Note

Common multiples of complete graphs and a 4-cycle

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Abstract

A graph $G$ is a common multiple of two graphs $H_1$ and $H_2$ if there exists a decomposition of $G$ into edge-disjoint copies of $H_1$ and also a decomposition of $G$ into edge-disjoint copies of $H_2$. In this paper, we consider the case where $H_1$ is the 4-cycle $C_4$ and $H_2$ is the complete graph with $n$ vertices $K_n$. We determine, for all positive integers $n$, the set of integers $q$ for which there exists a common multiple of $C_4$ and $K_n$ having precisely $q$ edges.

Keywords: Graph decomposition; Cycle decomposition; Graph design; Cycle system

1. Introduction

If $G$ and $H$ are graphs and $S$ is a set of subgraphs of $G$, all isomorphic to $H$, such that the edge set of $G$ is partitioned by the edge sets of the subgraphs of $S$, then $S$ is called an $H$-decomposition of $G$ and $G$ is said to be $H$-decomposable. If a graph $G$ is $H$-decomposable, then we write $H|G$ and say that $H$ divides $G$.

Given two graphs $H_1$ and $H_2$, one may ask for a graph $G$ that is a common multiple of $H_1$ and $H_2$ in the sense that both $H_1$ and $H_2$ divide $G$. Several authors have investigated the problem of finding least common multiples of pairs of graphs; that is, graphs of minimum size which are both $H_1$- and $H_2$-decomposable. The problem was introduced by Chartrand et al. in [3] and they showed that every two nonempty graphs have a least common multiple. The problem of finding the size of least common
Multiples of graphs has been studied for several pairs of graphs: cycles and stars [3,12], paths and complete graphs [9], pairs of cycles [7], and pairs of complete graphs [1]. Pairs of graphs having a unique least common multiple were investigated in [4] and least common multiples of digraphs were considered in [5].

If $G$ is a common multiple of $H_1$ and $H_2$ and $G$ has $q$ edges, then we call $G$ a $(q,H_1,H_2)$ graph. An obvious necessary condition for the existence of a $(q,H_1,H_2)$ graph is that $\sigma(H_1)\mid q$ and $\sigma(H_2)\mid q$, where $\sigma(H)$ denotes the number of edges in the graph $H$. This obvious necessary condition is not always sufficient. Hence, a natural question is: given two graphs $H_1$ and $H_2$, for which values of $q$ does there exist a $(q,H_1,H_2)$ graph? Here we give a complete solution to this problem in the case where $H_1$ is the 4-cycle $C_4$ and $H_2$ is a complete graph, see Theorem 1.2.

We require some additional notation. The complete graph with vertex set $\{v_1,v_2,\ldots,v_m\}$ will be denoted by $[v_1,v_2,\ldots,v_m]$ and the $m$-cycle $C_m$ with vertex set $\{v_1,v_2,\ldots,v_m\}$ and edges $\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_m,v_1\}$ will be denoted by $(v_1,v_2,\ldots,v_m)$. If $H$ and $G$ are graphs, and $H$ is a subgraph of $G$, then the graph obtained by removing the edges of $H$ from $G$ will be denoted by $G-H$. In particular, the graph obtained by removing the edges in a 1-factor from $K_n$ will be denoted by $K_n-F$. If $G_1$ and $G_2$ are graphs, then the union of $G_1$ and $G_2$, denoted $G_1 \cup G_2$, is the graph with vertex set $V(G_1 \cup G_2)=V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2)=E(G_1) \cup E(G_2)$. (We shall only be considering the union of edge-disjoint graphs.)

We shall make use of line graphs (see Lemma 1.9). The line graph of a graph $G$ is the graph $L(G)$ with vertex set $V(L(G))=E(G)$ and edge set $E(L(G))=\{(\{x,y\},\{x,z\}) : \{x,y\},\{x,z\} \in E(G)\}$. Several authors have investigated $C_4$-decompositions of the line graph of $K_n$ including Heinrich and Nonay [8] and Colby and Rodger [6].

We shall use equitable partial 4-cycle systems and the following theorem (Theorem 1.1) of Raines and Szaniszló [10]. A partial 4-cycle system of order $v$ is a set of edge-disjoint 4-cycles in $K_v$. The subgraph of $K_v$ whose edges occur in the partial 4-cycle system will be called the underlying graph. The leave of a partial 4-cycle system of order $v$ is the subgraph of $K_v$ whose edges do not occur in the partial 4-cycle system. In a partial 4-cycle system, if $r(x)$ denotes the number of 4-cycles containing the vertex $x$ and $|r(x)-r(y)| \leq 1$ for all vertices $x$ and $y$, then the partial 4-cycle system is said to be equitable.

**Theorem 1.1** (Raines and Szaniszló [10]). Let $M$ be the maximum number of 4-cycles in any partial 4-cycle system of order $v$. Then there exists an equitable partial 4-cycle system of order $v$ with $t$ 4-cycles for all $t$ in the range $1 \leq t \leq M$.

We also make use of maximum 4-cycle packings; that is, partial 4-cycle systems of order $v$ containing the maximum number of 4-cycles. Maximum 4-cycle packings have the following underlying graphs [11]:

- $K_v$ if $v \equiv 1 \pmod{8}$;
- $K_v-C_3$ if $v \equiv 3 \pmod{8}$;
• $K_v - H$ if $v \equiv 5 \,(\text{mod} \,8)$, where $H$ is any one of a 6-cycle, two $K_3$’s with a common vertex, or two vertex-disjoint $K_3$’s;
• $K_v - C_5$ if $v \equiv 7 \,(\text{mod} \,8)$;
• $K_v - F$ if $v \equiv 0, 2, 4, \text{ or } 6 \,(\text{mod} \,8)$.

If there exists a $(q, C_4, K_n)$ graph, then clearly we require that 4 divides $q$ and that $\binom{n}{2}$ divides $q$. Conditions (1)–(3) of our main theorem (Theorem 1.2) follow immediately from this and will be referred to as the obvious necessary conditions. The following two lemmas establish the remaining necessary conditions (see (4) and (5) of Theorem 1.2).

**Lemma 1.1.** If $n \equiv 5, 7 \,(\text{mod} \,8)$ and there exists a $(q, C_4, K_n)$ graph, then $q > 4 \binom{n}{2}$.

**Proof.** If $n \equiv 5 \,(\text{mod} \,8)$, then the obvious necessary condition for the existence of a $(q, C_4, K_n)$ graph $G$ is that $q \equiv 0 \,(\text{mod} \,2 \binom{n}{2})$. If $n \equiv 7 \,(\text{mod} \,8)$, then the obvious necessary condition for the existence of a $(q, C_4, K_n)$ graph $G$ is that $q \equiv 0 \,(\text{mod} \,4 \binom{n}{2})$.

First suppose that $n \equiv 5 \,(\text{mod} \,8)$, and that there exists a $(2 \binom{n}{2}, C_4, K_n)$ graph $G$. This is clearly impossible since such a graph $G$ consists of two copies of $K_n$ intersecting in at most one vertex (in order for $G$ to be $K_n$-decomposable), and $K_n$ is not $C_4$-decomposable.

Now suppose that $n \equiv 5$ or 7 (mod 8) and that there exists a $(4 \binom{n}{2}, C_4, K_n)$ graph $G$. Let the four copies of $K_n$ in a $K_n$-decomposition of $G$ be $G_1, G_2, G_3$ and $G_4$.

For $i = 1, 2, 3, 4$, let $S_i = \{v \in V(G_i) \mid v \in \bigcup_{j \neq i} V(G_j)\}$, and let $T_i = \{|u, v\} \mid u, v \in S_i\};$ that is, $S_i$ is the set of vertices in the graph $G_i$ which are also in at least one other copy of $K_n$, and $T_i$ is the set of edges which have both endpoints lying in more than one copy of $K_n$. Since two copies of $K_n$ can intersect in at most one vertex, $|S_i| \leq 3$ and $|T_i| \leq 3$ for $i = 1, 2, 3, 4$.

Now, a $C_4$-decomposition of $G$ consists of some “pure” $C_4$’s (lying completely within a copy of $K_n$) and some “mixed” $C_4$’s (whose edges are from at least two distinct copies of $K_n$). For $i = 1, 2, 3, 4$, let $L_i$ be the subgraph of $G_i$ that remains when all the pure $C_4$’s are removed from $G_i$. Note that $L_i$ is the leaf of a partial 4-cycle system of order $n$.

Since any mixed $C_4$ must contain edges from at least three distinct copies of $K_n$, it is clear that any $C_4$ with an edge in $L_i$ for some $i$, must have at least two edges in $T_1 \cup T_2 \cup T_3 \cup T_4$. It follows that $|E(L_1) \cup E(L_2) \cup E(L_3) \cup E(L_4)| \leq 2|T_1 \cup T_2 \cup T_3 \cup T_4|$, and hence that $|E(L_i)| \leq 2|T_i|$ for some $i$. It also follows that if $e \in E(L_i)$ for some $i$, then $e$ must be incident with some $v \in S_i$.

Thus, for some $i$, $|E(L_i)| \leq 2|T_i|$, and $|T_i| \leq 3$, so $|E(L_i)| \leq 6$. By considering the possible leaves of a partial 4-cycle system of order $n$, the graph $L_i$ must be one of the following graphs: a 6-cycle, two $K_3$’s with a common vertex, or two disjoint $K_3$’s (if $n \equiv 5 \,(\text{mod} \,8)$); or a 5-cycle (if $n \equiv 7 \,(\text{mod} \,8)$).

It is easily shown that for each of these graphs $L_i$, the conditions $|S_i| \leq 3$ and $|E(L_i)| \leq 2|T_i|$, force there to be an edge $e$ which is not incident with some $v \in S_i$. Hence, it is impossible for all the edges of each $L_i$ to occur in mixed $C_4$’s, and so no $(4 \binom{n}{2}, C_4, K_n)$ graph exists. \(\Box\)
Lemma 1.2. If $n$ is even and there exists a $(q, C_4, K_n)$ graph, then $q \geq (n+1)\left(\frac{n}{2}\right)$.

**Proof.** Let $G$ be a $(q, C_4, K_n)$ graph where $n$ is even. Since $C_4 | G$, we know that $2|\text{deg}(v)$ for all $v \in V(G)$ and since $K_n | G$, we know that $(n-1)|\text{deg}(v)$ for all $v \in V(G)$. Since $n - 1$ is odd, $2(n-1)|\text{deg}(v)$ for all $v \in V(G)$. That is, each vertex must be in at least two copies of $K_n$. Suppose one copy of $K_n$ is $[v_1, v_2, \ldots, v_n]$. Each of these vertices must occur in another copy of $K_n$ and so there are at least $n + 1$ copies of $K_n$ in total. □

We now construct the $(q, C_4, K_n)$ graphs required to prove Theorem 1.2. First note that if $n \equiv 1 (mod\ 8)$, then $C_4 | K_n$ and hence when $n \equiv 1 (mod\ 8)$, there exists a $(q, C_4, K_n)$ graph $G$ for all $q \equiv 0 (mod\ \left(\frac{n}{2}\right))$ (simply let $G$ be $q/\left(\frac{n}{2}\right)$ vertex-disjoint copies of $K_n$). Two main constructions are used to give the remaining congruence classes of $n (mod\ 8)$. Lemmas 1.7 and 1.9 deal with the cases $n$ is odd and $n$ is even, respectively. First we require a few preliminary lemmas.

Lemma 1.3. If $n$ is odd and there exists a $(\left(\frac{n}{2}\right), C_4, K_n)$ graph, then there exists a $(k\left(\frac{8n+n}{2}\right), C_4, K_{8n+n})$ graph for all non-negative integers $x$.

**Proof.** Let $n$ be odd, let $G_0$ be a $(k\left(\frac{n}{2}\right), C_4, K_n)$ graph and let $K_n(1), K_n(2), \ldots, K_n(k)$ be $k$ copies of $K_n$ in a $K_n$-decomposition of $G_0$. For $i=1, 2, \ldots, k$ let $V_i=\{v_1(i), v_2(i), \ldots, v_k(i)\}$ where $V_i \cap V_j = \emptyset$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Now let $H_i$ be the complete graph with vertex set $V(K_n(i)) \cup V_i$. Finally, let $G=H_1 \cup H_2 \cup \cdots \cup H_k$, and so $\{H_1, H_2, \ldots, H_k\}$ is a $K_{8n+n}$-decomposition of $G$. We now show that $G$ is $C_4$-decomposable. Since there exists a $C_4$-decomposition of $K_{8n+n} - K_n$ for all positive integers $x$ (see [2]), there exists a $C_4$-decomposition of $G - G_0$ (for $i = 1, 2, \ldots, k$, decompose $H_i$ into copies of $C_4$ leaving the edges of $K_n(i)$). Hence, since there exists a $C_4$-decomposition of $G_0$, $G$ is also $C_4$-decomposable. □

Lemma 1.3 allows us to construct all of the $(q, C_4, K_n)$ graphs that we require for $n \equiv 3, 5, 7 (mod\ 8)$ if we construct the required graphs for $n = 3, 5$ and 7. We now construct these graphs.

Lemma 1.4. For all $k \equiv 0 (mod\ 4)$, there exists a $(3k, C_4, K_3)$ graph.

**Proof.** Let $G$ be $k/4$ vertex-disjoint copies of the graph of the octahedron (that is, $K_{2,2,2}$ or $K_6 - F$). It is well known, and easy to see, that the graph of the octahedron is $C_4$-decomposable and $K_3$-decomposable. □

Lemma 1.5. For all even $k \geq 6$, there exists a $(10k, C_4, K_5)$ graph.

**Proof.** It is sufficient to construct a $(60, C_4, K_3)$ graph, an $(80, C_4, K_5)$ graph and a $(100, C_4, K_5)$ graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.
To construct a $(60, C_4, K_3)$ graph $G$, we let $G$ be the union of the following six edge-disjoint copies of $K_3$.

\[
\begin{align*}
[0, 1, 2, 3, 4] & \quad [0, 5, 6, 7, 8] & \quad [1, 5, 9, 10, 11] & \quad [2, 6, 9, 12, 13] \\
[3, 7, 10, 12, 14] & \quad [4, 8, 11, 13, 14]
\end{align*}
\]

A $C_4$-decomposition of $G$ is given by the following 15 copies of $C_4$.

\[
\begin{align*}
(0, 1, 2, 3) & \quad (0, 2, 4, 8) & \quad (0, 4, 1, 5) & \quad (0, 6, 5, 7) & \quad (1, 3, 4, 11) \\
(1, 9, 12, 10) & \quad (2, 6, 7, 12) & \quad (2, 9, 6, 13) & \quad (3, 7, 14, 10) & \quad (3, 12, 13, 14) \\
(4, 13, 11, 14) & \quad (5, 8, 7, 10) & \quad (5, 9, 10, 11) & \quad (6, 8, 14, 12) & \quad (8, 11, 9, 13)
\end{align*}
\]

To construct an $(80, C_4, K_3)$ graph $G$, we let $G$ be the union of the following eight edge-disjoint copies of $K_3$.

\[
\begin{align*}
[0, 1, 2, 3, 4] & \quad [0, 5, 6, 7, 8] & \quad [1, 5, 9, 10, 11] & \quad [2, 6, 9, 12, 13] \\
[3, 7, 14, 15, 16] & \quad [4, 8, 14, 17, 18] & \quad [10, 12, 15, 17, 19] & \quad [11, 13, 16, 18, 19]
\end{align*}
\]

A $C_4$-decomposition of $G$ is given by the following 20 copies of $C_4$.

\[
\begin{align*}
(0, 1, 2, 3) & \quad (0, 2, 4, 8) & \quad (0, 4, 1, 5) & \quad (0, 6, 5, 7) & \quad (1, 3, 15, 10) \\
(1, 9, 5, 11) & \quad (2, 6, 9, 12) & \quad (2, 9, 11, 13) & \quad (3, 4, 14, 7) & \quad (3, 14, 15, 16) \\
(4, 17, 14, 18) & \quad (5, 8, 17, 10) & \quad (6, 7, 15, 12) & \quad (6, 8, 18, 13) & \quad (7, 8, 14, 16) \\
(9, 10, 12, 13) & \quad (10, 11, 16, 19) & \quad (11, 18, 17, 19) & \quad (12, 17, 15, 19) & \quad (13, 16, 18, 19)
\end{align*}
\]

To construct a $(100, C_4, K_3)$ graph $G$, we let $G$ be the union of the following 10 edge-disjoint copies of $K_3$.

\[
\begin{align*}
[0, 1, 2, 3, 4] & \quad [0, 5, 6, 7, 8] & \quad [1, 5, 9, 10, 11] & \quad [2, 6, 9, 12, 13] \\
[3, 7, 10, 12, 14] & \quad [4, 15, 16, 17, 18] & \quad [8, 15, 19, 20, 21] & \quad [11, 16, 19, 22, 23] \\
[13, 17, 20, 22, 24] & \quad [14, 18, 21, 23, 24]
\end{align*}
\]

A $C_4$-decomposition of $G$ is given by the following 25 copies of $C_4$.

\[
\begin{align*}
(0, 1, 2, 3) & \quad (0, 2, 6, 5) & \quad (0, 4, 3, 7) & \quad (0, 6, 7, 8) & \quad (1, 3, 10, 5) \\
(1, 4, 16, 11) & \quad (1, 9, 12, 10) & \quad (2, 4, 17, 13) & \quad (2, 9, 6, 12) & \quad (3, 12, 7, 14) \\
(4, 15, 16, 18) & \quad (5, 7, 10, 9) & \quad (5, 8, 19, 11) & \quad (6, 8, 20, 13) & \quad (8, 15, 18, 21) \\
(9, 11, 22, 13) & \quad (10, 11, 23, 14) & \quad (12, 13, 24, 14) & \quad (14, 18, 23, 21) & \quad (15, 17, 16, 19) \\
(15, 20, 19, 21) & \quad (16, 22, 19, 23) & \quad (17, 18, 24, 20) & \quad (17, 22, 23, 24) & \quad (20, 21, 24, 22)
\end{align*}
\]

\[\square\]

**Lemma 1.6.** For all $k \equiv 0 \pmod{4}$, $k \geq 8$, there exists a $(21k, C_4, K_3)$ graph.
Proof. It is sufficient to construct a \((168, C_4, K_7)\) graph and a \((252, C_4, K_7)\) graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.

To construct a \((168, C_4, K_7)\) graph \(G\), we let \(G\) be the union of the following eight edge-disjoint copies of \(K_7\).

\[
\begin{align*}
[0, 1, 2, 3, 4, 5, 6] & \quad [0, 7, 8, 9, 10, 11, 12] & \quad [1, 7, 13, 14, 15, 16, 17] \\
[2, 8, 13, 18, 19, 20, 21] & \quad [3, 9, 14, 18, 22, 23, 24] & \quad [4, 10, 15, 19, 22, 25, 26] \\
[5, 11, 16, 20, 23, 25, 27] & \quad [6, 12, 17, 21, 24, 26, 27]
\end{align*}
\]

A \(C_4\)-decomposition of \(G\) is given by the following 42 copies of \(C_4\).

\[
\begin{align*}
(0, 1, 2, 3) & \quad (0, 2, 4, 5) & \quad (0, 4, 1, 6) & \quad (0, 7, 8, 9) & \quad (0, 8, 10, 11) \\
(0, 10, 7, 12) & \quad (1, 3, 4, 15) & \quad (1, 5, 2, 13) & \quad (1, 7, 9, 14) & \quad (1, 16, 7, 17) \\
(2, 6, 3, 18) & \quad (2, 8, 11, 20) & \quad (2, 19, 8, 21) & \quad (3, 5, 6, 24) & \quad (3, 9, 10, 22) \\
(3, 14, 16, 23) & \quad (4, 6, 12, 10) & \quad (4, 19, 10, 25) & \quad (4, 22, 15, 26) & \quad (5, 11, 9, 23) \\
(5, 16, 11, 25) & \quad (5, 20, 16, 27) & \quad (6, 17, 12, 21) & \quad (6, 26, 12, 27) & \quad (7, 11, 23, 14) \\
(7, 13, 14, 15) & \quad (8, 12, 9, 18) & \quad (8, 13, 18, 20) & \quad (9, 22, 14, 24) & \quad (10, 15, 17, 26) \\
(11, 12, 24, 27) & \quad (13, 15, 16, 17) & \quad (13, 16, 25, 19) & \quad (13, 20, 19, 21) & \quad (14, 17, 21, 18) \\
(15, 19, 22, 25) & \quad (17, 24, 21, 27) & \quad (18, 19, 26, 22) & \quad (18, 23, 22, 24) & \quad (20, 21, 26, 25) \\
(20, 23, 25, 27) & \quad (23, 24, 26, 27)
\end{align*}
\]

To construct a \((252, C_4, K_7)\) graph \(G\), we let \(G\) be the union of the following 12 edge-disjoint copies of \(K_7\).

\[
\begin{align*}
[0, 1, 2, 3, 4, 5, 6] & \quad [0, 7, 8, 9, 10, 11, 12] & \quad [1, 9, 13, 14, 15, 16, 17] \\
[2, 14, 18, 19, 20, 21, 22] & \quad [3, 19, 23, 24, 25, 26, 27] & \quad [4, 7, 24, 28, 29, 30, 31] \\
[8, 32, 33, 34, 35, 36, 37] & \quad [13, 35, 38, 39, 40, 41, 42] & \quad [18, 40, 43, 44, 45, 46, 47] \\
[23, 45, 48, 49, 50, 51, 52] & \quad [28, 32, 50, 53, 54, 55, 56] & \quad [33, 38, 43, 48, 53, 57, 58]
\end{align*}
\]

A \(C_4\)-decomposition of \(G\) is given by the following 63 copies of \(C_4\).

\[
\begin{align*}
(0, 1, 9, 10) & \quad (1, 2, 14, 15) & \quad (2, 3, 19, 20) & \quad (3, 4, 24, 25) & \quad (0, 4, 29, 7) \\
(8, 9, 13, 35) & \quad (13, 14, 18, 40) & \quad (18, 19, 23, 45) & \quad (23, 24, 28, 50) & \quad (7, 8, 32, 28) \\
(32, 33, 53, 54) & \quad (33, 34, 35, 38) & \quad (38, 39, 40, 43) & \quad (43, 44, 45, 48) & \quad (48, 49, 50, 53) \\
(0, 2, 4, 5) & \quad (0, 3, 1, 6) & \quad (0, 8, 10, 11) & \quad (0, 9, 7, 12) & \quad (1, 4, 6, 5) \\
(1, 13, 15, 16) & \quad (1, 14, 9, 17) & \quad (2, 5, 3, 6) & \quad (2, 18, 20, 21) & \quad (2, 19, 14, 22) \\
(3, 23, 25, 26) & \quad (3, 24, 19, 27) & \quad (4, 7, 24, 30) & \quad (4, 28, 29, 31) & \quad (7, 10, 12, 11)
\end{align*}
\]
Lemma 1.7. There exists a \((q, C_4, K_n)\) graph if

\[
\begin{align*}
(7,30,28,31) & \quad (8,11,9,12) & \quad (8,33,35,36) & \quad (8,34,32,37) & \quad (9,15,17,16) \\
(13,16,14,17) & \quad (13,38,40,41) & \quad (13,39,35,42) & \quad (14,20,22,21) & \quad (18,21,19,22) \\
(18,43,45,46) & \quad (18,44,40,47) & \quad (19,25,27,26) & \quad (23,26,24,27) & \quad (23,48,50,51) \\
(23,49,45,52) & \quad (24,29,30,31) & \quad (28,53,32,55) & \quad (28,54,50,56) & \quad (32,35,37,36) \\
(32,50,55,56) & \quad (33,36,34,37) & \quad (33,43,53,57) & \quad (33,48,38,58) & \quad (35,40,42,41) \\
(38,41,39,42) & \quad (38,53,58,57) & \quad (40,45,47,46) & \quad (43,46,44,47) & \quad (43,57,48,58) \\
(45,50,52,51) & \quad (48,51,49,52) & \quad (53,55,54,56) & \quad \Box
\end{align*}
\]

(1) \(q \equiv 0 \pmod{4} \binom{n}{2}\) and \(n \equiv 3 \pmod{8}\); or
(2) \(q \equiv 0 \pmod{2} \binom{n}{2}\), \(q \geq 6 \binom{n}{2}\) and \(n \equiv 5 \pmod{8}\); or
(3) \(q \equiv 0 \pmod{4} \binom{n}{2}\), \(q \geq 8 \binom{n}{2}\) and \(n \equiv 7 \pmod{8}\).

Proof. The result is true for \(n = 3, 5, 7\) by Lemmas 1.4, 1.5 and 1.6 respectively. Hence by applying Lemma 1.3, the result is true for all the required values of \(n\). \(\Box\)

We now construct the required \((q, C_4, K_n)\) graphs for the case \(n\) is even. Our constructions use \(C_4\)-decomposable \(n\)-regular graphs with \(k\) vertices, where \(k = q/\binom{n}{2}\). (So \(k\) is the number of copies of \(K_n\) in the \((q, C_4, K_n)\) graph.) The following lemma is an easy consequence of Theorem 1.1 and gives necessary and sufficient conditions for the existence of such graphs.

Lemma 1.8. There exists a \(C_4\)-decomposable \(n\)-regular graph \(G\) with \(k\) vertices if and only if \(k \geq n + 1\), \(n\) is even and \(nk \equiv 0 \pmod{8}\).

Proof. It is easy to see that the conditions are necessary. By Theorem 1.1, there exists an equitable partial 4-cycle system of order \(k\) containing \(nk/8\) copies of \(C_4\). Since the system is equitable, the vertices in the underlying graph differ in degree by at most 2 and so it follows that the graph is \(n\)-regular. \(\Box\)

Lemma 1.9. Suppose \(q \equiv (n+1) \binom{n}{2}\). Then there exists a \((q, C_4, K_n)\) graph if

\[
\begin{align*}
(1) & \quad q \equiv 0 \pmod{\binom{n}{2}} \text{ and } n \equiv 0 \pmod{8}; \text{ or} \\
(2) & \quad q \equiv 0 \pmod{2} \binom{n}{2} \text{ and } n \equiv 4 \pmod{8}; \text{ or} \\
(3) & \quad q \equiv 0 \pmod{4} \binom{n}{2} \text{ and } n \equiv 2, 6 \pmod{8}.
\end{align*}
\]

Proof. We construct a \(K_n\)-decomposable graph \(G\) in such a way that there is a 1-factor in each \(K_n\) and the union of the edges in these 1-factors is \(C_4\)-decomposable. The result then follows since there exists a \(C_4\)-decomposition of \(K_n - F\) for all even \(n\).

Let \(n\) and \(q\) be as in the statement of the lemma, let \(k = q/\binom{n}{2}\), and let \(G_0\) be an \(n\)-regular graph on \(k\) vertices such that \(G_0\) has a \(C_4\)-decomposition \(T_0\) (such a graph
exists by Lemma 1.8). Now let \( G \) be the line graph of \( G_0 \). That is, let \( V(G) = E(G_0) \) and \( E(G) = \{ \{x, y\}, \{x, z\} : \{x, y\}, \{x, z\} \in E(G_0) \} \). Then \( S = \{ \{x, y_1\}, \{x, y_2\}, \ldots, \{x, y_n\} : x \in V(G_0), \{x, y_1\}, \{x, y_2\}, \ldots, \{x, y_n\} \in V(G) \} \) is a \( K_n \)-decomposition of \( G \).

To see that \( C_4 \mid G \), we first note that the \( C_4 \)-decomposition \( T_0 \) of \( G_0 \) induces a 1-factor in each \( K_n \) if only if \( \{x, y_i\} \) and \( \{x, y_j\} \) are adjacent edges in a 4-cycle of \( T_0 \). We partition the edges of each \( K_n \) into copies of \( C_4 \) and its induced 1-factor and put all the copies of \( C_4 \) in \( T \). So now, the edges in the induced 1-factors are the only edges of \( G \) that have not been partitioned into copies of \( C_4 \). To complete the \( C_4 \)-decomposition \( T \) we let \((\{w, x\}, \{x, y\}, \{y, z\}, \{z, w\}) \in T\) for each 4-cycle \((w, x, y, z) \in T_0 \). \( \square \)

Combining the results of Lemmas 1.1, 1.2, 1.7 and 1.9, we have our main theorem.

**Theorem 1.2.** There exists a graph with \( q \) edges that is both \( C_4 \)-decomposable and \( K_n \)-decomposable if and only if

1. \( q \equiv 0 \pmod{\binom{n}{2}} \) when \( n \equiv 0, 1 \pmod{8} \);
2. \( q \equiv 0 \pmod{2 \binom{n}{2}} \) when \( n \equiv 4, 5 \pmod{8} \);
3. \( q \equiv 0 \pmod{4 \binom{n}{2}} \) when \( n \equiv 2, 3, 6, 7 \pmod{8} \);
4. \( q \geq (n + 1) \binom{n}{2} \) when \( n \) is even;
5. \( q > 4 \binom{n}{2} \) when \( n \equiv 5, 7 \pmod{8} \).

The following corollary gives the size of the least common multiple of \( C_4 \) and \( K_n \).

**Corollary 1.1.** Let \( q_0 \) be the number of edges in a least common multiple of \( C_4 \) and \( K_n \). Then

\[
q_0 = \begin{cases} 
\binom{n}{2} & \text{if } n \equiv 1 \pmod{8}, \\
4 \binom{n}{2} & \text{if } n \equiv 3 \pmod{8}, \\
6 \binom{n}{2} & \text{if } n \equiv 5 \pmod{8}, \\
8 \binom{n}{2} & \text{if } n \equiv 7 \pmod{8}, \\
(n + 1) \binom{n}{2} & \text{if } n \equiv 0 \pmod{8}, \\
(n + 2) \binom{n}{2} & \text{if } n \equiv 2, 4, 6 \pmod{8}.
\end{cases}
\]

**References**