n),1}(x^{\alpha_1} A_1, \ldots, x^{\alpha_n} A_n)\bar{y}(0)$, A_i is a matrix having ith row as $[a_{i1} \ldots a_{in}]$, and all other entries are zero and $\mathcal{E}_{(\alpha_1,\ldots,\alpha_n),1}(x^{\alpha_1} A_1, \ldots, x^{\alpha_n} A_n)$ denotes an extension of multivariate Mittag-Leffler function for matrix arguments. In particular the graphs (Figs. 1–4) of fractional oscillation equations and those (Figs. 5, 6) of Bagley–Torvik equation are in excellent agreement with those obtained using numerical methods [8].

The computations associated with the illustrative examples in this paper were carried out using Mathematica.

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