Partial Inner Product Spaces. II. Operators

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We study linear operators between nondegenerate partial inner product spaces and their relationships to selfadjoint operators in a "middle" Hilbert space.

1. INTRODUCTION

Partial inner product spaces [1] can be considered from two points of view. On the one hand they are, trivially, generalizations of inner product (and, more specially, Hilbert) spaces. On the other hand, the discussion and examples in [1] show that a nondegenerate partial inner product on a space \( V \) determines a (partially ordered) scale of intermediate spaces (called assaying subsets) densely embedded in each other and in the ambient space. (In an inner product space, the scale collapses to a single member.) Such scales are a standard tool in many problems of quantum mechanics and in partial differential equations. A partial inner product may then be viewed as a means of introducing a suitable family of auxiliary spaces around a given "physical" space \( V_0 \), with the aim of studying special classes of operators in \( V_0 \).

Both points of view appear in this paper, which begins the study of operators in (or between) nondegenerate partial inner product spaces. The definition, given in Section 2, reduces in Hilbert space to that of a bounded operator. Some properties of bounded operators are preserved in the general case (Section 3). However, the study of
products requires the machinery of "representatives" which we introduce, following [2], in Section 4 (compare [6]).

The second point of view predominates in Section 5 where we examine the problem of selfadjoint restrictions to a "middle Hilbert space" $V_0$ of operators in $V$ and make a short comparison with the theory of semibounded quadratic forms in Hilbert space. More examples are given in Section 6.

2. NATURAL DOMAINS

Let $V$ and $Y$ be nondegenerate partial inner product spaces, and $A$ a map from a subset $\mathcal{D} \subseteq V$ into $Y$. We say that $A$ is an operator with natural domain (or simply operator) if

(i) $\mathcal{D}$ is a nonempty union of assaying subsets of $V$.

(ii) The restriction of $A$ to any assaying subset $V_r$ contained in $\mathcal{D}$ is linear and continuous (for $\tau(V_r, V_r) \rightarrow \tau(Y, Y^*)$).

(iii) $A$ has no proper extension satisfying (i) and (ii), i.e., is maximal.

(A proper extension of $A$ satisfying (i) and (ii) would be a map $A'$ defined on a union of assaying subsets $\mathcal{D}' \supset \mathcal{D}$, coinciding with $A$ on $\mathcal{D}$, linear and continuous on every assaying subset in its domain).

The set of all operators with natural domain in $V$ and with range in $Y$ will be denoted by $\text{Op}(V, Y)$.

Given any $A \in \text{Op}(V, Y)$, its restriction to $V^\ast$ is continuous. Conversely

2.1. PROPOSITION. Given any $\tau(V^\ast, V) \rightarrow \tau(Y, Y^*)$-continuous linear map $\alpha$ from $V^\ast$ into $Y$, there exists one and only one $A \in \text{Op}(V, Y)$ having $\alpha$ as restriction to $V^\ast$.

Proof. (a) An extension of $\alpha$ will mean a map satisfying (i) and (ii) but not necessarily (iii). The family of all extensions of $\alpha$ carries a natural partial order (by inclusion of domains). It is easy to see that it satisfies the conditions of Zorn's lemma, and so has a maximal element. This proves the existence part.

(b) Let $A_1 \in \text{Op}(V, Y)$ and $A_2 \in \text{Op}(V, Y)$ have the same restriction to $V^\ast$. Let $\mathcal{D}_i$ be the domain of $A_i$ ($i = 1, 2$). Notice that $A_1$ and $A_2$ coincide not only on $V^\ast$, but on $\mathcal{D}_1 \cap \mathcal{D}_2$. Indeed, on each $V_r \subseteq \mathcal{D}_1 \cap \mathcal{D}_2$, the restrictions of $A_1$ and $A_2$ are continuous,
and $\mathcal{V}^*$ is dense on $\mathcal{V}_\iota$. Next, we see that $\mathcal{D}_1 = \mathcal{D}_2$; otherwise we could define an operator $A'_i$ with domain $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2$, equal to $A_i$ on $\mathcal{D}_i (i = 1, 2)$. This $A'_i$ would be a proper extension of $A_1$ and $A_2$, contradicting maximality.

**Corollary.** $\text{Op}(\mathcal{V}, \mathcal{Y})$ is isomorphic, as a vector space, to the space of all linear continuous maps from $\mathcal{V}^*$ to $\mathcal{Y}$. Operators can always be added.

In Section 3, we shall see that $\text{Op}(\mathcal{V}, \mathcal{V})$ is also a "partial *-algebra."

**Remark.** A natural domain $\mathcal{D}$ is a union of assaying subsets (each of which is a vector space); it need not, in general, be a vector space (it will be a vector space in most cases of interest). Linearity of an $A \in \text{Op}(\mathcal{V}, \mathcal{Y})$ has to be understood as linearity on each $\mathcal{V}_\iota \subseteq \mathcal{D}$. This creates no difficulties. This point will be discussed in another paper.

### 3. Adjoint

Let $\mathcal{V}$ and $\mathcal{Y}$ be nondegenerate partial inner product spaces; let $U_1, U_2$ be a dual pair in $\mathcal{V}$, and $Y_1, Y_2$ a dual pair in $\mathcal{Y}$. Let $\alpha$ be an arbitrary linear map from $U_1$ into $Y_1$ (no continuity assumed). It is clear that the sesquilinear form $\ell(y, v) = \langle y | \alpha v \rangle_y (y \in Y_2, v \in U_1)$ is continuous in $y$ for fixed $v$. Its continuity in the other argument is equivalent to the continuity of $\alpha$. Namely

**3.1. Lemma.** The following conditions are equivalent:

(i) $\alpha$ is continuous for $\tau(U_1, U_2) \to \tau(Y_1, Y_2)$

(i') $\alpha$ is continuous for $\sigma(U_1, U_2) \to \sigma(Y_1, Y_2)$

(ii) There exists a linear map $\beta$ from $Y_2$ into $U_2$, such that

$$\langle y | \alpha v \rangle_y = \langle \beta y | v \rangle_v$$

for all $y \in Y_2, v \in U_1$

(iii) The sesquilinear form

$$\ell(y, v) = \langle y | \alpha v \rangle_y$$

is separately continuous in each of its arguments, for the topologies $\tau(Y_2, Y_1), \tau(U_1, U_2)$.

**Proof.** See, e.g., [3].
We shall need mostly the following special case, which gives a convenient criterion for continuity on assaying subsets:

3.2. **Lemma.** Let \( r \in F(V) \) and \( q \in F(Y) \). Let \( \alpha \) be a linear mapping from \( V_r \) into \( Y_q \), and \( \beta \) a linear mapping from \( Y_q \) into \( V_r \) such that

\[
\langle y \mid \alpha v \rangle_Y = \langle \beta y \mid v \rangle_Y
\]

for all \( y \in Y_q \) and all \( v \in V_r \). Then \( \alpha \) is \( \tau(V_r, V_r) \rightarrow \tau(Y_q, Y_q) \)-continuous, and \( \beta \) is \( \tau(Y_q, Y_q) \rightarrow \tau(V_r, V_r) \)-continuous.

As an application, we prove

3.3. **Proposition.** Let \( A \) be a linear map of \( V \) into \( V \), such that

(i) \( A \) improves behaviour, i.e.,

\[
\{Af\}^* \supseteq \{f\}^* \quad \text{for every } f \in V.
\]

(ii) If \( f \) and \( g \) are compatible, then

\[
\langle g \mid Af \rangle = \langle Ag \mid f \rangle.
\]

Then, for every \( r \in F(V) \), one has \( AV_r \subseteq V_r \), and the restriction of \( A \) to \( V_r \) is \( \tau(V_r, V_r) \)-continuous.

**Proof.** By [1, Section 2], \( A \) maps every \( V_r \) into itself. Furthermore, \( \langle g \mid Af \rangle = \langle Ag \mid f \rangle \) for all \( f \in V_r, g \in V_r \) so that the preceding lemma applies.

We are now ready to define the adjoint of an arbitrary \( A \in \text{Op}(V, Y) \). We have seen in Section 2 that an \( A \in \text{Op}(V, Y) \) is fully determined by its restriction to \( V^* \). This restriction is \( \tau(V^*, V) \rightarrow \tau(Y, Y^*) \)-continuous and so gives rise to a sesquilinear form \( \delta(y, v) = \langle y \mid Av \rangle \) \((y \in Y^*, v \in V^*)\), separately continuous for \( \tau(V^*, Y) \) and \( \tau(Y^*, Y) \). This form determines a (unique) map \( \beta \) from \( Y^* \) into \( V \), defined by \( \delta(y, v) = \langle \beta y \mid v \rangle_Y \) for all \( v \in V^*, y \in Y^* \). Furthermore, \( \beta \) is \( \tau(Y^*, Y) \rightarrow \tau(V, V^*) \)-continuous. Consequently, \( \beta \) has a unique natural extension which we shall call the **adjoint of** \( A \) and denote by \( A^* \).

The correspondence \( A \leftrightarrow \delta \), given by \( \delta(y, v) = \langle y \mid Av \rangle_Y \) is a bijection between \( \text{Op}(V, Y) \) and the vector space \( \mathcal{B}(Y^*, V^*) \) of all separately continuous sesquilinear forms on \( Y^* \times V^* \). The correspondence \( A^* \leftrightarrow \delta \), given by \( \delta(y, v) = \langle A^*y \mid v \rangle \) is a bijection between \( \text{Op}(Y, V) \) and \( \mathcal{B}(Y^*, V^*) \).

To summarize:
3.4. Theorem. \( \text{Op}(V, Y) \) has a natural structure of vector space. The antilinear correspondence \( A \mapsto A^* \) is a bijection between \( \text{Op}(V, Y) \) and \( \text{Op}(Y, V) \). One has \( A^{**} = A \) for every \( A \in \text{Op}(V, Y) \). In particular, the correspondence \( A \leftrightarrow A^* \) is an involution in \( \text{Op}(V, V) \).

To say it simply: In adding operators and in taking adjoints one may proceed algebraically without any special precautions.

If \( A \in \text{Op}(V, V) \) satisfies \( A^* = A \), we say that it is symmetric in \( V \).

4. Representatives; Products

We have seen that the vector space structure of, say, \( \text{Op}(V, Y) \) is identical to that of the space \( \mathcal{B}(V^*, Y^*) \) of sesquilinear forms and to that of a space of continuous linear maps from \( V^* \) to \( Y \). One may wonder, then, whether there is any point in considering the natural extensions beyond \( V^* \).

The answer is yes, if one is interested in studying products of operators, as we shall see below.

Let \( A \in \text{Op}(V, Y) \). "Goodness" properties of \( A \) are conveniently described by the set \( \mathcal{J}(A) \) of all pairs \( \tau \in F(V), q \in F(Y) \) such that \( A \) maps \( V \) continuously into \( Y \), \( \tau(V_r, V_s) \) and \( \tau(Y_q, Y_d) \).

4.1. Notation. If \( AV_r \subseteq Y_q \) and if the restriction of \( A \) to \( V_r \) is \( \tau(V_r, V_s) \to \tau(Y_q, Y_d) \)-continuous we denote that restriction by \( A_{qr} \) and call it the \( \{r, q\} \)-representative of \( A \) (see [2]).

Thus \( \mathcal{J}(A) \) is the set of all pairs \( \{r, q\} \) for which a representative \( A_{qr} \) exists.

If \( \{r, q\} \in \mathcal{J}(A) \) then the representative \( A_{qr} \) is uniquely defined. If \( A_{qr} \) is any \( \tau(V_r, V_s) \to \tau(Y_q, Y_d) \)-continuous linear map from \( V_r \) to \( Y_q \), then there exists a unique \( A \in \text{Op}(V, Y) \) having \( A_{qr} \) as \( \{r, q\} \)-representative. This \( A \) can be defined by considering \( A_{qr} \) as a map from \( V^* \) to \( Y \) and then extending it to its natural domain.

A pair \( \{r, q\} \) belongs to \( \mathcal{J}(A) \) if and only if the pair \( \{q, \bar{r}\} \) belongs to \( \mathcal{J}(A^*) \). The representative \( A_{qr} \) is injective if and only if the representative \( (A^*)_{\bar{q} \bar{r}} \) has dense range (for \( \tau(V_r, V_s) \)).

Let \( V_r \) and \( V_r' \) be assaying subsets of \( V \). If \( V_r \supseteq V_r' \), denote by \( E_{r \to r'} \) the natural embedding operator from \( V_r \) into \( V_r' \), i.e., the \( \{r, r'\} \)-representative of the identity \( 1 \in \text{Op}(V, V) \). This \( E_{r \to r'} \) is continuous, of dense range, and manifestly injective.

If \( A \) maps continuously a \( V_r \) into a \( Y_q \), it also maps continuously
any smaller $V_{r'} \subseteq V_r$ into any larger $V_{q'} \supseteq V_q$. Consequently, it is convenient to introduce in the cartesian product $F(V) \times F(Y)$ a partial order by

$$\{r', q'\} \succ \{r, q\}$$

if and only if $r' \leq r$ and $q' \geq q$. By the continuity and injectivity of the natural embeddings $F_{rr'}$ and $F_{qq'}$ between assaying subsets, one has immediately

4.2. PROPOSITION. Let $\{r, q\} \in J(A)$, and let $\{r', q'\}$ be a successor of $\{r, q\}$ with respect to the partial order $\succ$. Then:

(i) $\{r', q'\}$ also belongs to $J(A)$, and

$$A_{q'r'} = E_{q'q} A_{qr} E_{rr'}$$

is the $\{r', q'\}$-representative of $A$.

(ii) If $A_{qr}$ is injective, so is $A_{q'r'}$.

(iii) If $A_{qr}$ has dense range, so has $A_{q'r'}$.

It will be convenient to call $A_{q'r'}$ a successor of $A_{qr}$.

We shall say that a representative $A_{qr}$ is invertible if it is bijective and has a continuous inverse. The second condition is automatically satisfied if $V_r$ and $Y_q$ are Fréchet (in particular Banach or Hilbert) spaces.

Any successor of an invertible representative is injective and has dense range. An invertible representative has in general no predecessors.

We shall now state the conditions under which a product $BA$ is defined. The main point (just as in the definition of partial inner product which is, in fact, a special case) is that the “goodness” of one multiplicand can compensate for the “badness” of the other.

Let $V^{(1)}$, $V^{(2)}$ and $V^{(3)}$ be nondegenerate partial inner product spaces; (some, or all, may coincide). Let $A \in \text{Op}(V_1, V_2)$ and $B \in \text{Op}(V_2, V_3)$. We say that the product $BA$ is defined if and only if there exist $r_1 \in F(V_1)$, $r_2 \in F(V_2)$, $r_3 \in F(V_3)$ such that $\{r_1, r_2\} \in J(A)$ and $\{r_2, r_3\} \in J(B)$. Then $B_{r_2r_3} A_{r_1r_2}$ is a continuous map from $V^{(1)}_{r_1}$ into $V^{(3)}_{r_3}$. It is the $\{r_1, r_3\}$-representative of a unique element of $\text{Op}(V_1, V_3)$ which will be denoted by $BA$ and called the product of $A$ and $B$. If $BA$ is defined, then $A^*B^*$ is also defined, and equal to $(BA)^*$. Similar definitions hold for products of more than two operators.
The product $CBA$ of $A \in \text{Op}(V^{(1)}, V^{(2)})$, $B \in \text{Op}(V^{(2)}, V^{(3)})$, and $C \in \text{Op}(V^{(3)}, V^{(4)})$ is defined whenever there exists a "chain" 
\[ \{r_1, r_2, r_3, r_4\} \] with \( \{r_1, r_2\} \in J(A) \), \( \{r_2, r_3\} \in J(B) \) and \( \{r_3, r_4\} \in J(C) \). The existence of $CBA$ does not follow from the existence of $CB$ and of $BA$.

If $A \in \text{Op}(V, Y)$ has an invertible representative, then there exists a $B \in \text{Op}(Y, I')$ such that $AB$ and $BA$ are defined, $AB = 1_Y$, and $BA = 1_Y$. This does not exclude the possibility of $A$ having a nontrivial null-space.

5. Selfadjoint Restrictions; Relationship to Quadratic Forms

Throughout this section we assume that $V$ has a “middle Hilbert space” $V_0$, i.e., that there is in $V$ an assaying subset $V_0$ such that $V_0 = (V_0)^*$ and that the restriction of the partial inner product to $V_0$ makes $V_0$ into a Hilbert space. This holds in [1, Examples 2, 3, 5, 6, 7, 8a, and 8b].

Given a symmetric operator $H$ in $\text{Op}(V, V)$, it is natural to ask whether $H$ has restrictions that are selfadjoint in $V_0$. (This is “dual” to the classical problem of extending symmetric operators from a domain in $V_0$.)

An answer is given by Theorem 5.1 below. It says that $H$ has a selfadjoint restriction provided some suitable representative of $H$ is invertible. This is essentially the KLMN theorem in (we believe) its natural setting.

Another obvious problem is: Given symmetric operators $T$ and $U$, study spectral properties of selfadjoint restrictions of $T + U$ (remember that there is no ambiguity in the definition of $T + U$). This corresponds to the basic problem in perturbation theory of selfadjoint operators.

5.1. Theorem. Let $H \in \text{Op}(V, V)$ and $H^* = H$ (in the sense of Section 3). Assume that there exists a $\lambda \in \mathbb{R}$ such that $H - \lambda$ has an invertible representative from a "small" $V_\gamma \subseteq V_0$ onto a "big" $V_s \supseteq V_0$. Then there exists a unique restriction of $H_{\text{op}}$ to a selfadjoint operator $\mathcal{H}$ in the Hilbert space $V_0$. The number $\lambda$ does not belong to the spectrum of $\mathcal{H}$. The domain of $\mathcal{H}$ is obtained by eliminating from $V_\gamma$ exactly the vectors $f$ that are mapped by $H_{\text{op}}$ beyond $V_0$ (i.e., satisfy $H_{\text{op}}f \notin V_0$).

1 For Kato, Lions, Lax, Milgram, Nelson. See [5, p. 4].
The resolvent \((\mathcal{H} - \lambda)^{-1}\) is compact (trace class, etc.) if and only if the natural embedding \(E_{sp}\) is compact (trace class, etc.).

Proof.

(a) **Lemma.** Let \(R \in \text{Op}(V, V)\) be symmetric. Assume that \(R\) has an injective representative \(R_{ps}\), with dense range from a ("big") \(V_s \supseteq V_0\) into a ("small") \(V_p \subseteq V_0\). Then the representative \(R_{00}\) has a selfadjoint inverse.

Indeed, \(R_{00}\) is a successor of \(R_{ps}\). By the results of Section 4 it is injective and of dense range. Furthermore, it is bounded and self-adjoint as the operator in the Hilbert space \(V_0\). So its inverse is selfadjoint on \(R_{00}V_0\).

(b) Existence of selfadjoint restriction: Define \(R_{ps}\) as the inverse of the invertible representative \(H_{sp} - \lambda E_{sp}\). Then \(R_{00} = E_{0p}R_{ps}E_{s0}\) is a restriction of \(R_{ps}\). Consequently its selfadjoint inverse \((R_{00})^{-1} = \mathcal{H} - \lambda\) is a restriction of \(H_{sp} - \lambda E_{sp}\).

We omit the easy proof of the remaining assertions.

We look now for conditions which ensure the existence of an invertible representative, needed in Theorem 5.1. If \(V_p\) and \(V_s\) are Hilbert spaces, we can use the following trivial criterion: If \(W\) is unitary from \(V_p\) onto \(V_s\), then all bounded operators \(A\) in the ball \(\| W - A \|_{sp} < 1\) (bound norm with respect to \(\| \cdot \|_p\) and \(\| \cdot \|_s\)) are invertible. (An obvious Banach generalization exists.) This gives, e.g.,

5.2. **Proposition.** Let \(T\) and \(U\) belong to \(\text{Op}(V, V)\), \(T^* = T\) and \(U^* = U\). Assume that there exists a \(V_p \subseteq V_0\) and a \(V_s \supseteq V_0\), both with Hilbert structures, such that \(T_{sp} - \lambda E_{sp}\) is unitary (from \(V_p\) onto \(V_s\)) and that \(\| U \|_{sp} < 1\) (bound norm with respect to \(\| \cdot \|_p\) and \(\| \cdot \|_s\)). Then \(H = T + U\) has a unique selfadjoint restriction, and all the other conclusions of Theorem 5.1 hold.

Remark. It is sometimes useful to treat \(T\) and \(U\) in a more symmetric fashion; take \(\lambda = 0\) and assume the existence of an operator \(K\) with unitary representative \(K_{sp}\) sufficiently close to both \(T_{sp}\) and \(U_{sp}\).

The above results can be made entirely explicit if \(T\) is given as a multiplication operator by a function \(t(p)\) on a measure space and if \(U\) is given as a kernel \(U(p, p')\) on the same space.

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2 Many assaying subsets have natural Hilbert structures. See Examples of [1]. The partial inner product determines only hilbertizable topologies on these assaying subsets; we are free to choose the metric.
The example below covers the Schrödinger equation and (with trivial modifications) the Dirac equation, in momentum representation.

5.3. Theorem. Let \( f(p) \) be real-valued and locally bounded function on \( \mathbb{R}^n \) (i.e., \( f \in L^\infty_{\text{loc}}(\mathbb{R}^n) \)). Assume that its range has a gap, i.e., that there exists a \( \lambda \in \mathbb{R} \) such that \( (f(p) - \lambda)^{-1} \) is a bounded function. Assume that \( U(p, p')(p, p' \in \mathbb{R}^n) \) is Hermitian and such that, for some \( r < 0 \), the kernel

\[
K(p, p') = (f(p) - \lambda)^{-(r/2)} U(p, p') (f(p') - \lambda)^r
\]

defines in \( L^2(\mathbb{R}^n; d^n p) \) an operator of bound norm \(< 1\). Then \( T + U \) (defined, say, in \( V = L^1_{\text{loc}}(\mathbb{R}^n; d^n p) \); see \([1, \text{example 5}]\)) has a selfadjoint restriction, and all the other conclusions of Theorem 5.1 hold.

Proof. A direct application of Proposition 5.2.

Related results can be found in \([5]\) and in \([7]\). We remark that results on closedness and on spectrum can be obtained without the assumption of symmetry.

The above results are about as strong as the ones obtained usually with the help of quadratic forms (but we do not have to assume semiboundedness). The relationship to quadratic forms is made more explicit in Proposition 5.5 below.

Let \( V_0 \) be a Hilbert space and \( B_{00} \) a bounded, positive, injective operator in \( V_0 \). Define \( V^{(b)} = V_1 \) as the completion of \( V_0 \) with respect to the norm \( \| f \|_1 = \| B_{00} f \|_0 = (f, f)_1 = (f, B_{00} f)^{1/2} \). Let \( V_1 \) be the range of \((B_{00})^{1/2}: V_1 = (B_{00})^{1/2} V_0 \). It is a Hilbert space with respect to the scalar product \((f, g)_1 = (B_{00}^{-1} f, B_{00}^{-1} g)_0\). Then \( V^{(b)} \) becomes a nondegenerate partial inner product space if \( f \neq g \) means: either both \( f \) and \( g \) belong to \( V_0 \), or at least one (say, \( g \)) belongs to \( V_1 \). In the first case we define \( \langle f | g \rangle = (f, g)_0 \). In the second case \( \langle f | g \rangle = \lim_{n \to \infty} (B_{00} f_n, B_{00}^{-1} g)_0 \) where \( \| f_n - f \|_1 \to 0, f_n \in V_0 \). The assaying subsets of \( V^{(b)} \) are exactly \( V_1, V_0 \) and \( V_1^{-1} \).

5.4. Proposition. Let \( R_{00} \) be a selfadjoint, bounded, and injective operator in \( V_0 \), and let \( R_{00} = W_{00} B_{00} \) be its polar decomposition, where \( W_{00} \) is unitary in \( V_0 \) and \( B_{00} \) is bounded. Let \( R \) be the operator in \( V^{(b)} \) having \( R_{00} \) as \( \{0, 0\} \)-representative. Then the natural domain of \( R \) is all of \( V^{(b)} \) and \( R \) has a unitary representative \( R_{11} \).

We omit the easy proof, which uses the fact that under our assumptions, \( W_{00} R_{00} = R_{00} W_{00} \).
5.5. Proposition. Let $q$ be a densely defined closed symmetric, quadratic form on a Hilbert space, such that for a $\lambda \in \mathbb{R}$, $q + \lambda$ is strictly positive. Let $V_1$ be the domain of $q$; it is a Hilbert space with the scalar product $(f, q)_1 = \lambda(f, g)_0 + q(f, g)$. Consider in $V_0$ the bounded positive injective operator $B_{00} = E_{01}(E_{01})^*$ where $E_{01}$ is the natural embedding of $V_1$ into $V_0$ and $(E_{01})^*$ is its adjoint (in the sense of bounded operators between Hilbert spaces). Consider the partial inner product space $V^{(a)}$, defined above. Let $H$ be the element of $\text{Op}(V^{(a)}, V^{(a)})$, defined by $H_{11} = \lambda E_{11} = (B_{11})^{-1}$. Then the selfadjoint restriction of $H_{11}$ is the selfadjoint operator associated to the form $q$ [4].

Remarks. (1) Form-bounds correspond to bounds in the $1 \rightarrow \overline{1}$ norm.

2) Proposition 5.4 covers more ground than Proposition 5.5, since it allows for a nontrivial $W_{00}$. This is related to so-called pseudo-Friedrichs extensions ([4, p. 341]).

6. Examples

(1) Let $V$ be arbitrary. Notice that $\mathbb{C}$, with the obvious inner product $\langle \xi | \eta \rangle = \xi \overline{\eta}$ is an inner product space. Hence both $\text{Op}(\mathbb{C}, V)$ and $\text{Op}(V, \mathbb{C})$ are well defined and antiisomorphic to each other. Elements of $\text{Op}(V, \mathbb{C})$ will be called regular linear functionals on $V$. Exactly as in [2], one shows that they are given precisely by the functionals

$$\langle g | : f \rightarrow \langle g | f \rangle \quad (f \in \{g\}^*)$$

The reasoning goes as follows:

(a) for every $f \in V$, define a map $|f\rangle: \mathbb{C} \rightarrow V$ by

$$|f\rangle: \xi \rightarrow \xi f$$

(b) notice that the correspondence $f \rightarrow |f\rangle$ is a linear bijection between $V$ and $\text{Op}(\mathbb{C}, V)$

(c) verify that the adjoint of $|f\rangle$ is $\langle f |$ as defined above.

As a result, $V$ has naturally a self-dual structure. In this respect, partial inner product spaces genuinely generalize Hilbert space, in contrast with schemes such as rigged Hilbert spaces. At this point we see that the algebraic structure of PIP-spaces is extremely simple, once one agrees to take as input the domain of the partial inner
product, and chooses Mackey topologies on every dual pair that shows up.

(2) Using the fact that \( \langle g | \in \text{Op}(V, \mathbb{C}) \) is defined exactly on \( \{g\}^* \), one verifies that:

(i) the product \( \langle g | \langle f \rangle \rangle \) is defined if and only if \( g \neq f \), and equals \( \langle g | f \rangle \).

(ii) the product \( \langle f \rangle \langle g \rangle \) is always defined; it is an element of \( \text{Op}(V, V) \), called a dyadic. Its action is given by:

\[
| f \rangle \langle g | h = \langle g | h \rangle f \quad (h \in \{g\}^*)
\]

The adjoint of \( | f \rangle \langle g | \) is \( | g \rangle \langle f | \). One constructs in the same way operators between different spaces and finite linear combinations of dyadics.

(3) If \( V \) is an inner product space, then an operator in \( V \) is a \( \tau(V, V) \)-continuous linear map. In particular, if \( V \) is a Hilbert space, \( \text{Op}(V, V) \) consists of all bounded operators in \( V \).

(4) Consider [1, Section 4, Example 21] (all sequences of complex numbers). To every \( A \in \text{Op}(V, V) \) there corresponds the infinite matrix \( a_{nm} = \langle e_n | Ae_m \rangle \) where \( e_1 = \{1, 0, \ldots\} \), \( e_2 = \{0, 1, 0, \ldots\} \), \ldots Conversely, an arbitrary infinite matrix \( \{a_{nm}\} \) gives rise to a unique \( A \in \text{Op}(V, V) \). Its natural domain is the union of all Köthe spaces \( V_r \) that have the property: All the row vectors \( \{a_{1n}\}, \{a_{2n}\}, \ldots \) of our matrix belong to \( V_r \).

(5) Let \( V = S' \) (tempered distributions). Elements of \( \text{Op}(S', S') \) are defined by tempered kernels. It should be interesting to study the corresponding natural domains.

If \( S' \) is realized in \( s' \), then the elements of \( \text{Op}(s', s') \) are given by "tempered matrices" satisfying either one of the equivalent conditions

\[
| a_{mn} | \leq C(1 + m)^N(1 + n)^{N'}
\]

\[
| a_{mn} | \leq C'(1 + m + n)^N
\]

for some \( C, C', N, N' > 0 \).

(6) In this example, we extend the machinery of Fock space (a symmetric tensor algebra over a Hilbert space) to partial inner product spaces (Compare [2]). Let \( (X, \mu) \) be a measure space. Consider, (as in [1, Sect. 4, Example 5]), the space

\[
V^{(1)} = L^{(1)}_{loe}(X; d\mu)
\]
with \( f_1 \not\sim g_1 \) iff \( \int |f_1 g_1| \, d\mu < \infty \) (\( f_1, g_1 \in V^{(1)} \)). Define \( V^{(n)} \) as
\[
V^{(n)} = L^{(1)}(X^n, d\mu_1 \cdots d\mu_n) \text{ with } f_n \not\sim g_n \text{ iff } \int_{X^n} |f_n g_n| \, d\mu_n < \infty, \text{ where } d\mu_n = d\mu \times \cdots \times d\mu.
\]
Finally, consider
\[
V = \Gamma(V^{(1)}),
\]
the space of all (i.e., arbitrary) sequences
\[
f = (f_0, \ldots, f_n, \ldots) \quad (f_0 \in V^{(0)} = \mathbb{C}, f_n \in V^{(n)}).
\]
Define compatibility and partial inner product by
\[
f \not\sim g \quad \text{iff} \quad (i) \quad f_n \not\sim g_n \quad \text{for all } n
\]
\[\quad \text{and} \quad (ii) \quad \sum_{n=0}^{\infty} \int_{X^n} |f_n g_n| \, d\mu_n < \infty.
\]
One verifies immediately that \( V^* \) consists of all sequences with finitely many nonzero \( f_n \), and \( f_n \in (V^{(n)})^* = L_c^{(1)}(X^n; d\mu_n) \).

On \( V \), we consider three kinds of operators:

1. The symmetrizer \( S = \oplus_{n=0}^{\infty} S_n \) where \( (S_n f_n)(k_1 \ldots k_n) = (1/n!) \sum_n f_n(k_{n_1}, \ldots, k_{n_k}) \). Then \( S \in \text{Op}(V, V) \), \( S^* = S \). Its natural domain is all of \( V \).

2. The number operator: \( N \), defined by \( (Nf)_n = nf_n \). One defines similarly \( \varphi(N) = \oplus \varphi(n) \mathbb{1}_n \) for an arbitrary sequence \( \{\varphi(n)\} \) of real numbers. Here, too, \( \varphi(N) \) is defined on all of \( V \), and \( \varphi(N)^* = \varphi(N) \).

3. The operators \( C^+(f) \) and \( C(f) \) (\( f \in V \)) defined by
\[
[C^+(f)g]_n = \sum_{j=1}^{n} f_j \otimes g_1
\]
\[
(f_j \otimes g_1)(k_1, \ldots, k_j, k_{j+1}, \ldots, k_{j+r}) = f_j(k_1, \ldots, k_j) g_1(k_{j+1}, \ldots, k_{j+r})
\]
\[
C(f) = (C^+(f))^*.
\]
Again, \( C^+(f) \in \text{Op}(V, V) \) is everywhere defined, but \( C(f) \) is not unless \( f \in V^* \). Its natural domain can be found by methods used in [2].

It follows that the product
\[
a^+(f) = N^{1/2}SC^+(f)S
\]
belongs to \( \text{Op}(V, V) \) and has \( V \) as natural domain. We may also introduce
\[
a(f) = [a^+(f)]^*
\]
and the free field operator \( A(f) = a^+(f) + a(f) \).
Consider in particular the case where $X$ is the positive hyperboloid of mass $m$ in Minkowski space, and $d \mu = d^3k/k_0$ the invariant measure on $X$. If we take in particular $f = f_x$, where $f_x(k) = (2\pi)^{-3/2} e^{ikz} k \in X, z \in \mathbb{C}^4$, then the operator $A(x) \equiv A(f_x)$ is the (nonsmeared) free field at the (real or complex) point $x$. From here on, one can study Wick products, analyticity in $z$, etc., along the lines of [2]. In particular, the asymmetry in the domains of $a^+(f)$ and $a(f)$ shows that Wick-ordered products are better behaved (i.e., have a larger $\mathcal{J}(\cdot)$) than other products.

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