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Stability of delay equations via Lyapunov functions ☆

Luis Barreira *, Claudia Valls

Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

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ABSTRACT

The importance of Lyapunov functions is well known. In the general setting of nonautonomous linear delay equations $v' = L(t)v_t$, we show how to characterize completely the existence of a nonuniform exponential contraction or of a nonuniform exponential dichotomy in terms of Lyapunov functions. This includes uniform exponential behavior as a very special case, and it provides an alternative (usually simpler and particularly more direct) approach to verify the existence of exponential behavior or to obtain the robustness of the dynamics under sufficiently small perturbations.

1. Introduction

We consider a linear delay equation

$$v' = L(t)v_t. \tag{1}$$

Let $T(t,s)$ be the associated evolution operator (see Section 2). Eq. (1) is said to admit a *nonuniform exponential contraction* if there exist constants $a < 0$, $D > 0$ and $\varepsilon \geqslant 0$ such that

$$\|T(t,s)\| \leqslant De^{a(t-s)+\varepsilon|s|}, \quad t \geqslant s. \tag{2}$$

The more general notion of nonuniform exponential dichotomy corresponds to the existence of nonuniform exponential behavior along stable and unstable directions. Using the so-called Lyapunov regularity theory, we can show that the *nonuniform* exponential behavior is very common when compared to what happens in the uniform setting. Essentially, the existence of nonzero Lyapunov exponents leads to a nonuniform exponential behavior. We refer to the books [1,4] for detailed discussions. We also refer to the book [10] for a discussion of the geometric theory in the infinite-dimensional setting, with emphasis on delay differential equations (see also Section 7.5 for a discussion of the consequences of nonuniform hyperbolicity).

Our objective in this note is to characterize the existence of a *nonuniform* exponential contraction or of a *nonuniform* exponential dichotomy in terms of Lyapunov functions. The importance of Lyapunov functions is well established, particularly in the study of the stability of solutions of differential equations, both in the finite- and in the infinite-dimensional settings. It goes back to the seminal work of Lyapunov in his 1892 thesis, republished most recently in [13]. Among the first accounts of the theory are the books by LaSalle and Lefschetz [12], Hahn [9], and Bhatia and Szegö [7]. Unfortunately, there exists no general method to construct explicitly Lyapunov functions for a given dynamics. On the other hand, to know that there exists a Lyapunov function may be quite helpful for the stability theory. Our work can be partly seen as a development of somewhat related approaches in the books by Dalec'kiĭ and Kreĭn [8] and Massera and Schäffer [14], which go back to

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* Corresponding author.
E-mail addresses: barreira@math.ist.utl.pt (L. Barreira), cvalls@math.ist.utl.pt (C. Valls).

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Lyapunov in the finite-dimensional setting, although they only consider the case of uniform exponential behavior. We refer to the book by Mitropolsky, Samoilenko and Kulik [15] for a detailed study of Lyapunov functions in the particular case of uniform exponential dichotomies for ordinary differential equations.

2. Setup

Given $r \geqslant 0$ (the delay), we consider the Banach space $C = C([-r, 0], \mathbb{R}^n)$ of continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$ with the norm

$$\|\phi\| = \sup\{\|\phi(\theta)\|: -r \leqslant \theta \leqslant 0\}.$$

Given linear operators $L(t): C \to \mathbb{R}^n$ for $t \in \mathbb{R}$ such that $(t, v) \mapsto L(t)v$ is continuous, we consider Eq. (1), where v' denotes the right-hand derivative and where $v_t(\theta) = v(t + \theta)$, $\theta \in [-r, 0]$. We assume that there exists $\kappa > 0$ such that

$$\int_t^{t+r} \|L(\tau)\|\, d\tau \leqslant \kappa(1 + |t|) \tag{3}$$

for any $t \in \mathbb{R}$. Then for each $(s, \phi) \in \mathbb{R} \times C$ there is a unique solution $v_t(\cdot, s, \phi)$, $t \geqslant s$ of Eq. (1) with $v_s(\cdot, s, \phi) = \phi$ (see for example [11]). We define the evolution operator $T(t, s): C \to C$ for $t \geqslant s$ by

$$T(t, s)\phi = v_t(\cdot, s, \phi).$$

It is sometimes necessary to extend the domain of $T(t, s)$ to a space that contains some discontinuous functions. For this we write $L(t)$ in the form

$$L(t)\phi = \int_{-r}^{0} d_\theta[\eta(t, \theta)]\phi(\theta) \tag{4}$$

for some $n \times n$ matrices $\eta(t, \theta)$ that are measurable in $(t, \theta) \in \mathbb{R} \times \mathbb{R}$ and continuous from the left in θ. Now we set

$$m_\eta(t) = \operatorname{Var} \eta(t, \cdot),$$

where Var denotes the total variation in $[-r, 0]$. We note that $\|L(t)\| = m_\eta(t)$. Now let $\hat{C}$ be the space of functions $\phi : [-r, 0] \to \mathbb{R}^n$ such that for each $s \in [-r, 0]$ there exist the limits

$$\lim_{\theta \to s^-} \phi(\theta) \quad \text{and} \quad \lim_{\theta \to s^+} \phi(\theta),$$

and ϕ is right-continuous at s, i.e., $\lim_{\theta \to s^+} \phi(\theta) = \phi(s)$. We also consider the supremum norm in C. Each linear operator $L(t)$ can be extended to $\hat{C}$ using the integral in (4), provided that the Riemann–Stieltjes sums take the value $[\eta(t, b) - \eta(t, a)]\phi(b^-)$ for each subinterval $[a, b] \subset [-r, 0]$ (so that points at which both ϕ and $\eta(t, \cdot)$ have discontinuities cause no problem). Moreover, for each $(s, \phi) \in \mathbb{R} \times \hat{C}$ there is a unique solution $v_t(\cdot, s, \phi) \in \hat{C}$ for $t \geqslant s$ with $v_s(\cdot, s, \phi) = \phi$ of the integral equation obtained from (1) (see [11] for a related discussion). The corresponding evolution operator $\hat{T}(t, s): \hat{C} \to \hat{C}$ is defined for $t \geqslant s$ by

$$\hat{T}(t, s)\phi = v_t(\cdot, s, \phi).$$

We note that $\hat{T}(t, s)|C = T(t, s)$ and $\hat{T}(t, s)\hat{C} \subset C$ for $t \geqslant s + r$.

3. Lyapunov functions and exponential contractions

We consider in this section the special case of exponential contractions. We recall that Eq. (1) is said to admit a *nonuniform exponential contraction* (in C) if there exist constants $a < 0$, $D > 0$ and $\varepsilon \geqslant 0$ satisfying (2). It is shown in [2] that if Eq. (1) admits a nonuniform exponential contraction in C, then there exists $\hat{D} > 0$ such that

$$\|\hat{T}(t, s)\| \leqslant \hat{D} e^{a(t-s)+(\varepsilon+\kappa)|s|}, \quad t \geqslant s,$$

with κ as in (3). This shows that Eq. (1) admits a nonuniform exponential contraction in C if and only if it admits a nonuniform exponential contraction in $\hat{C}$. Thus, in the case of (linear) contractions, it is sufficient to consider the space C.

We give several examples of nonuniform exponential contractions, starting with the simpler case of ordinary differential equations.

Example 1. We first recall that any linear ordinary differential equation $v' = A(t)v$ in $\mathbb{R}^n$, where the $n \times n$ matrix $A(t)$ varies continuously with $t \in \mathbb{R}$, can be seen as a linear delay equation. Namely, defining linear operators $L(t): C \to \mathbb{R}^n$ by $L(t)\phi = A(t)\phi(0)$, Eq. (2) becomes

$$v' = L(t)v_t = A(t)v_t(0) = A(t)v.$$

Now we consider the scalar equation

$$v' = (-\omega - at\sin t)v, \tag{5}$$

where $\omega > a > 0$ are real parameters. It is easy to verify that each solution $v(t)$ of (5) satisfies

$$v(t) = e^{-\omega t + \omega s + at\cos t - as\cos s - a\sin t + a\sin s} v(s) \tag{6}$$

for any $t, s \in \mathbb{R}$. Writing (6) in the form

$$v(t) = e^{(-\omega + a)(t-s) + at(\cos t - 1) - as(\cos s - 1) + a(\sin s - \sin t)} v(s) \tag{7}$$

for $t, s \geqslant 0$ we obtain

$$\left|v(t)\right| \leqslant e^{2a} e^{(-\omega+a)(t-s)+2as} \left|v(s)\right|. \tag{8}$$

For $t \geqslant 0$ and $s \leqslant 0$ it also follows from (7) that

$$\left|v(t)\right| \leqslant e^{2a} e^{(-\omega+a)(t-s)} \left|v(s)\right|, \tag{9}$$

and finally, for $s \leqslant t \leqslant 0$ we obtain

$$\left|v(t)\right| \leqslant e^{2a} e^{(-\omega+a)(t-s)+2a|t|} \left|v(s)\right| \leqslant e^{2a} e^{(-\omega+a)(t-s)+2a|s|} \left|v(s)\right|. \tag{10}$$

It follows from (8), (9), and (10) that

$$\left|v(t)\right| \leqslant e^{2a} e^{(-\omega+a)(t-s)+2a|s|} \left|v(s)\right|$$

for every $t \geqslant s$, and thus Eq. (5) admits a nonuniform exponential contraction. Moreover, setting $t = 2k\pi$ and $s = (2l-1)\pi$ with $k, l \in \mathbb{N}$ we obtain

$$v(t) = e^{(-\omega+a)(t-s)+2as} v(s).$$

This shows that the contraction is not uniform, in the sense that the constant ε in (2) cannot be taken equal to zero.

It is also easy to show that the nonuniform exponential contraction for Eq. (5) induces a nonuniform exponential contraction for Eq. (1), in the norm of C. Indeed,

$$\frac{\|T(t,s)v_s\|}{\|v_s\|} = \frac{\sup\{|(T(t,s)v_s)(\theta)|:\ \theta \in [-r,0]\}}{\sup\{|v_s(\theta)|:\ \theta \in [-r,0]\}} = \frac{\sup\{|v(t+\theta)|:\ \theta \in [-r,0]\}}{\sup\{|v(s+\theta)|:\ \theta \in [-r,0]\}},$$

and since

$$\left|v(t+\theta)\right| \leqslant e^{2a} e^{(-\omega+a)(t-s)+2a|s+\theta|} \left|v(s+\theta)\right|,$$

we obtain

$$\frac{\|T(t,s)v_s\|}{\|v_s\|} \leqslant e^{2a} e^{(-\omega+a)(t-s)+2a|s|+2ar}.$$

Hence,

$$\left\|T(t,s)\right\| \leqslant e^{2a(1+r)} e^{(-\omega+a)(t-s)+2a|s|}, \quad t \geqslant s.$$

The following is an example of nonuniform exponential contraction given by a delay equation which is not an ordinary differential equation.

Example 2. We consider the linear delay equation

$$v' = (-\omega - at\sin t)v + \delta e^{-2a|t|} L(t)v_t \tag{11}$$

with $\omega > a > 0$ and $\delta > 0$, for any linear operators $L(t): C \to \mathbb{R}$ such that $\sup_{t\in\mathbb{R}} \|L(t)\| < \infty$. One can use the variation-of-constants formula to show that given ω and a, for any sufficiently small $\delta > 0$ Eq. (11) admits a nonuniform exponential contraction. More precisely, if $\tilde{T}(t,s)$ is the evolution operator associated to Eq. (11), then

$$\left\|\tilde{T}(t,s)\right\| \leqslant e^{2a} e^{(-\omega+a+\delta e^{2a})(t-s)+2a|s|}$$

for every $t \geqslant s$. We refer to [3] for a related discussion.

We also present examples of other nature.

Example 3. We consider the scalar delay equation

$$v' = av + bv(t-\tau) \tag{12}$$

for some constants $a, b \in \mathbb{R}$ and $\tau > 0$. We assume that all roots in $\mathbb{C}$ of the characteristic equation

$$\lambda = a + be^{-\tau\lambda} \tag{13}$$

have negative real part. Then it follows from the theory of representation of solutions of Eq. (12) by series of exponentials $e^{\lambda t}$ with λ a root of (13) (see for example [6]) that Eq. (12) admits a nonuniform exponential contraction. We note that it is a uniform exponential contraction.

To obtain further nonuniform exponential contractions, which need not be uniform, one can use combinations of the above constructions.

Example 4. Given constants $a, b \in \mathbb{R}$ and $\tau > 0$, we assume that all roots in $\mathbb{C}$ of the characteristic equation (13) have negative real part, and we consider the delay equation

$$v' = av + bv(t-\tau) + \delta e^{-\varepsilon|t|} L(t) v_t \tag{14}$$

for some constants $\delta, \varepsilon \geqslant 0$ and some linear operators $L(t): C \to \mathbb{R}$ such that $\sup_{t \in \mathbb{R}} \|L(t)\| < \infty$. Then, for each ε, provided that δ is sufficiently small Eq. (14) admits a nonuniform exponential contraction.

Example 5. We also consider the equation

$$\begin{aligned} v' &= (-\omega - at\sin t)v, \\ w' &= aw + bw(t-\tau) \end{aligned} \tag{15}$$

for some constants $\omega > a > 0$, with the same hypothesis on the roots of the characteristic equation (13) as in Example 3. Then Eq. (15) admits a nonuniform exponential contraction that is not uniform.

Now we introduce the notion of Lyapunov function. Consider a continuous function $V: \mathbb{R} \times C \to \mathbb{R}_0^-$, and assume that there exist $\gamma > 0$ and $\delta \geqslant 0$ such that

$$\|x\| \leqslant \left|V(t,x)\right| \leqslant \frac{e^{\delta|t|}}{\gamma}\|x\| \tag{16}$$

for every $t \in \mathbb{R}$ and $x \in C$. We say that V is a *strict Lyapunov function* for Eq. (1) if there exists $\theta \in (0,1)$ such that

$$\left|V\big(t, T(t,s)x\big)\right| \leqslant \theta^{t-s}\left|V(s,x)\right| \tag{17}$$

for every $t \geqslant s$ and $x \in C$.

The following result gives an optimal characterization of nonuniform exponential contractions in terms of the existence of strict Lyapunov functions.

Theorem 1. *The following properties are equivalent*:

1. *Eq.* (1) *admits a nonuniform exponential contraction*;
2. *there is a strict Lyapunov function for Eq.* (1).

Proof. We follow the proof of Theorem 4 in [5]. We first assume that V is a strict Lyapunov function for Eq. (1). By (16) and (17), for every $t \geqslant s$ and $x \in C$ we have

$$\left\|T(t,s)x\right\| \leqslant \left|V\big(t, T(t,s)x\big)\right| \leqslant \theta^{t-s}\left|V(s,x)\right| \leqslant \frac{e^{\delta|s|}}{\gamma}\theta^{t-s}\|x\|.$$

Therefore, Eq. (1) admits a nonuniform exponential contraction with $a = \log\theta$, $\varepsilon = \delta$ and $D = 1/\gamma$. Now we assume that Eq. (1) admits a nonuniform exponential contraction, and we construct explicitly a strict Lyapunov function. For each $t \in \mathbb{R}$ and $x \in C$ set

$$V(t,x) = -\sup\left\{\left\|T(\tau,t)x\right\|e^{-a(\tau-t)}\colon \tau \geqslant t\right\}. \tag{18}$$

It follows from (2) that

$$\left|V(t,x)\right| = \sup\left\{\left\|T(\tau,t)x\right\|e^{-a(\tau-t)}\colon\ \tau \geqslant t\right\} \leqslant De^{\varepsilon|t|}\|x\|,$$

and setting $t=\tau$ we obtain $|V(t,x)| \geqslant \|x\|$. This establishes (16) with $\delta=\varepsilon$ and $\gamma = 1/D$. Moreover, for each $t \geqslant s$ we have

$$\begin{aligned}\left|V\left(t, T(t,s)x\right)\right| &= \sup\left\{\left\|T(\tau,t)T(t,s)x\right\|e^{-a(\tau-t)}\colon\ \tau \geqslant t\right\}\\ &= e^{a(t-s)}\sup\left\{\left\|T(\tau,s)x\right\|e^{-a(\tau-s)}\colon\ \tau \geqslant t\right\}\\ &\leqslant e^{a(t-s)}\sup\left\{\left\|T(\tau,s)x\right\|e^{-a(\tau-s)}\colon\ \tau \geqslant s\right\}\\ &= e^{a(t-s)}\left|V(s,x)\right|\end{aligned}$$

and (17) holds with $\theta = e^a < 1$. This completes the proof. □

For each nonuniform exponential contraction, the proof of Theorem 1 can be readily used to construct explicitly a strict Lyapunov function. Namely, a strict Lyapunov function is given by (18). We illustrate this with Example 1.

Example 6. For Eq. (5), using the explicit formula for the evolution operator in (7), a strict Lyapunov function $V:\mathbb{R}\times C \to \mathbb{R}_0^-$ is given by

$$\begin{aligned}V(t,v_t) &= -\sup\left\{\left\|T(\tau,t)v_t\right\|e^{(\omega-a)(\tau-t)}\colon\ \tau \geqslant t\right\}\\ &= -\sup\left\{\left|v(\tau+\theta)\right|e^{(\omega-a)(\tau-t)}\colon\ \tau \geqslant t,\ \theta\in[-r,0]\right\}\\ &= -\sup\left\{b(\tau,t,\theta)v_t(\theta)\colon\ \tau \geqslant t,\ \theta\in[-r,0]\right\},\end{aligned}$$

where

$$b(\tau,t,\theta) = e^{a(\tau+\theta)[\cos(\tau+\theta)-1]-a(t+\theta)[\cos(t+\theta)-1]+a[\sin(t+\theta)-\sin(\tau+\theta)]}. \tag{19}$$

We emphasize that this is not a Lyapunov function $\tilde{V}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}_0^-$ for the associated ordinary differential equation, but instead a Lyapunov function for the delay equation in the whole space C.

In the case of differentiable Lyapunov functions we have an alternative characterization of the strictness property. Given $V:\mathbb{R}\times C\to\mathbb{R}_0^-$ differentiable, let

$$\dot{V}(t,x) = \frac{d}{dh}V\left(t+h, T(t+h,t)x\right)_{|h=0}.$$

It is shown in [5] that if $V(t,x) < 0$ for every $t\in\mathbb{R}$ and $x\neq 0$, then V is a strict Lyapunov function for Eq. (1) if and only if

$$\dot{V}(t,x) \geqslant V(t,x)\log\theta, \quad t\in\mathbb{R},\ x\in C.$$

4. Nonuniform exponential dichotomies

We consider in this section the case of exponential dichotomies, and we obtain a characterization of the dichotomies in terms of Lyapunov functions. We say that Eq. (1) admits a *nonuniform exponential dichotomy* in $\hat{C}$ if there are projections $P(t):\hat{C}\to\hat{C}$ for $t\in\mathbb{R}$ and constants $a<0<b$, $\varepsilon\geqslant 0$ and $D>0$ such that for every $t,s\in\mathbb{R}$ with $t\geqslant s$:

1. $$T(t,s)P(s) = P(t)T(t,s); \tag{20}$$
2. setting $Q(t) = \mathrm{Id} - P(t)$, the map $T(t,s)Q(s) : Q(s)\hat{C} \to Q(t)\hat{C}$ is invertible;
3. $$\left\|T(t,s)P(s)\right\| \leqslant De^{a(t-s)+\varepsilon|s|} \tag{21}$$
   and
   $$\left\|T(t,s)^{-1}Q(t)\right\| \leqslant De^{-b(t-s)+\varepsilon|t|}. \tag{22}$$

The following are examples of nonuniform exponential dichotomies.

Example 7. Repeating arguments in Example 1 we can easily show that the equation

$$\begin{aligned}v' &= (-\omega - at\sin t)v,\\ w' &= (\omega + at\sin t)w\end{aligned} \tag{23}$$

admits a nonuniform exponential dichotomy provided that $\omega > a > 0$. Moreover, the dichotomy is not uniform, in the sense that the constant ε in (21) and (22) cannot be taken equal to zero.

We also present a dichotomy version of Example 3.

Example 8. We consider the delay equation (12) for some constants $a, b \in \mathbb{R}$ and $\tau > 0$. We assume that no root of the characteristic equation (13) has real part equal to zero. Then Eq. (12) admits a nonuniform exponential dichotomy. We note that it is a uniform exponential dichotomy.

In a similar manner to that for contractions one can also consider sufficiently small linear perturbations of nonuniform exponential dichotomies (we refer to [3,4] for related discussions).

Given a function $V : \hat{C} \to \mathbb{R}$ we consider the cones

$$C^u(V) = \{0\} \cup V^{-1}(0, +\infty) \quad \text{and} \quad C^s(V) = \{0\} \cup V^{-1}(-\infty, 0).$$

We say that a continuous function $V : \mathbb{R} \times \hat{C} \to \mathbb{R}$ is a *strict Lyapunov function* for Eq. (1) if:

1. setting $V_\tau = V(\tau, \cdot)$, for each $\tau \in \mathbb{R}$ there exist subspaces $E^u_\tau \subset C^u(V_\tau)$ and $E^s_\tau \subset C^s(V_\tau)$ such that $\hat{C} = E^u_\tau \oplus E^s_\tau$, and
$$T(t, \tau)E^u_\tau = E^u_t \quad \text{and} \quad T(t, \tau)E^s_\tau = E^s_t, \quad t \geqslant \tau;$$
2. there exist $\gamma > 0$ and $\delta \geqslant 0$ such that for every $t \in \mathbb{R}$ we have
$$\left|V(t, x)\right| \leqslant \gamma e^{\delta|t|}\|x\|, \quad x \in \hat{C}, \tag{24}$$
and
$$\left|V(t, x)\right| \geqslant \frac{1}{\gamma} e^{-\delta|t|}\|x\|, \quad x \in E^u_t \cup E^s_t; \tag{25}$$
3. there exist $\theta \in (0, 1)$ and $\mu > 1$ with $\log\theta + \delta < 0$ and $\log\mu - \delta > 0$ such that for each $\tau \in \mathbb{R}$ and $x \in \hat{C}$ we have
$$V\big(t, T(t, \tau)x\big) \geqslant \mu^{t-\tau} V(\tau, x), \quad t \geqslant \tau,\ x \in E^u_\tau, \tag{26}$$
and
$$\left|V\big(t, T(t, \tau)x\big)\right| \leqslant \theta^{t-\tau}\left|V(\tau, x)\right|, \quad t \geqslant \tau,\ x \in E^s_\tau. \tag{27}$$

We detail the consequences of the existence of a strict Lyapunov function.

Theorem 2. *Assume that*:

1. *Eq.* (1) *has a strict Lyapunov function*;
2. *for each* $t \geqslant \tau$ *the operator* $T(t, \tau)|E^u_\tau$ *is invertible from* E^u_τ *to* E^u_t.

Then for every $t, \tau \in \mathbb{R}$ *with* $t \geqslant \tau$ *we have*

$$\left\|T(t, \tau)\middle|E^s_\tau\right\| \leqslant \gamma^2 e^{(\log\theta+\delta)(t-\tau)+2\delta|\tau|},$$

and

$$\left\|T(t, \tau)^{-1}\middle|E^u_t\right\| \leqslant \gamma^2 e^{-(\log\mu-\delta)(t-\tau)+2\delta|t|}.$$

Proof. By (24), (25) and (27), for every $x \in E^s_\tau$ and $t \geqslant \tau$ we have

$$\left\|T(t, \tau)x\right\| \leqslant \gamma e^{\delta|t|}\theta^{t-\tau}\left|V(\tau, x)\right| \leqslant \gamma^2 e^{\delta|t|}\theta^{t-\tau}e^{\delta|\tau|}\|x\| \leqslant \gamma^2 e^{(\delta+\log\theta)(t-\tau)}e^{2\delta|\tau|}\|x\|.$$

Similarly, it follows from (24), (25) and (26) that

$$\left\|T(t, \tau)x\right\| \geqslant \frac{e^{-\delta|t|}}{\gamma}\mu^{t-\tau}V(\tau, x) \geqslant \frac{e^{-\delta|t|}}{\gamma^2}\mu^{t-\tau}e^{-\delta|\tau|}\|x\| \geqslant \frac{1}{\gamma^2}e^{(\log\mu-\delta)(t-\tau)}e^{-2\delta|t|}\|x\|$$

for every $t \geqslant \tau$ and $x \in E^u_\tau$. Hence,

$$\left\|T(t, \tau)^{-1}x\right\| \leqslant \gamma^2 e^{-(\log\mu-\delta)(t-\tau)}e^{2\delta|t|}\|x\|$$

for every $t \geqslant \tau$ and $x \in E^s_t$. This completes the proof of the theorem. □

We also give a criterion for the existence of nonuniform exponential dichotomies. We note that by the definition of strict Lyapunov function, for each $t \in \mathbb{R}$ there exist projections

$$P(t): \hat{C} \to E_t^s \quad \text{and} \quad Q(t): \hat{C} \to E_t^u$$

satisfying (20).

Theorem 3. *If Eq.* (1) *has a strict Lyapunov function and there exist constants* $c, \alpha > 0$ *such that*

$$\|P(t)\|, \|Q(t)\| \leqslant ce^{\alpha|t|} \quad \text{for } t \in \mathbb{R}, \tag{28}$$

then Eq. (1) *admits a nonuniform exponential dichotomy.*

Proof. Since

$$\|T(t,\tau)P(\tau)\| \leqslant \|T(t,\tau)|E_\tau^s\| \cdot \|P(\tau)\|,$$

and

$$\|T(t,\tau)^{-1}Q(t)\| \leqslant \|T(t,\tau)^{-1}|E_t^u\| \cdot \|Q(t)\|,$$

it follows from Theorem 2 and (28) that Eq. (1) admits a nonuniform exponential dichotomy. □

The following example shows that condition (28) is unavoidable.

Example 9. We modify Eq. (23) in $\mathbb{R}^2$ as follows. Given a C^1 function $\beta: \mathbb{R} \to (0, \pi/2)$ we consider the matrix

$$R(t) = \begin{pmatrix} 1 & \cos\beta(t) \\ 0 & \sin\beta(t) \end{pmatrix}.$$

Since the entries of $R(t)$ are bounded we have

$$\sup_{t\in\mathbb{R}} \|R(t)\| < \infty. \tag{29}$$

One can easily verify that if $\tilde{T}(t,\tau)$ is the evolution operator of Eq. (23) in $\mathbb{R}^2$, then

$$U(t,\tau) = R(t)\tilde{T}(t,\tau)R(\tau)^{-1} \tag{30}$$

is the evolution operator of the equation $x' = B(t)x$, where

$$B(t) = \left[R'(t) + R(t)A(t)\right]R(t)^{-1}.$$

Now we set

$$E_t^u = R(t)\left(\{0\} \times \mathbb{R}\right) \quad \text{and} \quad E_t^s = R(t)\left(\{0\} \times \mathbb{R}\right)$$

for each $t \in \mathbb{R}$. Since Eq. (23) admits a nonuniform exponential dichotomy, it follows from (29) and (30) that

$$\|U(t,\tau)|E_\tau^s\| = \|R(t)\tilde{T}(t,\tau)|\left(\mathbb{R} \times \{0\}\right)\| \leqslant D'e^{a(t-\tau)+\varepsilon|\tau|},$$

and

$$\|U(t,\tau)^{-1}|E_t^u\| = \|R(t)\tilde{T}(t,\tau)|\left(\mathbb{R} \times \{0\}\right)\| \leqslant D'e^{-b(t-\tau)+\varepsilon|t|}$$

for every $t \geqslant \tau$ and some constant $D' > 0$. On the other hand, since the subspaces E_t^u and E_t^s are generated respectively by the vectors $(\cos\beta(t), \sin\beta(t))$ and $(1, 0)$, we have $\beta(t) = \angle(E_t^s, E_t^u)$, and hence,

$$\|P(t)\| = \|Q(t)\| = \frac{1}{2\sin(\beta(t)/2)}.$$

Therefore, the norms of the projections $P(t)$ and $Q(t)$ can be made arbitrarily large by making $\beta(t)$ arbitrarily small, and in particular there exists a function β such that condition (28) fails. We note that even though the above construction was carried out for an ordinary differential equation, a similar construction can be done in the space C.

Finally, we construct a strict Lyapunov function for each nonuniform exponential dichotomy.

Theorem 4. *If Eq.* (1) *admits a nonuniform exponential dichotomy with $a+\delta>0$ and $b-\delta>0$, then it has a strict Lyapunov function.*

Proof. We follow arguments in the proof of Theorem 3 in [5]. For completeness we include the full details. For each $t\in\mathbb{R}$ and $x\in\hat{C}$, set

$$V(t,x)=-V^s(t,y)+V^u(t,z), \tag{31}$$

with $x=y+z$, $y\in F_t^s=P(t)\hat{C}$, and $z\in F_t^u=Q(t)\hat{C}$, where

$$V^s(t,y)=\sup\bigl\{\bigl\|T(r,t)y\bigr\|e^{-a(r-t)}\colon r\geqslant t\bigr\},$$

and

$$V^u(t,z)=\sup\bigl\{\bigl\|T(r,t)z\bigr\|e^{b(t-r)}\colon r\leqslant t\bigr\}.$$

By (21) with $t=\tau$ we have

$$\bigl|V(t,x)\bigr|\leqslant V^s(t,y)+V^u(t,z)\leqslant De^{\varepsilon|t|}\bigl(\|y\|+\|z\|\bigr)\leqslant 2D^2e^{2\varepsilon|t|}\|x\|.$$

Moreover, if $x_\tau\in F_\tau^s$ and $t\geqslant\tau$, then since $b>a$, we obtain

$$\begin{aligned}\bigl|V\bigl(t,T(t,\tau)x\bigr)\bigr|&=V^s\bigl(t,T(t,\tau)y\bigr)-V^u\bigl(t,T(t,\tau)z\bigr)\\&=\sup\bigl\{\bigl\|T(r,t)T(t,\tau)y\bigr\|e^{-a(r-t)}\colon r\geqslant t\bigr\}-\sup\bigl\{\bigl\|T(r,t)T(t,\tau)z\bigr\|e^{b(t-r)}\colon r\leqslant t\bigr\}\\&=e^{a(t-\tau)}\sup\bigl\{\bigl\|T(r,\tau)y\bigr\|e^{-a(r-\tau)}\colon r\geqslant t\bigr\}-e^{b(t-\tau)}\sup\bigl\{\bigl\|T(r,\tau)z\bigr\|e^{b(\tau-r)}\colon r\leqslant t\bigr\}\\&\leqslant e^{a(t-\tau)}\bigl(\sup\bigl\{\bigl\|T(r,\tau)y\bigr\|e^{-a(r-\tau)}\colon r\geqslant\tau\bigr\}-\sup\bigl\{\bigl\|T(r,\tau)z\bigr\|e^{b(\tau-r)}\colon r\leqslant\tau\bigr\}\bigr)\\&=e^{a(t-\tau)}\bigl|V(\tau,x)\bigr|,\end{aligned}$$

and (27) holds with $\theta=e^a$. Similarly, if $x\in F_\tau^u$ and $t\geqslant\tau$, then

$$\begin{aligned}V\bigl(t,T(t,\tau)x\bigr)&=-V^s\bigl(t,T(t,\tau)y\bigr)+V^u\bigl(t,T(t,\tau)z\bigr)\\&=-e^{a(t-\tau)}\sup\bigl\{\bigl\|T(r,\tau)y\bigr\|e^{-a(r-\tau)}\colon r\geqslant t\bigr\}+e^{b(t-\tau)}\sup\bigl\{\bigl\|T(r,\tau)z\bigr\|e^{b(\tau-r)}\colon r\leqslant t\bigr\}\\&\geqslant-e^{a(t-\tau)}\sup\bigl\{\bigl\|T(r,\tau)y\bigr\|e^{-a(r-\tau)}\colon r\geqslant\tau\bigr\}+e^{b(t-\tau)}\sup\bigl\{\bigl\|T(r,\tau)z\bigr\|e^{b(\tau-r)}\colon r\leqslant\tau\bigr\}\\&\geqslant e^{b(t-\tau)}\bigl(-\sup\bigl\{\bigl\|T(r,\tau)y\bigr\|e^{-a(r-\tau)}\colon r\geqslant\tau\bigr\}+\sup\bigl\{\bigl\|T(r,\tau)z\bigr\|e^{b(\tau-r)}\colon r\leqslant\tau\bigr\}\bigr)\\&=e^{b(t-\tau)}V(\tau,x),\end{aligned}$$

and (26) holds with $\mu=e^b$.

Now we establish (25) with $\delta=0$. If $x\in F_\tau^s$, then

$$\begin{aligned}\bigl|V(\tau,x)\bigr|&\geqslant\bigl|V(\tau,x)\bigr|-\bigl|V\bigl(\tau+1,T(\tau+1,\tau)x\bigr)\bigr|\\&=V^s(\tau,y)-V^s\bigl(\tau+1,T(\tau+1,\tau)y\bigr)-V^u(\tau,z)+V^u\bigl(\tau+1,T(\tau+1,\tau)z\bigr).\end{aligned}\tag{32}$$

Moreover,

$$\begin{aligned}V^s(\tau,y)-V^s\bigl(\tau+1,T(\tau+1,\tau)y\bigr)&=\sup\bigl\{\bigl\|T(r,\tau)y\bigr\|e^{-a(r-\tau)}\colon r\geqslant\tau\bigr\}-e^a\sup\bigl\{\bigl\|T(r,\tau)y\bigr\|e^{-a(r-\tau)}\colon r\geqslant\tau+1\bigr\}\\&\geqslant\bigl(1-e^a\bigr)\sup\bigl\{\bigl\|T(r,\tau)y\bigr\|e^{-a(r-\tau)}\colon r\geqslant\tau\bigr\}\\&\geqslant\bigl(1-e^a\bigr)\|y\|,\end{aligned}$$

and

$$\begin{aligned}-V^u(\tau,z)+V^u\bigl(\tau+1,T(\tau+1,\tau)z\bigr)&=e^b\sup\bigl\{\bigl\|T(r,\tau)z\bigr\|e^{b(\tau-r)}\colon r\leqslant\tau+1\bigr\}-\sup\bigl\{\bigl\|T(r,\tau)z\bigr\|e^{b(\tau-r)}\colon r\leqslant\tau\bigr\}\\&\geqslant\bigl(e^b-1\bigr)\sup\bigl\{\bigl\|T(r,\tau)z\bigr\|e^{b(\tau-r)}\colon r\leqslant\tau\bigr\}\\&\geqslant\bigl(e^b-1\bigr)\|z\|.\end{aligned}$$

Setting $\eta=\min\{1-e^a,e^b-1\}$, it follows from (32) that

$$\bigl|V(\tau,x)\bigr|\geqslant\eta\bigl(\|y\|+\|z\|\bigr)\geqslant\eta\|x\|.$$

Similarly, if $x\in F_\tau^u$, then

$$\begin{aligned} V(\tau,x) &\geqslant V(\tau,x) - V\big(\tau-1, T(\tau-1,\tau)x\big) \\ &= -V^s(\tau,y) + V^s\big(\tau-1, T(\tau-1,\tau)y\big) + V^u(\tau,z) - V^u\big(\tau-1, T(\tau-1,\tau)z\big). \end{aligned} \tag{33}$$

Also,

$$\begin{aligned} &-V^s(\tau,y) + V^s\big(\tau-1, T(\tau-1,\tau)y\big) \\ &\quad = -\sup\big\{\big\|T(r,\tau)y\big\|e^{-a(r-\tau)}\colon r \geqslant \tau\big\} + e^{-a}\sup\big\{\big\|T(r,\tau)y\big\|e^{-a(r-\tau)}\colon r \geqslant \tau-1\big\} \\ &\quad \geqslant \big(e^{-a}-1\big)\sup\big\{\big\|T(r,\tau)y\big\|e^{-a(r-\tau)}\colon r \geqslant \tau\big\} \\ &\quad \geqslant \big(e^{-a}-1\big)\|y\|, \end{aligned}$$

and

$$\begin{aligned} &V^u(\tau,z) - V^u\big(\tau-1, T(\tau-1,\tau)z\big) \\ &\quad = \sup\big\{\big\|T(r,\tau)z\big\|e^{b(\tau-r)}\colon r \leqslant \tau\big\} + e^{-b}\sup\big\{\big\|T(r,\tau)z\big\|e^{b(\tau-r)}\colon r \leqslant \tau-1\big\} \\ &\quad \geqslant \big(1-e^{-b}\big)\sup\big\{\big\|T(r,\tau)z\big\|e^{b(\tau-r)}\colon r \leqslant \tau\big\} \\ &\quad \geqslant \big(1-e^{-eb}\big)\|z\|. \end{aligned}$$

Setting $\eta' = \min\{e^{-a}-1, 1-e^{-b}\}$, it follows from (33) that

$$V(\tau,x) \geqslant \eta'\big(\|y\|+\|z\|\big) \geqslant \eta'\|x\|.$$

This completes the proof of the theorem. □

We can use the construction in the proof of Theorem 4 to provide an explicit strict Lyapunov function for any nonuniform exponential dichotomy. We illustrate this with Example 7.

Example 10. We use the formula in (31) to construct explicitly a strict Lyapunov function for Eq. (23). Namely, writing $x = (v, w)$ we take

$$V(t,x) = -V^s(t,v) + V^u(t,w),$$

where

$$V^s(t,v) = \sup\big\{b(\tau,t,\theta)v(\theta)\colon \tau \geqslant t,\ \theta \in [-r,0]\big\},$$

and

$$V^u(t,w) = \sup\big\{w(\theta)/b(\tau,t,\theta)\colon \tau \leqslant t,\ \theta \in [-r,0]\big\},$$

with $b(\tau,t,\theta)$ as in (19).

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