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The Partition Polynomial of a Finite Set System

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We introduce the partition polynomial of a finite set system, which generalizes the matching polynomial of a graph, and elucidate some of its properties. Among these are its connections with the matching polynomial and with a generalized chromatic polynomial and various structural conditions which imply that the partition polynomial has only real roots. The properties of the partition polynomial with respect to composition of set systems also prove interesting; the main result is an extension of the Heilmann–Lieb Theorem to this context. © 1991 Academic Press, Inc.

0. INTRODUCTION

In 1946 Kaplansky and Riordan introduced the notion of *rook polynomials* in their exposition of the *ménage problem* and other problems of enumerating permutations with restricted positions [22, 26]. It was later conjectured that all the roots of any rook polynomial are real [12], and a proof of this was supplied by Nijenhuis [25]. Somewhat earlier, Heilmann and Lieb [19] had defined the *matching polynomial* of a graph, which reduces to the rook polynomial just when the graph is bipartite, and had proved that for any graph the matching polynomial has only real roots. Subsequently, many investigations have been made into the properties of rook and matching polynomials [5–17, 19, 25].

We present here a third level of generality, by defining a *partition polynomial* for any finite set system. Informally, the partition polynomial of $\mathscr{F} \subset 2^V$ is the rank-generating function for those partitions π of V such that each block of π is in $\mathscr{F}$. This polynomial reduces to the matching polynomial when the set system is the collection of vertices and edges of a graph.

The motivation for this definition is an effort to determine how generally

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a result of the Heilmann–Lieb type can hold. Although in general a partition polynomial need not have only nonpositive roots, we obtain various conditions which imply this property. (Since the coefficients of a partition polynomial are nonnegative, all its real roots are nonpositive.) For the first class of examples we use an argument due to C. Godsil to reduce certain kinds of partition polynomials to matching polynomials and apply the theorem of Heilmann and Lieb. This class contains, among other set systems, all order complexes of finite partially ordered sets. A second class of examples is obtained by investigating the connection between the partition polynomial and a generalized *chromatic polynomial* of a finite set system; we find that the independence complexes of supersolvable graphs have partition polynomials with only nonpositive roots.

In addition to these rather specialized results, our main results concern the whole class of set systems which have partition polynomials with only nonpositive roots. Denoting this class by $\mathscr{R}$, we show that $\mathscr{R}$ is closed under the operations of composition into any graph. Composition is defined precisely below, but intuitively one substitutes a set system for each vertex of the graph and joins them along each edge of the graph. Indeed, composition into any set system may be defined similarly, but in general the composite need not be a member of $\mathscr{R}$, even when each of its arguments is.

In Theorem 4.1 we show that for any set system $\mathscr{F}$ the partition polynomial of a composition into $\mathscr{F}$ is determined by the structure of $\mathscr{F}$ and by the partition polynomials of its arguments. The point is that in order to calculate the partition polynomial of a composite one does not need complete information about its arguments, one needs to know only their partition polynomials. Thus we have a transformation $\Phi_{\mathscr{F}}: \mathbf{R}[x]^{V(\mathscr{F})} \to \mathbf{R}[x]$ which takes the set of partition polynomials of the arguments to the partition polynomial of their composition into $\mathscr{F}$.

To prove the main result we define a subclass $\mathscr{R}^*$ of $\mathscr{R}$ by a certain condition of $\Phi_{\mathscr{F}}$ and show that both $\mathscr{R}$ and $\mathscr{R}^*$ are closed by taking compositions into members of $\mathscr{R}^*$. Finally, we show in Theorem 4.5 that every one-dimensional simplicial complex (i.e., graph) is a member of $\mathscr{R}^*$. This can be regarded as an extension of the Heilmann–Lieb Theorem.

On the basis of the above examples one is led to the conjecture that the partition polynomial of every finite simplicial complex has only nonpositive roots. However, this is false! A computer program written to calculate and factor the partition polynomials of skeleta of simplices has provided counterexamples, the smallest being the 2-skeleton of the 8-simplex.

The organization of the paper is as follows. Section 1 contains a brief outline of the method of interlacing roots, which is our tool for showing that a polynomial has only real roots. Inequalities obtaining among the coefficients of such a polynomial are also presented in this section.

Section 2 contains the definition of the partition polynomial and some of its simple properties, as well as its relation to the matching polynomial, and a presentation of the first class of examples of set systems in $\mathscr{R}$ (Theorem 2.5). In Section 3 we examine the connection between the partition polynomial and a generalized chromatic polynomial and provide a second class of examples of set systems in $\mathscr{R}$ (Theorem 3.5). Section 4 contains the main results, regarding the effect of composition of set systems on the partition polynomial. We conclude in Section 5 with an explanation of the results of the computer program which calculated and factored the partition polynomials of skeleta of simplices and make some remarks on possible avenues for future research.

Finally, I thank Chris Godsil, Gian-Carlo Rota, and Richard Stanley for several interesting conversations and for their helpful comments on an earlier draft of this article. My thanks also to the referee, who spotted a minor error in an earlier form of Theorem 4.1 and whose helpful criticism led to many other improvements.

1. The Method of Interlacing Roots

The central topic of this article is the question of when the partition polynomial has only real roots. This property implies many inequalities among the coefficients, and in particular the following propositions hold. When the coefficients of such a polynomial have combinatorial meaning, as they do in our case, these inequalities are particularly interesting.

Proposition 1.1 (Newton's Inequalities [20, (51)]). *If $p(x)=a_n x^n + \cdots + a_0$ has only real roots and $a_n \neq 0$ then*

$$\frac{a_k^2}{\binom{n}{k}^2} \geqslant \frac{a_{k-1}}{\binom{n}{k-1}} \cdot \frac{a_{k+1}}{\binom{n}{k+1}}$$

for $1 \leqslant k \leqslant n-1$. Consequently $a_0, ..., a_n$ is logarithmically concave and hence unimodal in absolute value.

Proposition 1.2 [2, Sect. 7.III]. *The polynomial $p(x)=a_n x^n + \cdots + a_0$ with $a_n > 0$ has only nonpositive roots if and only if every finite square submatrix of the $\mathbf{Z} \times \mathbf{Z}$-indexed matrix $M(p)$ defined by*

$$(M(p))_{ij} = \begin{cases} a_{j-i} & \text{if } 0 \leqslant j-i \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

has nonnegative determinant. (That is, $M(p)$ is "totally nonnegative.")

One useful technique for proving that a polynomial has only real roots is the method of interlacing roots. We say that a polynomial p *interlaces* a polynomial q if both p and q have only real roots, $\deg q = 1 + \deg p$, and the roots $\xi_1 \leqslant \cdots \leqslant \xi_n$ of p and $\theta_1 \leqslant \cdots \leqslant \theta_{n+1}$ of q satisfy

$$\theta_1 \leqslant \xi_1 \leqslant \theta_2 \leqslant \cdots \leqslant \xi_n \leqslant \theta_{n+1}.$$

We also say that p and q *alternate* if for sufficiently large ζ, either p interlaces $(x-\zeta)\,q$ or q interlaces $(x-\zeta)\,p$. If a polynomial p has only real roots we denote its largest root by $\Lambda(p)$.

The next two lemmas are easy consequences of the Intermediate Value Theorem.

LEMMA 1.3. *Suppose p interlaces q and the leading coefficients of p and q are of the same sign. Then $f = \alpha p + \beta q$ has only real roots for any $\alpha, \beta \in \mathbf{R}$, and if $\beta \neq 0$ then p interlaces f, and q and f alternate. Furthermore, if $\alpha\beta < 0$ then $\Lambda(q) \leqslant \Lambda(f)$, and if $\alpha\beta > 0$ then $\Lambda(f) \leqslant \Lambda(q)$.*

LEMMA 1.4. *If p_1 and p_2 both interlace q and the leading coefficients of p_1 and p_2 are of the same sign, then $p_1 + p_2$ interlaces q.*

Throughout the paper we use D to denote the differentiation operator d/dx. The next lemma is a special case of Rolle's Theorem.

LEMMA 1.5. *If p has only real roots then Dp interlaces p.*

Proof. Let $p = \alpha \prod_{i=1}^{n} (x - \xi_i)$. Then $Dp = \sum_{i=1}^{n} \hat{p}_i$, where $\hat{p}_i = p/(x-\xi_i)$ for $1 \leqslant i \leqslant n$. Each $\hat{p}_i$ interlaces p, and they all have leading coefficient α. Hence by Lemma 1.4, Dp interlaces p. ∎

Proposition 1.6 provides a characterization of those polynomials which interlace a given polynomial with only real roots. Richard Askey, Mourad Ismail, and Paul Nevai inform me that it is part of the "folklore" of orthogonal polynomials, perhaps due to Laguerre, that they have the properties in Proposition 1.6 (see Theorem 3.3.5 in [29]). In the 1930s Krein showed that any two polynomials whose roots (strictly) interlace can be extended to a sequence of orthogonal polynomials (see [23, 30]). This suffices to prove Proposition 1.6, but for completeness we provide a short direct proof.

PROPOSITION 1.6. *Let $p = \prod_{i=1}^{n} (x - \xi_i)$, where $\xi_i \in \mathbf{R}$ for $1 \leqslant i \leqslant n$, and let $\hat{p}_i = p/(x - \xi_i)$ for $1 \leqslant i \leqslant n$. Then a monic polynomial q interlaces p if and* *only if it is a convex combination of* $\hat{p}_1, \ldots, \hat{p}_n$. *That is, if and only if there are real* $\alpha_i \geqslant 0$ *for* $1 \leqslant i \leqslant n$ *with* $\alpha_1 + \cdots + \alpha_n = 1$ *and* $q = \sum_{i=1}^n \alpha_i \hat{p}_i$.

Proof. If q is a convex combination of the $\hat{p}_i$ then Lemma 1.4 implies that q interlaces p, just as in the proof of Lemma 1.5. For the converse we proceed by induction on $n = \deg p$. The basis $n = 1$ or $n = 2$ is easily checked. Now suppose the result holds for $\deg p < n$, and let monic q interlace p. If $q = (1/n)\, Dp$ then we are done since then $q = (1/n) \sum_i \hat{p}_i$.

Otherwise, $q \neq (1/n)\, Dp$, and we claim that there is an $\varepsilon \in [0, 1)$ such that p and $f_\varepsilon = q - (\varepsilon/n)\, Dp$ have a common root, and f_ε has only real roots. To see this there are two cases. Firstly, if f_ε has nonreal roots for some $\varepsilon \in [0, 1)$ then let $\beta = \inf\{\varepsilon \in [0, 1) : f_\varepsilon \text{ has nonreal roots}\}$. Then f_β has only real roots and at least one double root. Since the roots of f_ε vary continuously with ε, $f_0 = q$, and q interlaces p, it follows that for some $\varepsilon \in [0, \beta]$, p and f_ε have a common root. Secondly, if f_ε has only real roots for all $\varepsilon \in [0, 1)$ consider f_ε as $\varepsilon \to 1^-$. The degree of f_1 is strictly less than $\deg f_0$, so at least one of the roots of f_ε must tend to $+\infty$ or to $-\infty$ as $\varepsilon \to 1^-$. Since $f_0 = q$ interlaces p it follows that p and f_ε have a common root for some $\varepsilon \in [0, 1)$.

Now let γ be the least ε as in the preceeding paragraph, and suppose that ξ_n is a common root of p and f_γ. Writing $g = f_\gamma/(x - \xi_n)$ we find that g interlaces $\hat{p}_n$. By the induction hypothesis we have a convex combination $(1-\gamma)^{-1} f_\gamma = \sum_{i=1}^{n-1} \delta_i \hat{p}_i$. Thus, putting $\delta_n = 0$, we get the convex combination

$$q = \sum_{i=1}^{n} \left((1-\gamma)\, \delta_i + \frac{\gamma}{n} \right) \hat{p}_i.$$

This completes the induction step and the proof. ∎

2. The Partition Polynomial and the Matching Polynomial

Let V be a set of n elements called *vertices*. A collection of subsets $\mathscr{F} \subset 2^V$ is called a *set system* if $\varnothing \in \mathscr{F}$ and $V = \bigcup \mathscr{F}$. In view of the second condition we may omit explicit reference to V, although it is convenient to use the notation $V(\mathscr{F}) = \bigcup \mathscr{F}$. A *simplicial complex* is a set system $\mathscr{K}$ with the property that if $S \in \mathscr{K}$ and $S' \subset S$ then $S' \in \mathscr{K}$.

Recall that a *partition of a set* V is a collection π of pairwise disjoint nonempty subsets of V such that $\bigcup \pi = V$. We say that a *partition of a set system* $\mathscr{F}$ is a partition π of the underlying set $V(\mathscr{F})$ which also satisfies $\pi \subset \mathscr{F}$. Note that this terminology is ambiguous since we might want to partition $\mathscr{F}$ as a set, that is, in the first sense. However, we never do this, and "partition of $\mathscr{F}$" is always to be understood in the second sense.

Let $p_k(\mathscr{F})$ denote the number of partitions of $\mathscr{F}$ with exactly k parts. The *partition polynomial* of $\mathscr{F}$ is defined to be

$$\rho(\mathscr{F}; x) = \sum_k p_k(\mathscr{F})\, x^k.$$

Note that $\rho(\{\varnothing\}; x) = 1$, while $\rho(\mathscr{F}; 0) = 0$ if $\mathscr{F} \neq \{\varnothing\}$. Also, since the coefficients $p_k(\mathscr{F})$ are nonnegative, every real root of $\rho(\mathscr{F}; x)$ is nonpositive.

For example, suppose that $\mathscr{F} = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, b, c\}\}$; then $\rho(\mathscr{F}; x) = x^3 + x^2 + x$, which clearly has nonreal roots. Nonetheless, we find many set systems for which the partition polynomial has only nonpositive roots.

There is a recursive formula from which one can calculate $\rho(\mathscr{F}; x)$ for any set system $\mathscr{F}$. For any $v \in V(\mathscr{F})$ let the *star* v be

$$\operatorname{st}(v) = \{S \in \mathscr{F} : v \in S\}$$

and for any $S \in \mathscr{F}$ let the *deletion* of S be

$$\mathscr{F}_S = \{S' \in \mathscr{F} : S' \cap S = \varnothing\}.$$

Finally, let the *core* of $\mathscr{F}$ be

$$C(\mathscr{F}) = \{S \in \mathscr{F} : V(\mathscr{F}_S) = V(\mathscr{F}) \backslash S\}.$$

Note that if $\{\{v\} : v \in V(\mathscr{F})\} \subset \mathscr{F}$ then $C(\mathscr{F}) = \mathscr{F}$. (This condition is sufficient but not necessary.)

PROPOSITION 2.1. *For any vertex-disjoint set systems $\mathscr{F}$ and $\mathscr{G}$ and any $v \in V(\mathscr{F})$:*

(a) $\rho(\mathscr{F}; x) = \rho(C(\mathscr{F}); x)$ *if* $V(C(\mathscr{F})) = V(\mathscr{F})$; *otherwise* $\rho(\mathscr{F}; x) = 0$.

(b) $\rho(\mathscr{F}; x) = x \cdot \sum_S \rho(\mathscr{F}_S; x)$, *where the summation is over all* $S \in \operatorname{st}(v) \cap C(\mathscr{F})$.

(c) $D\rho(\mathscr{F}; x) = \sum_S \rho(\mathscr{F}_S; x)$, *where the summation is over all* $\varnothing \neq S \in C(\mathscr{F})$.

(d) $\rho(\mathscr{F} \cup \mathscr{G}; x) = \rho(\mathscr{F}; x) \cdot \rho(\mathscr{G}; x)$.

Proof. For part (a) note that if π is partition of $\mathscr{F}$ then in fact $\pi \subset C(\mathscr{F})$. Part (b) is proved by summing the contributions to $\rho(\mathscr{F}; x)$ for those partitions of $\mathscr{F}$ which use a given $S \in \operatorname{st}(v)$. Part (c) is proved by counting in two ways the pairs (S, π) where $S \in \pi$ and π is a partition of $\mathscr{F}$. Part (d) is immediate from the definitions. ∎

Let $G=(V, E)$ be a graph with n vertices, and let $m_k(G)$ denote the number of matchings in G with exactly k edges. The matching polynomial of G is

$$\mu(G; x)=\sum_k m_k(G)(-1)^k x^{n-2k}.$$

This polynomial has the following remarkable property, first noted by Heilmann and Lieb [19].

PROPOSITION 2.2 (Heilmann–Lieb Theorem). *For any graph G, $\mu(G; x)$ has only real roots.*

For an excellent survey of the theory of the matching polynomial, see [9]. In addition, [19] gives applications to physics, and [15, 16] give applications to theoretical chemistry.

We use the matching polynomial in a slightly different form. Let the *modified matching polynomial* of G be

$$\tilde{\mu}(G; x)=\sum_k m_k(G)\, x^{n-k}.$$

We note that

$$\tilde{\mu}(G; -x^2)=(-x)^n\, \mu(G; x)$$

and since $m_k(G)\geqslant 0$ for all k we conclude that $\tilde{\mu}(G; x)$ has only nonpositive roots if and only if $\mu(G; x)$ has only real roots.

If $G=(V, E)$ is a graph we may associate a set system with G as follows:

$$\mathscr{K}(G)=\{\varnothing\}\cup\{\{v\} : v\in V\}\cup E.$$

It is immediate from the definitions that $\rho(\mathscr{K}(G); x)=\tilde{\mu}(G; x)$; in this way the partition polynomial generalizes the matching polynomial.

Note that the set system $\mathscr{K}(G)$ associated with the graph G is a one-dimensional simplicial complex, and conversely any one-dimensional simplicial complex determines a graph. (We are ignoring the trivial case in which $E(G)=\varnothing$ and $\mathscr{K}(G)$ is zero-dimensional.) Thus we have the following reformulation of Proposition 2.2.

PROPOSITION 2.3. *For any one-dimensional simplical complex $\mathscr{K}$, $\rho(\mathscr{K}; x)$ has only nonpositive roots.*

Later, we derive Proposition 2.3 as a corollary of Theorem 4.5.

The fact that $\tilde{\mu}(G; x)$ has only nonpositive roots for any graph G gives another class of examples of set systems $\mathscr{F}$ for which $\rho(\mathscr{F}; x)$ has only nonpositive roots, as follows. Let $G=(V, A)$ be a directed graph with n vertices, and let $g_k(G)$ be the number of arc-induced subgraphs B of G which have exactly k arcs and are such that $\text{indegree}_B(v) \leqslant 1$ and $\text{outdegree}_B(v) \leqslant 1$ for each $v \in V$. Let

$$\gamma(G; x)=\sum_k g_k(G)\, x^{n-k}.$$

LEMMA 2.4. *For any directed graph $G=(V, A)$, $\gamma(G; x)$ has only nonpositive roots.*

Proof. Define a graph $H=(W, E)$ by letting $W=V\times\{0,1\}$ and $\{(v,0),(w,1)\}\in E$ if and only if $(v,w)\in A$. Then $g_k(G)=m_k(H)$, and so $x^n\gamma(G; x)=\tilde{\mu}(H; x)=\rho(\mathscr{K}(H); x)$. The result now follows from Proposition 2.3. ∎

The construction in this proof appears in [24, (4.31)], but is such a folklore result that its origin is difficult to trace. The proof of Lemma 2.4, and consequently that of Theorem 2.5, is due to Chris Godsil (private communication).

If $G=(V, A)$ is a directed graph we may define a set system $\mathscr{P}=\mathscr{P}(G)\subset 2^V$ as follows: $S\in\mathscr{P}$ if and only if $S=\{v_1, ..., v_l\}$, where $(v_i, v_{i+1})\in A$ for $1\leqslant i\leqslant l-1$. We call $\mathscr{P}(G)$ the *path system* of G. For a partially ordered set (*poset*) P, we denote the path system of its Hasse diagram by $\Sigma(P)$ and call it the *system of saturated chains* of P. The path system P itself (with $(u, v)\in A$ iff $u<v$ in P), denoted by $\Delta(P)$, is called the *order complex* of P. The order complex of any poset is a simplicial complex.

If the directed graph G has no circuits, then $\gamma(G; x)=\rho(\mathscr{P}(G); x)$. Thus we have established the following consequence of Lemma 2.4.

THEOREM 2.5 (Godsil). *For any directed graph G without circuits, $\rho(\mathscr{P}(G); x)$ has only nonpositive roots. In particular, if P is a poset then both $\rho(\Sigma(P); x)$ and $\rho(\Delta(P); x)$ have only nonpositive roots.*

When the directed graph G is "sparse" we have a detailed factorization of $\rho(\mathscr{P}(G); x)$, brought to my attention by Richard Stanley. Denote by P_n the graph which is a path on n vertices, and denote by C_n the graph which is a cycle on n vertices. Then

$$\rho(\mathscr{K}(P_n); x)=\sum_{k\geqslant n/2}\binom{k}{n-k}x^k$$

and

$$\rho(\mathscr{K}(C_n); x)=\sum_{k\geqslant n/2}\frac{n}{k}\binom{k}{n-k}x^k.$$

Now let $G=(V, A)$ be a directed graph with n vertices and without circuits and such that $\mathrm{indegree}_G(v)\leqslant 2$ and $\mathrm{outdegree}_G(v)\leqslant 2$ for every $v\in V$. Under this hypothesis the graph H associated with G in the proof of Lemma 2.4 is a disjoint union of paths and even cycles. Now $x^n\rho(\mathscr{P}(G); x)=\rho(\mathscr{K}(H); x)$, which by Proposition 2.1(d) factors as a product of polynomials of the forms $\rho(\mathscr{K}(P_m); x)$ and $\rho(\mathscr{K}(C_{2m}); x)$.

3. The Connection with Chromatic Polynomials

The chromatic polynomial of a graph $G=(V, E)$ may be defined to be

$$\chi(G; x)=\sum_k p_k(\mathscr{I})(x)_k,$$

where $\mathscr{I}=\mathscr{I}(G)\subset 2^V$ is the simplicial complex of independent sets of vertices of G and

$$(x)_k=x(x-1)\cdots(x-k+1).$$

We call $\mathscr{I}(G)$ the *independence complex* of G.

Chapters 8 to 14 of Biggs [3] contain an excellent account of the chromatic polynomial and some related polynomials. In addition, $\chi(G; x)$ can be interpreted as a rank-generating function of a broken-circuit complex [31] and as the characteristic polynomial (in the sense of Rota) of a lattice of contractions [27, Sect. 9].

For an arbitrary finite system $\mathscr{F}$ we define the *chromatic polynomial* of $\mathscr{F}$ to be

$$\chi(\mathscr{F}; x)=\sum_k p_k(\mathscr{F})(x)_k.$$

Accordingly, we henceforth write $\chi(\mathscr{I}(G); x)$ for the usual chromatic polynomial of the graph G. The following proposition, justifying the terminology, is immediate.

Proposition 3.1. *For any set system $\mathscr{F}$ and any integer $l\geqslant 1$ the number of functions $f\colon V(\mathscr{F})\to\{1, ..., l\}$ such that $f^{-1}(i)\in\mathscr{F}$ for each $1\leqslant i\leqslant l$ is exactly $\chi(\mathscr{F}; l)$.*

Note that the condition $\varnothing\in\mathscr{F}$ is essential for Proposition 3.1.

As the connection between $\chi(\mathscr{F}; x)$ and $\rho(\mathscr{F}; x)$ is rather obviously the $\mathbf{R}$-linear transformation $T\colon \mathbf{R}[x]\to\mathbf{R}[x]$ defined by $T(x)_k=x^k$ and linear extension, we turn now to an investigation of this operator. Lemma 3.2 is a mundane calculation.

LEMMA 3.2. *For any polynomial* $p \in \mathbf{R}[x]$ and $\xi \in \mathbf{R}$,

(a) $T(x-\xi)\,p = [x(1+D)-\xi]\,Tp$

(b) $Tp = p(x(1+D))\,1$.

Proof. (a) Let $p = \sum_k \alpha_k (x)_k$. Then $(x-\xi)\,p = \sum_k \alpha_k [(x)_{k+1} + (k-\xi)(x)_k]$, so that $T(x-\xi)\,p = [x + xD - \xi]\,Tp$, as was to be shown. Now (b) follows from (a) by induction on $\deg p$. ▮

We would like to know the effect of T on the location of the roots of polynomials. The next result gives a condition on the roots of a polynomial p sufficient to imply that Tp has only nonpositive roots. We also see that the chromatic polynomial of (the independence complex of) a supersolvable graph satisfies these conditions.

LEMMA 3.3. (a) *If* $f \in \mathbf{R}[x]$ *has only nonpositive roots and* $\xi \in \mathbf{R}$ *then* $g = [x(1+D)-\xi]\,f$ *has only real roots.*

(b) *Let* m *denote the multiplicity of* 0 *as a root of* f. *Then* g *has only nonpositive roots if and only if* $\xi \leqslant m$.

(c) *Furthermore, the multiplicity of* 0 *as a root of* g *is* m *if* $\xi \neq m$, *and is at least* $m+1$ *if* $\xi = m$.

Proof. (a) Since f has only nonpositive roots, Df interlaces f, by Lemma 1.5. Hence $(1+D)\,f$ has only nonpositive roots, f alternates with $(1+D)\,f$, and $\Lambda((1+D)\,f) \leqslant \Lambda(f)$, by Lemma 1.3. Therefore, f interlaces $x(1+D)\,f$, so that g has only real roots, by Lemma 1.3 again.

(b) and (c). Let $f = \sum_k \alpha_k x^k$ and $g = \sum_k \beta_k x^k$. Now g has only nonpositive roots if and only if $\beta_k \geqslant 0$ for all k, and one can check that $\beta_k = \alpha_{k-1} + (k-\xi)\,\alpha_k$. Since m is the multiplicity of 0 as a root of f, $\alpha_k = 0$ for $k < m$ and $\alpha_m \neq 0$, and since f has only nonpositive roots, all $\alpha_k \geqslant 0$. Now $\beta_k = 0$ for $k < m$, and $\beta_m \geqslant 0$ if and only if $\xi \leqslant m$, with $\beta_m = 0$ if and only if $\xi = m$. This proves (c). Also, for $k > m$ and $\xi \leqslant m$ we have $\beta_k \geqslant 0$, which proves (b). ▮

PROPOSITION 3.4. *Suppose that* $p \in \mathbf{R}[x]$ *has only real roots and is such that whenever* $p(\xi) = 0$ *and* $\xi \in (a, a+1]$ *for some integer* $a \geqslant 0$, *then* $p(a) = 0$ *also. It follows that* Tp *has only nonpositive roots.*

Proof. We proceed by induction on $\deg p$. If $\deg p = 1$ then $Tp = p$ and the result is clear since the hypothesis ensures that the root of p must be nonpositive. Further assume as part of the inductive hypothesis that the multiplicity of 0 as a root of Tp is at least $\max\{0, 1+b\}$, where b is the largest integer root of p. For the induction step, let ξ be the largest root of

p, and let $\hat{p} = p/(x-\xi)$. Thus $Tp = [x(1+D)-\xi]\, T\hat{p}$ by Lemma 3.2(a). Now $\hat{p}$ satisfies the hypothesis of the proposition, so by induction $T\hat{p}$ has only nonpositive roots and the multiplicity of 0 has a root of $T\hat{p}$ is at least $\max\{0, 1+\lceil \xi - 1 \rceil\}$, where $\lceil \eta \rceil$ denotes the least integer not less than η. This multiplicity is at least ξ, so by parts (a) and (b) of Lemma 3.3, Tp has only nonpositive roots. Finally, by Lemma 3.3(c), the multiplicity of 0 as a root of Tp is at least $\max\{0, 1+\lceil \xi - 1 \rceil\}$ if ξ is not an integer and is at least $\max\{0, 1+\xi\}$ if ξ is an integer. This completes the induction step and the proof. ∎

A graph $G = (V, E)$ is *supersolvable* if its vertex-set V may be ordered $v_1, v_2, ..., v_n$ so that for $1 \leqslant i \leqslant n$ the neighbors of v_i among $v_1, ..., v_{i-1}$ induce a complete subgraph of G. This concept is a special case of supersolvability of lattices, as developed by Stanley in [28].

Suppose that G is a supersolvable graph and that V has been ordered $v_1, ..., v_n$ as in the definition. Let the number of neighbors of v_i among $v_1, ..., v_{i-1}$ be denoted by c_i for $1 \leqslant i \leqslant n$. It is not hard to see that the chromatic polynomial of (the independence complex of) G is

$$\chi(\mathscr{I}(G); x) = \prod_{i=1}^{n} (x - c_i).$$

We are now ready to prove the main result of this section.

THEOREM 3.5. *For any supersolvable graph G, $\rho(\mathscr{I}(G); x)$ has only nonpositive roots.*

Proof. Since $\rho(\mathscr{I}(G); x) = T\chi(\mathscr{I}(G); x)$ we need only check that $\chi(\mathscr{I}(G); x)$ satisfies the hypothesis of Proposition 3.4. All the roots of $\chi(\mathscr{I}(G); x)$ are nonnegative integers, and if l denotes the chromatic number of G then $(x)_l$ divides $\chi(\mathscr{I}(G); x)$ and its largest root is $l-1$. Thus the hypothesis is satisfied, and the proof follows. ∎

We conclude this section by noting that the definition of $\chi(\mathscr{F}; x)$ may be inverted to yield

$$p_k(\mathscr{F}) = \frac{1}{k!} \sum_{l \geqslant 0} \binom{k}{l} (-1)^{k-l} \chi(\mathscr{F}; l).$$

As an application of these ideas, let V be a set of n elements and let π be any partition of V, and denote by $S(\pi, k)$ the number of partitions π' of V into exactly k parts, such that $\pi \wedge \pi' = \hat{0}$ in the lattice of partitions of

V (cf. [1, p. 13]). We recover the Stirling numbers of the second kind $S(n, k)$ when $\pi = \hat{0}$. We leave as an exercise the proof that

$$S(\pi, k) = \frac{1}{k!} \sum_{l \geqslant 0} \binom{k}{l} (-1)^{k-l} \prod_{B \in \pi} (l)_{\# B}$$

and that the polynomial $\sum_k S(\pi, k)\, x^k$ has only nonpositive roots. The second assertion may be proved in two ways, by appeal to either Theorem 2.5 or Theorem 3.5. In the case $\pi = \hat{0}$ the formula for $S(n, k)$ is classical (see [4, p. 204] or [26, p. 43]), while the statement about the roots of $\sum_k S(n, k)\, x^k$ appears in [18]. Unimodality of $S(n, k)$ (for fixed n) appears in [1, p. 91].

4. Composition of Set Systems

In this section we examine composition of set systems, which in the language of species [21] is a natural transformation from the composition of species $\mathscr{S}\mathscr{S}[\mathscr{S}\mathscr{S}]$ to $\mathscr{S}\mathscr{S}$, where $\mathscr{S}\mathscr{S}$ denotes the species of set systems. Its interest for our purposes is that the partition polynomial of the "composite" depends on the structure of its "arguments" only through their partition polynomials. Of course, the structure of the set system into which the arguments are composed determines how these polynomials are to be "mixed."

Let $\mathscr{F}$ be a set system and let $\mathbf{G} = \{\mathscr{G}_v : v \in V(\mathscr{F})\}$ be a collection of pairwise vertex-disjoint set systems indexed by $V(\mathscr{F})$. The *composition of* $\mathbf{G}$ *into* $\mathscr{F}$, denoted by $\mathscr{F}[\mathbf{G}]$, is defined as follows. Let $U(\mathbf{G}) = \bigcup \{V(\mathscr{G}_v) : v \in V(\mathscr{F})\}$. Then $S \subset U(\mathbf{G})$ is an element of $\mathscr{F}[\mathbf{G}]$ if and only if both

(i) $S \cap V(\mathscr{G}_v) \in \mathscr{G}_v$ for each $v \in V(\mathscr{F})$, and

(ii) $\{v \in V(\mathscr{F}) : S \cap V(\mathscr{G}_v) \neq \varnothing\} \in \mathscr{F}$.

The vertex-set of $\mathscr{F}[\mathbf{G}]$ is of course defined by $V(\mathscr{F}[\mathbf{G}]) = \bigcup \mathscr{F}[\mathbf{G}]$.

Since $\varnothing \in \mathscr{F}[\mathbf{G}]$ this clearly defines a set system, but in general there are some pathologies for which $V(\mathscr{F}[\mathbf{G}]) \neq U(\mathbf{G})$. Consider the set system $\mathscr{F} = \{\varnothing, \{1\}, \{1, 2\}\}$: if $\mathscr{G}_1 = \{\varnothing\}$ and $\mathscr{G}_2 \neq \{\varnothing\}$ then $U(\mathbf{G}) = V(\mathscr{G}_2) \neq \varnothing$, while $\mathscr{F}[\mathbf{G}] = \{\varnothing\}$. This deficiency does not occur if either $\{\varnothing\} \notin \mathbf{G}$ or (as in all our examples) $\{\{v\} : v \in V(\mathscr{F})\} \subset \mathscr{F}$.

One can check that if $\mathscr{F}$ is a simplicial complex and $\mathbf{G}$ is a set of pairwise vertex-disjoint simplicial complexes indexed by $V(\mathscr{F})$ then $\mathscr{F}[\mathbf{G}]$ is also a simplicial complex. Furthermore, if $\mathscr{F}$ and all $\mathscr{G}_v$ are path systems (or order complexes, or independence complexes, respectively) then $\mathscr{F}[\mathbf{G}]$ is a path system (or order complex, or independence complex). However, the property of being a system of saturated chains is not preserved by composition; the composite will merely be a path system.

For any set S in a composite set system $\mathscr{F}[\mathbf{G}]$ we define the *support* of S to be $\operatorname{supp} S = \{v \in V(\mathscr{F}) : S \cap V(\mathscr{G}_v) \neq \varnothing\}$. For a partition π of $\mathscr{F}[\mathbf{G}]$ we define the *signature* of π to be the function $f_\pi : \mathscr{F} \to \mathbf{N}$ given by $f_\pi(S) = \#\{B \in \pi : \operatorname{supp} B = S\}$. (We use $\mathbf{N}$ to denote the set of natural numbers.)

We now exhibit a transformation $\Phi_{\mathscr{F}} : \mathbf{R}[x]^{V(\mathscr{F})} \to \mathbf{R}[x]$ which takes the collection of partition polynomials of the arguments $\mathbf{G}$ to the partition polynomial of the composite $\mathscr{F}[\mathbf{G}]$. Let $E_0 : \mathbf{R}[x] \to \mathbf{R}$ denote the *evaluation at* 0 operator $E_0 p = p(0)$.

For $\mathbf{p} = \{p_v : v \in V(\mathscr{F})\} \in \mathbf{R}[x]^{V(\mathscr{F})}$ define

$$\Phi_{\mathscr{F}}(\mathbf{p}) = \sum_f \tau(f) \prod_{v \in V(\mathscr{F})} x^{\bar{f}(v)} E_0 D^{\bar{f}(v)} p_v,$$

where the summation is over all functions $f : \mathscr{F} \to \mathbf{N}$ such that $f(\varnothing) = 0$, and

$$\tau(f) = \left(\prod_{S \in \mathscr{F}} x^{f(S)(\# S - 1)} f(S)! \right)^{-1}$$

and

$$\bar{f}(v) = \sum_{S \in \operatorname{st}(v)} f(S).$$

THEOREM 4.1. *Let $\mathscr{F}$, $\mathbf{G}$, and $\Phi_{\mathscr{F}}$ be as above. Then*

$$\rho(\mathscr{F}[\mathbf{G}]; x) = \Phi_{\mathscr{F}}(\rho(\mathscr{G}_v; x) : v \in V(\mathscr{F})).$$

Proof. The coefficient of x^k on the left-hand side is by definition $p_k(\mathscr{F}[\mathbf{G}])$. A little reflection shows that on the right-hand side the coefficient of x^k is

$$\sum_f \frac{\prod_{v \in V(\mathscr{F})} \bar{f}(v)!\, p_{\bar{f}(v)}(\mathscr{G}_v)}{\prod_{S \in \mathscr{F}} f(S)!},$$

where the summation is over all $f : \mathscr{F} \to N$ such that $f(\varnothing) = 0$ and $\sum_{S \in \mathscr{F}} f(S) = k$. Thus to prove the theorem it suffices to show that for all $k \in \mathbf{N}$ these coefficients are equal.

For any $f : \mathscr{F} \to \mathbf{N}$ with $f(\varnothing) = 0$ let $\mathscr{T}_f$ denote the set of all partitions π of $\mathscr{F}[\mathbf{G}]$ with signature $f_\pi = f$, and for $k \in \mathbf{N}$ let $\mathscr{T}_k = \bigcup_f \mathscr{T}_f$, where the union is over all $f : \mathscr{F} \to \mathbf{N}$ with $f(\varnothing) = 0$ and $\sum_{S \in \mathscr{F}} f(S) = k$. Thus $p_k(\mathscr{F}[\mathbf{G}]) = \#\mathscr{T}_k$. For $f : \mathscr{F} \to \mathbf{N}$ with $f(\varnothing) = 0$ and $v \in V(\mathscr{F})$, let $\mathcal{O}_f^v$ denote the set of ordered partitions of $\mathscr{G}_v$ into $\bar{f}(v)$ blocks; hence $\#\mathcal{O}_f^v = \bar{f}(v)!\, p_{\bar{f}(v)}(\mathscr{G}_v)$.

Now fix an arbitrary total order $\prec$ on the members of $\mathscr{F}$. For each $f: \mathscr{F} \to \mathbf{N}$ with $f(\varnothing)=0$ we define a function $\Upsilon_f: \prod_{v \in V(\mathscr{F})} \mathcal{O}_f^v \to \mathscr{T}_f$ as follows. Given a collection of ordered partitions $\vec{\pi}_v \in \mathcal{O}_f^v$ for $v \in V(\mathscr{F})$ let $\sigma = \Upsilon_f(\vec{\pi}_v : v \in V(\mathscr{F}))$; then σ is constructed as follows. For each $v \in V(\mathscr{F})$, the star of v is totally ordered by $\prec$, say $\mathrm{st}(v) = \{S_1^v \prec \cdots \prec S_t^v\}$ (where t depends on v). Now assign the first $f(S_1^v)$ blocks of $\vec{\pi}_v$ to S_1^v, the next $f(S_2^v)$ blocks of $\vec{\pi}_v$ to S_2^v, and so on. Do this for each $v \in V(\mathscr{F})$. Thus, for each $S \in \mathscr{F}$ and each $v \in S$ there is an ordered $f(S)$-tuple of blocks of $\vec{\pi}_v$ assigned to S; let these be denoted by $(B_1^{S,v}, ..., B_{f(S)}^{S,v})$. Now form the unions $B_i^S = \bigcup_{v \in S} B_i^{S,v}$ for all $S \in \mathscr{F}$ and all $1 \leqslant i \leqslant f(S)$. These are all members of $\mathscr{F}[\mathbf{G}]$, and $\sigma = \{B_i^S : S \in \mathscr{F}$ and $1 \leqslant i \leqslant f(S)\}$ is the desired partition of $\mathscr{F}[\mathbf{G}]$. Note that σ has signature $f_\sigma = f$, as required.

The function Υ_f is in fact surjective. To see this, consider any $\sigma \in \mathscr{T}_f$ and let $\vec{\sigma}$ be any ordering of σ such that if block B_i precedes block B_j in $\vec{\sigma}$ then $\mathrm{supp}\, B_i \preccurlyeq \mathrm{supp}\, B_j$ in the fixed total order $\prec$ on $\mathscr{F}$. Now construct ordered partitions $\vec{\pi} \in \mathcal{O}_f^v$ by letting $\vec{\pi}_v = \{B \cap V(\mathscr{G}_v) : B \in \sigma\} \backslash \{\varnothing\}$ with the order on the blocks induced from that on $\vec{\sigma}$. It is easily seen that $\sigma = \Upsilon_f(\vec{\pi}_v : v \in V(\mathscr{F}))$. In fact. for any $\sigma \in \mathscr{T}_f$, $\# \Upsilon_f^{-1}(\sigma) = \prod_{S \in \mathscr{F}} f(S)!$ since there are exactly this many choices for the ordered partition $\vec{\sigma}$ obtained from σ above, and each of these ordered partitions will result in a different set of ordered partitions $\vec{\pi}_v$ in $\mathcal{O}_f^v$.

Consequently,

$$\# \mathscr{T}_f = \frac{\prod_{v \in V(\mathscr{F})} \bar{f}(v)!\, p_{\bar{f}(v)}(\mathscr{G}_v)}{\prod_{S \in \mathscr{F}} f(S)!}.$$

Since $p_k(\mathscr{F}[\mathbf{G}]) = \sum_f \# \mathscr{T}_f$, where the summation is over all $f: \mathscr{F} \to \mathbf{N}$ with $f(\varnothing)=0$ and $\sum_{S \in \mathscr{F}} f(S) = k$, the proof follows. ∎

For many set systems $\mathscr{F}$ some of the terms in the summation defining $\Phi_{\mathscr{F}}$ may be collected, yielding a simpler formula. Let $W(\mathscr{F}) = \{v \in V(\mathscr{F}) : \{v\} \in \mathscr{F}\}$ be the set of *proper vertices* of $\mathscr{F}$.

Proposition 4.2. *For any set system $\mathscr{F}$, the transformation $\Phi_{\mathscr{F}}$ is also expressed by the formula*

$$\Phi_{\mathscr{F}}(\mathbf{p}) = \sum_h \tau(h) \left(\prod_{v \notin W(\mathscr{F})} x^{\bar{h}(v)} E_0 D^{\bar{h}(v)} p_v \right) \left(\prod_{v \in W(\mathscr{F})} x^{\bar{h}(v)} D^{\bar{h}(v)} p_v \right)$$

in which the summation is over all functions $h: \mathscr{F} \to \mathbf{N}$ such that $h(\varnothing)=0$ and $h(\{v\})=0$ for all $v \in W(\mathscr{F})$, and $\tau(h)$ and $\bar{h}(v)$ have meanings as in the definition of $\Phi_{\mathscr{F}}$.

Proof. We calculate using the definition of $\Phi_{\mathscr{F}}$. Let $f\colon \mathscr{F} \to \mathbf{N}$ be as in the definition of $\Phi_{\mathscr{F}}$, let $h\colon \mathscr{F} \to \mathbf{N}$ be as in the statement of the proposition, and let $g\colon \mathscr{F} \to \mathbf{N}$ satisfy $g(S) = 0$ unless $\# S = 1$. The functions f, g, h will vary subject to these conditions in the summations below. Now any f can be written uniquely as $f = g + h$ for functions of these forms. Hence

$$\begin{aligned}\Phi_{\mathscr{F}}(\mathbf{p}) &= \sum_f \tau(f) \prod_{v \in V(\mathscr{F})} x^{\bar{f}(v)} E_0 D^{\bar{f}(v)} p_v \\ &= \sum_h \tau(h) \sum_g \frac{\tau(g+h)}{\tau(h)} \left(\prod_{v \notin W(\mathscr{F})} x^{\bar{h}(v)} E_0 D^{\bar{h}(v)} p_v \right) \\ &\quad \times \left(\prod_{v \in W(\mathscr{F})} x^{\bar{g}(v) + \bar{h}(v)} E_0 D^{\bar{g}(v) + \bar{h}(v)} p_v \right)\end{aligned}$$

because $\bar{g}(v) = 0$ when $v \notin W(\mathscr{F})$. Now for any g and h,

$$\frac{\tau(g+h)}{\tau(h)} = \frac{\prod_{S \in \mathscr{F}} x^{h(S)(\# S - 1)} h(S)!}{\prod_{S \in \mathscr{F}} x^{(g(S) + h(S))(\# S - 1)} (g(S) + h(S))!} = \prod_{v \in W(\mathscr{F})} \frac{1}{\bar{g}(v)!}$$

by the conditions on g and h. Thus we may continue:

$$\begin{aligned}\Phi_{\mathscr{F}}(\mathbf{p}) &= \sum_h \tau(h) \left(\prod_{v \notin W(\mathscr{F})} x^{\bar{h}(v)} E_0 D^{\bar{h}(v)} p_v \right) \\ &\quad \times \left(\sum_g \prod_{v \in W(\mathscr{F})} x^{\bar{h}(v)} \frac{x^{\bar{g}(v)}}{\bar{g}(v)!} E_0 D^{\bar{g}(v)} D^{\bar{h}(v)} p_v \right) \\ &= \sum_h \tau(h) \left(\prod_{v \notin W(\mathscr{F})} x^{\bar{h}(v)} E_0 D^{\bar{h}(v)} p_v \right) \left(\prod_{v \in W(\mathscr{F})} x^{\bar{h}(v)} \sum_{j_v \geq 0} \frac{x^{j_v}}{j_v!} E_0 D^{j_v} D^{\bar{h}(v)} p_v \right).\end{aligned}$$

But for polynomial $p \in \mathbf{R}[x]$,

$$\sum_{j \geq 0} \frac{x^j}{j!} E_0 D^j p = p,$$

so that

$$\Phi_{\mathscr{F}}(\mathbf{p}) = \sum_h \tau(h) \left(\prod_{v \notin W(\mathscr{F})} x^{\bar{h}(v)} E_0 D^{\bar{h}(v)} p_v \right) \left(\prod_{v \in W(\mathscr{F})} x^{\bar{h}(v)} D^{\bar{h}(v)} p_v \right).$$

This completes the proof. ∎

As an example of the formula of Proposition 4.2, consider composition into the set system $\mathscr{B} = 2^{\{1,2\}}$. (For simplicial complexes $\mathscr{K}_1$ and $\mathscr{K}_2$ the composition $\mathscr{B}[\mathscr{K}_1, \mathscr{K}_2]$ is the familiar *join* of $\mathscr{K}_1$ and $\mathscr{K}_2$.) The functions

$h: \mathscr{B} \to \mathbf{N}$ in the statement of Proposition 4.2 are nonzero only on the element $\{1, 2\}$ of $\mathscr{B}$. Thus we have

$$\Phi_{\mathscr{B}}(p_1, p_2) = \sum_{k \geqslant 0} \frac{x^k}{k!} (D^k p_1)(D^k p_2).$$

This is much simpler than the formula defining $\Phi_{\mathscr{B}}(p_1, p_2)$.

We now turn to consideration of the question of when $\Phi_{\mathscr{F}}$ preserves nonpositivity of roots. As mentioned in the Introduction, we let $\mathscr{R}$ denote the class of set systems for which the partition polynomial has only nonpositive roots. Also, a set system $\mathscr{F}$ is a member of the class $\mathscr{R}^*$ if and only if the following conditions hold:

(o*) $W(\mathscr{F}) = V(\mathscr{F})$, or equivalently $\{\{v\} : v \in V(\mathscr{F})\} \subset \mathscr{F}$.

(i*) For any $\mathbf{p} \in \mathbf{R}[x]^{V(\mathscr{F})}$ such that p_v has only nonpositive roots for each $v \in V(\mathscr{F})$, the polynomial $\Phi_{\mathscr{F}}(\mathbf{p})$ has only nonpositive roots.

(ii*) If $\mathbf{p}$ and $\mathbf{q}$ are both as in condition (i*) and there is some $w \in V(\mathscr{F})$ such that q_w interlaces p_w and $q_v = p_v$ for $v \neq w$, then $\Phi_{\mathscr{F}}(\mathbf{q})$ interlaces $\Phi_{\mathscr{F}}(\mathbf{p})$.

(Clause (o*) is used to avoid the pathological deficiency $V(\mathscr{F}[\mathbf{G}]) \neq U(\mathbf{G})$ noted above.)

For a simple example of a set system in $\mathscr{R}^*$, let V be a finite set and let $\mathscr{E}_V = \{\varnothing\} \cup \{\{v\} : v \in V\}$. From Proposition 4.2 it follows that $\Phi_{\mathscr{E}_V}(p_v : v \in V) = \prod_{v \in V} p_v$, and so $\mathscr{E}_v$ is a member of $\mathscr{R}^*$.

PROPOSITION 4.3. (a) *$\mathscr{R}^*$ is a subclass of $\mathscr{R}$.*

(b) *$\mathscr{R}$ is closed under $\mathscr{F}[\cdot]$ for all $\mathscr{F} \in \mathscr{R}^*$.*

(c) *$\mathscr{R}^*$ is closed under $\mathscr{F}[\cdot]$ for all $\mathscr{F} \in \mathscr{R}^*$.*

Proof. For part (a) note that for any set system $\mathscr{F}$, $\rho(\mathscr{F}; x) = \Phi_{\mathscr{F}}(\mathbf{x})$, where $\mathbf{x}$ is that element of $\mathbf{R}[x]^{V(\mathscr{F})}$ for which $x_v = x$ for all $v \in V(\mathscr{F})$, and so (a) follows from property (i*).

Part (b) follows easily from Theorem 4.1 and the definitions of $\mathscr{R}$ and $\mathscr{R}^*$.

Part (c): Let $\mathscr{F} \in \mathscr{R}^*$, $\mathbf{G} \subset \mathscr{R}^*$, and $\mathscr{H} = \mathscr{F}[\mathbf{G}]$. By (o*) for $\mathscr{F}$ and each $\mathscr{G}_v$, $\mathscr{H}$ also satisfies (o*). Now

$$\Phi_{\mathscr{H}}(p_v : v \in V(\mathscr{H})) = \Phi_{\mathscr{F}}(\Phi_{\mathscr{G}_v}(p_u : u \in V(\mathscr{G}_v)) : v \in V(\mathscr{F}))$$

and clauses (i*) and (ii*) of the definition of $\mathscr{R}^*$ are easily verified for $\mathscr{H}$. ∎

We are almost ready to prove our main theorem, which gives a class of examples of set systems in $\mathscr{R}^*$. Lemma 4.4 gives two properties of $\Phi_{\mathscr{F}}$

which are be needed in the proof. Two definitions are necessary for its statement. Given a set system $\mathscr{F}$ and a vertex $w \in V(\mathscr{F})$, the *link* of w is

$$\mathrm{lk}(w) = \{S \backslash \{w\} : S \in \mathrm{st}(w)\}.$$

Given $\mathbf{q} \in \mathbf{R}[x]^{V(\mathscr{F})}$ and $L \subset V(\mathscr{F})$ define $D_L \mathbf{q}$ by

$$D_L q_v = \begin{cases} q_v & \text{if } v \notin L, \\ Dq_v & \text{if } v \in L. \end{cases}$$

LEMMA 4.4. *Let $\mathscr{F}$ be a set system, and let $\Phi_{\mathscr{F}}$ be the associated transformation.*

(a) *$\Phi_{\mathscr{F}}$ is $\mathbf{R}$-multilinear.*

(b) *If $\mathbf{p}, \mathbf{q} \in \mathbf{R}[x]^{V(\mathscr{F})}$ are such that for some $w \in V(\mathscr{F})$,*

$$p_v = \begin{cases} q_v & \text{if } v \neq w, \\ xq_w & \text{if } v = w, \end{cases}$$

then

$$\Phi_{\mathscr{F}}(\mathbf{p}) = x \cdot \sum_{L \in \mathrm{lk}(w)} \Phi_{\mathscr{F}}(D_L \mathbf{q}).$$

Proof. Part (a) is obvious from the definition of $\Phi_{\mathscr{F}}$. For part (b) we calculate as follows:

$$\begin{aligned} \Phi_{\mathscr{F}}(\mathbf{p}) &= \sum_f \tau(f) \left(\prod_{v \neq w} x^{f(v)} E_0 D^{f(v)} q_v \right) (x^{f(w)} E_0 D^{f(w)} x q_w) \\ &= \sum_f \tau(f) \left(\prod_{v \neq w} x^{f(v)} E_0 D^{f(v)} q_v \right) \\ &\quad \times x^{f(w)} (E_0 x D^{f(w)} q_w + E_0 f(w)\, D^{f(w)-1} q_w) \\ &= \sum_f \tau(f)\, f(w) \left(\prod_{v \neq w} x^{f(v)} E_0 D^{f(v)} q_v \right) x^{f(w)} E_0 D^{f(w)-1} q_w \\ &= \sum_{S \in \mathrm{st}(w)} \sum_f \tau(f)\, f(S) \left(\prod_{v \neq w} x^{f(v)} E_0 D^{f(v)} q_v \right) x^{f(w)} E_0 D^{f(w)-1} q_w. \end{aligned}$$

In this formula we may neglect the term corresponding to (S, f) whenever $f(S) = 0$. For each term corresponding to (S, f) with $f(S) > 0$ define $g: \mathscr{F} \to \mathbf{N}$ by $g(S') = f(S')$ if $S' \neq S$, and $g(S) = f(S) - 1$. Then $\tau(g) = \tau(f)\, f(S)\, x^{\# S - 1}$, and for fixed $S \in \mathrm{st}(w)$, as f ranges over all functions with $f(S)>0$ the corresponding g ranges over all functions $g\colon \mathscr{F}\to \mathbf{N}$ with $g(\varnothing)=0$. Thus we may continue the calculation

$$\begin{aligned}
\Phi_{\mathscr{F}}(\mathbf{p}) &= \sum_{S\in \mathrm{st}(w)} \sum_{g} \tau(g)\, x^{1-\#S} \left(\prod_{v\notin S} x^{\bar{g}(v)} E_0 D^{\bar{g}(v)} q_v\right) \\
&\quad \times \left(\prod_{v\in S\setminus\{w\}} x^{\bar{g}(v)+1} E_0 D^{\bar{g}(v)+1} q_v\right) (x^{\bar{g}(w)+1} E_0 D^{\bar{g}(w)} q_w) \\
&= x\cdot \sum_{L\in \mathrm{lk}(w)} \sum_{g} \tau(g) \prod_{v\in V(\mathscr{F})} x^{\bar{g}(v)} E_0 D^{\bar{g}(v)} D_L q_v \\
&= x\cdot \sum_{L\in \mathrm{lk}(w)} \Phi_{\mathscr{F}}(D_L \mathbf{q}).
\end{aligned}$$

This completes the proof. ∎

THEOREM 4.5. *Every one-dimensional simplicial complex $\mathscr{K}$ is a member of $\mathscr{R}^*$.*

Proof. Clause (o*) clearly holds for $\mathscr{K}$, so we need only check (i*) and (ii*). By Lemma 4.4(a) it suffices to consider only monic polynomials. We proceed to verify both clauses (i*) and (ii*) simultaneously by induction on $\deg \mathbf{p} = \sum_{v\in V(\mathscr{K})} \deg p_v$. Of course, we are assuming that p_v has only nonpositive roots, for all $v\in V(\mathscr{K})$.

If $\deg \mathbf{p}=1$ then by the definition of $\Phi_{\mathscr{K}}$, or from Proposition 4.2, $\Phi_{\mathscr{K}}(\mathbf{p}) = x-\xi$ for some nonpositive ξ, and the result is easily checked. Now assume that if $\deg \mathbf{p}<n$ then both clauses (i*) and (ii*) hold, and let $\deg \mathbf{p}=n$.

We begin by verifying clause (i*) when $\deg \mathbf{p}=n$. Let w be any vertex of $V(\mathscr{K})$ with $\deg p_w>0$, let ξ be any root of p_w, and define $\mathbf{q}$ by the equation

$$q_v = \begin{cases} p_v & \text{if } v\neq w, \\ p_w/(x-\xi) & \text{if } v=w. \end{cases}$$

Now from Lemma 4.4, and since $D_\varnothing \mathbf{q}=\mathbf{q}$, we have

$$\Phi_{\mathscr{K}}(\mathbf{p}) = (x-\xi)\,\Phi_{\mathscr{K}}(\mathbf{q}) + x\cdot \sum_{\varnothing\neq L\in \mathrm{lk}(w)} \Phi_{\mathscr{K}}(D_L\mathbf{q}).$$

Since $\mathscr{K}$ is one-dimensional, each $\varnothing\neq L\in \mathrm{lk}(w)$ is a singleton $L=\{u\}$ for some $u\in V(\mathscr{K})$, so that

$$D_L q_v = \begin{cases} q_v & \text{if } v\neq u, \\ Dq_u & \text{if } v=u. \end{cases}$$

Thus, for each $\varnothing = L \in \mathrm{lk}(w)$, $D_L\mathbf{q}$ and $\mathbf{q}$ are related as in clause (ii*) of the definition of $\mathscr{R}^*$, by Lemma 1.5.

Now $\deg \mathbf{q} = n-1$ and the induction hypothesis applies. Thus $\Phi_{\mathscr{K}}(\mathbf{q})$ and $\Phi_{\mathscr{K}}(D_L\mathbf{q})$ have only nonpositive roots (for all $\varnothing \neq L \in \mathrm{lk}(w)$) by clause (i*) of the induction hypothesis. From clause (ii*) we also know that for all $\varnothing \neq L \in \mathrm{lk}(w)$, $\Phi_{\mathscr{K}}(D_L\mathbf{q})$ interlaces $\Phi_{\mathscr{K}}(\mathbf{q})$. By Lemma 1.4 it follows that

$$\sum_{\varnothing \neq L \in \mathrm{lk}(w)} \Phi_{\mathscr{K}}(D_L\mathbf{q})$$

interlaces $\Phi_{\mathscr{K}}(\mathbf{q})$. By Lemma 1.3 we now deduce that

$$Q = \Phi_{\mathscr{K}}(\mathbf{q}) + \sum_{\varnothing \neq L \in \mathrm{lk}(w)} \Phi_{\mathscr{K}}(D_L\mathbf{q})$$

alternates with $\Phi_{\mathscr{K}}(\mathbf{q})$, and $\Lambda(Q) \leqslant \Lambda(\Phi_{\mathscr{K}}(\mathbf{q}))$. Consequently, $\Phi_{\mathscr{K}}(\mathbf{q})$ interlaces xQ, and Lemma 1.3 now implies that $\Phi_{\mathscr{K}}(\mathbf{p}) = xQ - \xi\Phi_{\mathscr{K}}(\mathbf{q})$ has only real roots and $\Phi_{\mathscr{K}}(\mathbf{q})$ interlaces $\Phi_{\mathscr{K}}(\mathbf{p})$. In fact, since $\xi \leqslant 0$, Lemma 1.3 implies that $\Lambda(\Phi_{\mathscr{K}}(\mathbf{p})) \leqslant \Lambda(xQ)$, so that $\Phi_{\mathscr{K}}(\mathbf{p})$ actually has only nonpositive roots. Thus we have succeeded in verifying clause (i*) of the induction step.

To verify clause (ii*) of the induction step, let $\mathbf{p}$ and $\mathbf{q}$ be related as in that part of the definition of $\mathscr{R}^*$: $\mathbf{p}$ and $\mathbf{q}$ have only nonpositive roots and there is some $w \in V(\mathscr{K})$ such that q_w interlaces p_w and $q_v = p_v$ for $v \neq w$.

Suppose that $p_w = \prod_{i=1}^{d} (x - \xi_i)$. Since q_w interlaces p_w, we have $d > 0$. By Proposition 1.6 we have $q_w = \sum_{i=1}^{d} \alpha_i \hat{p}_{wi}$, where $\alpha_i \geqslant 0$ and $\hat{p}_{wi} = p_w/(x - \xi_i)$ for $1 \leqslant i \leqslant d$. By Lemma 4.4(a) it follows that

$$\Phi_{\mathscr{K}}(\mathbf{q}) = \sum_{i=1}^{d} \alpha_i \Phi_{\mathscr{K}}(\hat{\mathbf{p}}_i),$$

where

$$(\hat{\mathbf{p}}_i)_v = \begin{cases} p_v & \text{if } v \neq w, \\ \hat{p}_{wi} & \text{if } v = w. \end{cases}$$

Now for each $1 \leqslant i \leqslant d$, $\hat{\mathbf{p}}_i$ is related to $\mathbf{p}$ just as $\mathbf{q}$ was related to $\mathbf{p}$ in the first part of the induction step. It follows that $\Phi_{\mathscr{K}}(\hat{\mathbf{p}}_i)$ interlaces $\Phi_{\mathscr{K}}(\mathbf{p})$ for all $1 \leqslant i \leqslant d$. Lemma 1.4 now implies that $\Phi_{\mathscr{K}}(\mathbf{q})$ interlaces $\Phi_{\mathscr{K}}(\mathbf{p})$, which completes the proof of clause (ii*) of the induction step. The theorem is proved. ∎

The Heilmann–Lieb Theorem (Proposition 2.3) now follows immediately from Theorem 4.5 and Proposition 4.3(a).

The set system $\mathscr{B} = 2^{\{1,2\}}$ defined above is thus found to be a member of R^*. For disjoint sets V and W one has $2^{V \cup W} = \mathscr{B}[2^V, 2^W]$, and it follows by induction on $\# V$ using Proposition 4.3(c) that 2^V is a member of $\mathscr{R}^*$ for any finite set V. Using the facts that 2^V and $\mathscr{E}_V$ are in $\mathscr{R}^*$, as an exercise the reader may now strengthen the result that for any partition π of a set V, the polynomial $\sum_k S(\pi, k)\, x^k$ has only nonpositive roots.

5. Concluding Remarks

As noted in the Introduction, not all simplicial complexes have partition polynomials with only nonpositive roots. Denote by Δ_k^n the k-skeleton of the n-simplex, i.e., the set system consisting of all subsets of at most $k+1$ elements from an $(n+1)$-set. Applying Proposition 2.1(b) one obtains the recursion relation

$$\rho(\Delta_k^n; x) = x \cdot \sum_{j=0}^{k} \binom{n}{j} \rho(\Delta_l^{n-1-j}; x),$$

where $l = \min\{n-1-j, k\}$. This provides a rapid way to calculate these polynomials.

Using this method, a double-precision FORTRAN program was able to calculate $\rho(\Delta_k^n; x)$ for all (n, k) in the range $0 \leqslant k \leqslant n \leqslant 23$ before round-off error became significant. The polynomials were then factored using a standard subroutine from the FORTRAN NAG library. The results show that the majority of the Δ_k^n in the indicated range are in the class $\mathscr{R}$, the exceptional pairs (n, k) being $(8, 2)$ through $(16, 2)$ and $(21, 3)$ and $(22, 3)$, each with one complex conjugate pair, and $(17, 2)$ through $(23, 2)$, each with two complex conjugate pairs. Mysteriously, the polynomial for $(23, 3)$ seems to have only nonpositive roots, but the result might not be accurate for such large n. Of course, Δ_0^n, Δ_1^n, and Δ_n^n are in $\mathscr{R}^*$ for all n, as seen in Section 4. It would be very interesting to know the rate of growth of the function $f(n) = \max\{k : \Delta_k^n \notin \mathscr{R}\}$.

There seems to be some evidence that the order complex of any poset is not only in $\mathscr{R}$ (Godsil's Theorem), but is also even in $\mathscr{R}^*$. Firstly, any composition of path systems into an order complex (more generally, into any path system) is a path system, hence is a member of $\mathscr{R}$. Secondly, any poset P of rank 2 determines a bipartite graph $G = (P, E)$ by putting $\{u, v\} \in E$ if and only if $u \neq v$ and u and v are comparable in P. Since $\Delta(P) = \mathscr{K}(G)$ it follows from Theorem 4.5 that $\Delta(P)$ is in $\mathscr{R}^*$. Consequently any poset which can be obtained as an iterated composition of rank 2 posets is a member of $\mathscr{R}^*$. (Note: Composition of posets is *not* the composition in Section 4, rather it is a natural transformation from $\mathscr{PO}[\mathscr{PO}]$ to $\mathscr{PO}$, where $\mathscr{PO}$ is the species of finite posets. The construction of the order complex of a poset gives a natural transformation from $\mathscr{PO}$ to $\mathscr{SS}$ which commutes with the compositions on the two different species.) Proving that any order complex is in $\mathscr{R}^*$ would substantially enrich the class of set systems known to be in $\mathscr{R}$.

The matching polynomial has a more general formulation in which the edges of the graph are given positive weights and a matching is given weight equal to the product of the weights of the edges it contains. This weighted matching polynomial still has only real roots, for any weighted graph. I suspect that a similar phenomenon holds for partition polynomials (when the members of a set system are given positive weights), but perhaps not for arbitrary weight functions. Indeed, one can see that in the construction in Lemma 2.4, a weight function on H will determine a weight function on $\mathscr{P}(G)$, but not all weight functions on $\mathscr{P}(G)$ arise in this way. It is more difficult to see how to extend Theorem 3.5 to the weighted case. Theorem 4.5, on the other hand, holds true for arbitrarily weighted $\mathscr{K}$. (One needs to define the effect of composition on the weight functions in an appropriate way, which then generalizes the definition of $\Phi_{\mathscr{F}}$, and then the proof is a simple generalization of the one given.)

Throughout this study, only some elementary results concerning the location of the roots of polynomials have been used. Indeed, in contrast with the generality of the definition of the partition polynomial, our results seem constrained by the method of interlacing roots. Perhaps more sophisticated techniques from analysis can establish more profound results in this area.

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