Functions Positive-Definite on $\mathbb{R}^3$ and the Heisenberg Group*

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In this paper functions that are simultaneously positive-definite on $\mathbb{R}^3$ and the Heisenberg group are considered. The set of cyclic vectors associated to infinite-dimensional representations of the Heisenberg group, realized in $L^2(\mathbb{R})$, which give rise to such functions is investigated. It is seen to consist precisely of exponentials of quadratic polynomials for which the leading coefficient has negative real part.

In this paper we consider functions which are simultaneously positive-definite with respect to two distinct group structures on three-dimensional Euclidean space. The two group structures in question are the usual vector group structure on $\mathbb{R}^3$ and the group structure obtained by identifying the three-dimensional Heisenberg group, $N_3$, with its Lie algebra via the exponential map. The relevant facts concerning the Heisenberg group are summarized later in the paper.

Before embarking on the study of the problem at hand, some general remarks on this and related problems are in order. Firstly, simultaneously positive-definite distributions are easy to find. Indeed, if $N$ is a general nilpotent Lie group, identified with its Lie algebra via exp, then any linear unitary character of a subgroup (such characters arise naturally in the Kirillov theory) defines a distribution on $N$ that is simultaneously positive-definite. Further, Schiffman [8] has shown that a central distribution on $N$ is positive-definite if and only if it is simultaneously positive-definite. If $N$ is a Heisenberg group (in general of dimension $2n + 1$) then Howe [5] has given criteria for positive-definiteness of functions on $N$ that permits the construction of many simultaneously positive-definite functions. More concretely perhaps, it follows easily from Bochner's theorem and [3] (the calculation in [3] is done only for $N_3$ but is easily extended to the general

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2n + 1 dimensional Heisenberg group) that even convolution powers of the Gaussian density function on the general Heisenberg group are simultaneously positive-definite.

Although the results presented in this study are, by appropriate choice of coordinates, easily extended to the general Heisenberg group, we shall restrict our attention to $N_3$. $N_3$ is most naturally present as the group of $3 \times 3$ unipotent matrices. This presentation immediately leads to a realization of $N_3$ as $\mathbb{R}^3$ with the group multiplication law given as

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy').$$

For our purposes, however, we prefer to identify $N_3$ with $\mathbb{R}^3$ in a more canonical manner.

The Lie algebra, $L$, of $N_3$ is $\mathbb{R}^3$ with the Lie bracket being given by

$$[(x, y, z), (x', y', z')] = (0, 0, xy' - x'y) \quad (1)$$

The exponential map, $\exp$, from $L$ to $N_3$ is a global diffeomorphism which is also a group isomorphism if $L$ is furnished with the Campbell–Hausdorff group law:

$$C - H(l, l') = \exp^{-1}(\exp l \exp l')$$

$$= l + l' + \frac{1}{2}[l, l'].$$

Note that since $N_3$ is two-step nilpotent the higher-order bracket terms usually appearing in the Campbell-Hausdorff formula are identically zero. A further feature of the exponential map, important in our setting, is the fact that it carries Lebesgue measure on $L$ to Haar measure on $N_3$.

Throughout the sequel we identify $N_3$ with $L = \mathbb{R}^3$ as above and think of $N_3$ as being $\mathbb{R}^3$ with the following explicit group multiplication law:

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)). \quad (2)$$

We shall make use of the unitary dual, $\hat{N}_3$, of the Heisenberg group. $\hat{N}_3$ consists of two classes of representations, which, in the coordinates that we have chosen for $N_3$, may be realized as follows:

1. For every pair of real numbers $(a, b)$ there is a linear unitary character of $N_3$:

$$(x, y, z) \mapsto \exp(i(ax + by)).$$

2. For every non-zero real number $\lambda$ there is a representation $\pi^\lambda$, realized in $L^2(\mathbb{R})$, acting according to the following prescription:

$$\pi^\lambda(x, y, z)f(t) = \exp(i(z + ty + \frac{1}{2}xy)f(t + x).$$
We shall make use of a few standard facts concerning positive-definite functions \[1, 4, 7\]

1. Every measurable positive-definite function is continuous and is bounded in modulus by its value at the identity.

2. (Bochner's Theorem) Continuous positive-definite functions on locally compactly abelian groups are precisely the Fourier–Stieltjes transforms of finite, Borel measures. If the positive-definite function in question is integrable, then, by the Fourier inversion theorem, the corresponding measure is absolutely continuous with respect to Haar measure.

3. The product of two positive-definite functions is positive-definite.

4. Continuous positive-definite functions, via the identification of $L^\infty$ with the dual of $L^1$, are canonically identified with continuous positive functionals on the $L^1$ convolution algebra of the group.

We use "\(\hat{\cdot}\)" to denote Fourier transform and "\(\ast\)" to denote convolution.

As is well known (e.g., \([1]\)), given any locally compact group and a positive-definite function $\gamma$ on $G$, there exists a unitary representation $\pi$ of $G$ and a cyclic vector $f$ for $\pi$ so that

$$
\gamma(g) = \langle \pi(g)f, f \rangle, \quad g \in G.
$$

The correspondence is such that $\gamma$ is an extreme point of the set of positive-definite functions normalized to take the value 1 at the identity of $G$ if and only if $f$ is a unit vector and $\pi$ is irreducible. This observation permits the use of Choquet-type integral representations in the study of positive-definite functions on $G$, once the unitary dual, $\hat{G}$, of $G$ is explicitly known, for instance in one proof of Bochner's theorem.

Let $P_{N_3}, P_{R^3}$ denote the set of normalized positive-definite functions on $N_3, R^3$, respectively. Ideally we would like to completely characterize $P = P_{N_3} \cap P_{R^3}$. Since we do not at this time have complete knowledge of the extreme points of $P$, we are unable to involve Choquet theory to affect the desired characterization. We have, however, been able to identify a number of extreme points of $P$.

Clearly, any extreme point of $P_{N_3}$ or $P_{R^3}$, which happens to lie in $P$ is also an extreme point of $P$. Thus, one sees that the linear unitary characters of $N_3$, being also characters of $R^3$ are extreme points of $P$. In fact, only those characters of $R^3$ that are identically 1 on the center of $N_3$ are $N_3$-positive-definite. In addition to the characters of $N_3$, we shall see that the infinite-dimensional representations of $N_3$ give rise to functions $\gamma$ in $P$ according to (3) for certain choices of cyclic vectors $f$. We shall in fact classify all such $f$. 
Fix \( \lambda \in \mathbb{R} \), \( \lambda \neq 0 \), and consider a function \( \gamma \) as specified by (3) in the case \( G = N \) and \( \pi = \pi^1 \). Then for \( f \in L^2(\mathbb{R}) \),

\[
\gamma(x, y, z) = \int_{\mathbb{R}} \exp(i\lambda(z + ty + xy/2))f(t + x)f(t)dt
\]

\[
= \exp(i\lambda z) \int_{\mathbb{R}} \exp(i\lambda ty)f(t + x/2)f(t - x/2)dt.
\]

\( \gamma \) will be \( N \)-positive-definite for any \( f \in L^2(\mathbb{R}) \). We henceforth concentrate on the determination of those \( f \) for which the resulting \( \gamma \) is \( \mathbb{R}^3 \)-positive-definite. Multiplying the last expression in ( ) by \( \exp(-i\lambda z) \) and replacing \( y \) by \( y/\lambda \) and \( x \) by \( 2x \) we see that \( \gamma \) is \( \mathbb{R}^3 \)-positive-definite if and only if the function \( H \) defined by

\[
H(x, y) = \int_{\mathbb{R}} \exp(ity)f(t + x)f(t - x)dt
\]

is \( \mathbb{R}^2 \)-positive-definite.

**Proposition 1.** Let \( H \) be given by (4). Assume that \( f \) is both square integrable and continuous. Then \( H \) is positive-definite on \( \mathbb{R}^2 \) and only if for every \( t \in \mathbb{R} \) the map \( F_t \) defined by

\[
F_t(x) = f(t + x)f(t - x)
\]

is positive-definite on \( \mathbb{R} \).

**Proof.** If \( F_t \) is positive-definite for every \( t \) then it is obvious that \( H \) is \( \mathbb{R}^2 \)-positive-definite. Conversely, if \( H \) is \( \mathbb{R}^2 \)-positive-definite then \( H = \mu \) for some positive Borel measure \( \mu \) on \( \mathbb{R}^2 \). Using the technique of disintegration of a measure [2], we can write \( \mu \) as \( \int_{\mathbb{R}} v_t \, d\sigma(t) \), where \( v_t \) is a probability measure on \( \mathbb{R}^2 \) concentrated on \( \mathbb{R} \times \{ t \} \) and \( \sigma \) is a positive Borel measure on \( \mathbb{R} \). Each \( F_t \) is then the Fourier transform of \( v_t \) and the proposition follows.

**Remark.** The continuity of \( f \) is needed only in order to conclude that \( F_t = \hat{\nu} \) for every \( t \). If \( f \) is assumed to be only \( L^2 \) then we are able to conclude only that \( F_t \) is positive-definite for almost every \( t \). However, this is enough for our purposes. If \( \phi(x) = \exp(-x^2) \), then the calculation given in part (a) of Proposition (6), to come, shows that \( f * \phi \) satisfies the hypotheses of Proposition (1), since the convolution of two \( L^2 \) functions is continuous. If we can classify all \( f * \phi \) resulting in a positive-definite \( H \), then we can easily classify the \( f \).

Let

\[
S = \{ f \in L^2(\mathbb{R}) \mid x \rightarrow f(t + x)f(t - x) \text{ is positive-definite for all } t \in \mathbb{R} \}.
\]
The preceding proposition and remark show that the classification of those $L^2$ functions $f$ for which the corresponding $\gamma$ of (3) is $\mathbb{R}^3$-positive-definite for some, hence all, $\lambda \neq 0$ is equivalent to the characterization of $S$.

We shall eventually show that $S$ consists of exponentials of quadratic polynomials with complex coefficients, the leading coefficient having negative real part. Before proceeding with the proof of this fact, however, we digress briefly to indicate why one might anticipate such a result.

First, note that the class of functions alluded to above is contained in $S$, as one may easily verify. Secondly note that the composition of a positive-definite function with a group automorphism is again positive-definite. For the problem at hand, we are thus led to the consideration of maps which are simultaneously automorphisms of $N_3$ and $\mathbb{R}^3$. Such automorphisms are simply automorphisms of $\mathbb{R}^3$ that preserve the Lie bracket (1) and are canonically identified with matrices of the form

$$\begin{pmatrix}
\alpha_1 & a_2 & 0 \\
b_1 & \beta_2 & 0 \\
c_1 & d_2 & \eta
d\end{pmatrix}$$

for which $\alpha_1 b_2 - \alpha_2 b_1 = \eta \neq 0$.

If $\sigma$ is one of the automorphisms described above, then for each $\lambda \neq 0, \pi^\lambda \circ \sigma$ is a unitary representation of $N_3$ that is equivalent to $\pi^{1n}$. Hence there is a unitary intertwining operator $B$, depending on $\lambda$ and $\sigma$, acting in $L^2(\mathbb{R})$ so that for $f \in L^2(\mathbb{R}), g \in N_3$,

$$\langle \pi^\lambda \circ (g)f, f \rangle = \langle \pi^{1n}(g)Bf, Bf \rangle.$$

The set $S$ must be invariant under all such $B$.

Consider now matrices of the following forms:

$$(1) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2) \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3) \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(4) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e & d & 1 \end{pmatrix} \quad (5) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta \end{pmatrix},$$

where the only constraints on the various real constants are that $a \neq 0$, $\eta \neq 0$. Matrices of the forms (1)–(3) are naturally identified via their principle minors with a generating set for $SL_2(\mathbb{R})$ [6]. Knowing this it is straightforward to verify that the five listed families of matrices generate the full group of automorphisms on which our attention is at present focused.
Corresponding to each of the matrices listed, for fixed $\lambda$, one verifies by direct calculation that the associated intertwining operator, denoted generically by $B$, acts on an $L^2$ function $f$ as follows:

\begin{align*}
(1) \quad & Bf(t) = \sqrt{\lambda} \tilde{f}(\lambda t), \\
(2) \quad & Bf(t) = \sqrt{q} f(at), \\
(3) \quad & Bf(t) = \exp(i\lambda(b/2) t^2)f(t), \\
(4) \quad & Bf(t) = \exp(i\lambda ct)f(t - d), \\
(5) \quad & Bf(t) = f(t).
\end{align*}

Since $S$ must contain all exponentials of quadratic polynomials with leading coefficients having negative real part, and since this class of functions is invariant under transformations of the above types, one is lead to the, perhaps optimistic, conjecture that such functions exhaust $S$. As we now proceed to demonstrate, this conjecture is correct.

Let

\[ f(x) = c \exp(-ax^2 + bx), \]

where $a, b, c \in \mathbb{R}$, and $\text{Re}(a) > 0$. Then

\[ f(t + x)f(t - x) = |c|^2 \exp[2 \text{Re}(at^2) + 2 \text{Re}(bt) + 2 \text{Re}(a)x^2 + i(4 \text{Im}(at) + 2 \text{Im}(b)) x] \]

for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. Although ultimately our interest lies only in the case in which $t$ is real, it is necessary at this time to note that the extension of $f$ to $\mathbb{C}$ is obvious and that for every complex $t$ the map $x \to f(t + x)f(t - x)$ is positive-definite on $\mathbb{R}$. Accordingly we set

\[ S_\mathbb{C} = \{ f: \mathbb{C} \to \mathbb{C} \mid x \to f(t + x)f(t - x) \text{ is in } L^2(\mathbb{R}) \text{ and } x \to f(t + x)f(t - x) \text{ is positive-definite on } \mathbb{R} \text{ for all } t \in \mathbb{C} \}. \]

Having disposed of the necessary preliminary observations, we are ready to begin in earnest the characterization of $S$.

**Proposition 2.** Let $f \in S$. Then $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$. ($C_0(\mathbb{R})$ denotes the set of continuous functions on $\mathbb{R}$ that vanish at infinity.)

**Proof.** By Bochner's theorem there is a function $W: \mathbb{R}^2 \to \mathbb{R}$ such that $W(t, \cdot) \in L^1(\mathbb{R})$, $W \geq 0$ and

\[ f(t + x)f(t - x) = \int_{\mathbb{R}} \exp(ify)W(t, y) dy \]
for all $t, x \in \mathbb{R}$. Also,

$$\int_{\mathbb{R}} |f(t)|^2 \, dt = \int_{\mathbb{R}^2} W(t, y) \, dy \, dt.$$ 

Hence $W \in L^2(\mathbb{R}^2)$. Note that measurability of $W$ is not a problem since $f = \hat{W}$ for some $W \in L^2(\mathbb{R})$ and $W = \chi_w \ast \overline{\chi_w}$, where $\chi_t(x) = \exp(-ixt)$.

Replacing $f$ by a translate of $f$ if necessary, and normalizing, we may assume that $f(0) = 1$. We are of course ignoring the trivial case $f \equiv 0$. Then for all $u, w \in \mathbb{R}$,

$$f(u)f(w) = \int_{\mathbb{R}} \exp\left(-iy\frac{u-w}{2}\right) W\left(\frac{u+w}{2}, y\right) \, dy.$$ 

Taking $w = 0$, we see that

$$f(u) = \int_{\mathbb{R}} \exp\left(-iy\frac{u}{2}\right) W\left(\frac{u}{2}, y\right) \, dy.$$ 

Now,

$$\int_{\mathbb{R}} |f(u)| \, du \leq \int_{\mathbb{R}^2} W\left(\frac{u}{2}, y\right) \, dy \, du < \infty.$$ 

Therefore $f \in L^1(\mathbb{R})$ and $\hat{f} \in C_0(\mathbb{R})$. Since $S$ is invariant under Fourier transform, and the action of the Fourier transform on $L^2(\mathbb{R})$ is periodic of period 4, it follows that $f \in C_0(\mathbb{R})$. This completes the proof of the proposition.

The above proposition may be applied to the map $x \rightarrow f(t + x)$ for $f \in S_\varepsilon$ and $t \in \varepsilon$ to conclude that $f$ is continuous and integrable along horizontal lines in the complex plane. Together with the fact that the product of positive-definite functions is positive-definite the above results furnishes proof of the following corollary.

**Corollary 3.** $S$ and $S_\varepsilon$ are closed under pointwise multiplication.

Since $S$ is a semigroup under pointwise multiplication and is invariant under Fourier transform, $S$ is also a semigroup under convolution. We exploit this fact in order to bring the methods of complex analysis to bear on our problem.

For $f \in S$, $\phi \in S_\varepsilon$, define

$$f \ast \phi(t) = \int_{\mathbb{R}} \phi(t - x)f(x) \, dx.$$
for \( t \in \mathbb{C} \). This is not quite a classical convolution product, but it is very close.

**Proposition 4.** Let \( \phi(t) = \exp(-t^2), \ t \in \mathbb{C} \). Suppose that \( f \in S \). Then

(a) \( f * \phi \in S_\mathbb{C} \),

(b) \( f * \phi \) is entire and of order 2.

**Proof.** (a) For fixed \( t \in \mathbb{C} \), the map \( x \to \phi(t + x) \) lies in \( S \). Since \( S \) is a convolution semigroup it follows that \( f * \phi(t + \cdot) \in S \) or in other words that \( f * \phi \in S_\mathbb{C} \).

(b) The usual device of viewing \( f * \phi \) as a vector-valued integral in the space of entire functions with the topology of uniform convergence on compacta makes clear the fact that \( f * \phi \) is entire. The inequality

\[
|\exp(-(t - x)^2)| \leq \exp(2|t|^2)
\]

implies that

\[
|f * \phi(t)| \leq \exp(2|t|^2) \|f\|_1.
\]

This completes the proof of the proposition.

**Theorem 5.** Let \( f \in S \). Then there are complex numbers \( a, b, c \) with \( \text{Re}(a) > 0 \) so that

\[
f(x) = c \exp(-ax^2 + bx), \quad x \in \mathbb{R}.
\]  

**Proof.** If \( f \equiv 0 \) there is nothing to prove. Assume \( f \not\equiv 0 \). Then letting \( \phi(t) = \exp(-t^2), \ t \in \mathbb{C} \), we have by the preceding proposition that \( f * \phi \) is entire, of order 2, and belongs to \( S_\mathbb{C} \). Also \( f * \phi \not\equiv 0 \).

We claim that \( f * \phi \) is nowhere 0. To prove this we assume to the contrary that \( f * \phi(t_0) = 0 \) for some \( t_0 \in \mathbb{C} \). Then the map \( x \to f * \phi(t_0 + x)f * \phi(t_0 - x) \) is positive-definite on \( \mathbb{R} \) and 0 for \( x = 0 \), and hence must be identically 0. The analytic function \( f * \phi \) is therefore 0 on some sequence contained in the horizontal line through \( t_0 \) and converging to \( t_0 \). Thus \( f * \phi \) is identically 0, a contradiction.

Since \( f * \phi \) is nowhere 0, \( f * \phi = \exp(h) \) for some entire function \( h \). According to Proposition 4, \( h \) grows at most as rapidly as a quadratic polynomial and therefore must in fact be a polynomial of degree at most 2. The integrability condition satisfied by \( f * \phi \) implies that \( h \) must be a quadratic polynomial with negative leading coefficient. Restricting \( f * \phi \) to \( \mathbb{R} \) and considering \( f * \phi \) we see that \( f \) must be of the prescribed form. This completes the proof of the theorem.
COROLLARY 6. If $\gamma$ as given by (4.2) is $\mathbb{R}^3$-positive-definite then $f$ is almost everywhere equal to a function of the form (5).

Proof: If $f$ is as hypothesized and $\phi$ is as in the proof of the above theorem, then $f \ast \phi \in \mathcal{S}$. It now follows as above that $f$ is of the required form.

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