Fibrewise suspension and Lusternik–Schnirelmann category

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Abstract

Since Iwase disproved the Ganea conjecture the question became to find a characterization of the spaces \( X \) which satisfy the Ganea conjecture, i.e. for which the equality \( \text{cat}(X \times S^k) = \text{cat} X + 1 \) holds for any \( k \geq 1 \). Recently Scheerer et al. (H. Scheerer, D. Stanley, D. Tanré, Fibrewise localization applied to Lusternik–Schnirelmann category, Israel J. Math. (2002) to appear.) have introduced an approximation of the category, denoted by \( Q \text{cat} \), and have conjectured that, for a CW-complex \( X \) of finite dimension, we have \( \text{cat}(X \times S^k) = \text{cat} X + 1 \) for any \( k \geq 1 \) if and only if \( Q \text{cat} X = \text{cat} X \). In this paper, we establish the formula \( Q \text{cat}(X \times S^k) = Q \text{cat} X + 1 \) and deduce from this that if \( Q \text{cat} X = \text{cat} X \) then \( X \) satisfies the Ganea conjecture. In other words, a first direction of the conjecture of Scheerer et al. (H. Scheerer, D. Stanley, D. Tanré, Fibrewise localization applied to Lusternik–Schnirelmann category, Israel J. Math. (2002) to appear.) is proved. Using this new sufficient condition for a space to satisfy the Ganea conjecture, we prove that any \((r-1)\)-connected CW-complex \( X \) with \( r \text{cat}(X) \geq 3 \) and \( \text{dim}(X) \leq 2r \text{cat}(X) - 3 \) satisfies the Ganea conjecture. This shows for example that the Lie group \( Sp(3) \) satisfies the Ganea conjecture. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The Lusternik–Schnirelmann category of a topological space \( X \), denoted by \( \text{cat} X \), is the least integer \( n \) for which \( X \) can be covered by \( n + 1 \) open sets each of which is contractible in \( X \). In [4], Iwase constructed the first example of a space \( X \) such that \( \text{cat}(X \times S^k) = \text{cat} X \). This disproved the...
long-standing conjecture of Ganea on the equality \( \text{cat}(X \times S^k) = \text{cat}X + 1 \) which was supposed to hold for any \( X \) and \( k \geq 1 \). After Iwase’s work it would be interesting to have a complete characterization of the spaces \( X \) for which \( \text{cat}(X \times S^k) = \text{cat}X + 1 \). In [11], Scheerer et al. define an approximation of the \( L \)-\( S \) category, denoted by \( Q \text{cat} \) and ask if this invariant could detect if a finite CW-complex \( X \) satisfies the Ganea conjecture. In other words, is it true that the equality \( \text{cat}(X \times S^k) = \text{cat}X + 1 \) (for any \( k \geq 1 \)) holds for a finite CW-complex \( X \) if and only if \( Q \text{cat}X = \text{cat}X \)?

In this paper, one direction of this equivalence is proved. More precisely, we show that, if \( Q \text{cat}X = \text{cat}X \) for a finite CW-complex, then \( \text{cat}(X \times S^k) = \text{cat}X + 1 \) for any \( k \geq 1 \). This will follow from the fact that, unlike the category, the invariant \( Q \text{cat} \) satisfies the formula \( Q \text{cat}(X \times S^k) = Q \text{cat}X + 1 \) for any finite CW-complex \( X \) and for any \( k \geq 1 \).

From this sufficient condition for a space to satisfy the Ganea conjecture it is possible to exhibit new classes of spaces satisfying the Ganea conjecture. In particular, we prove that any \((r - 1)\)-connected CW-complex such that \( r \text{cat}(X) \geq 3 \) and \( \dim X \leq 2r \text{cat}(X) - 3 \) satisfies the Ganea conjecture. This improves the result of Rudyak asserting that any \((r - 1)\)-connected closed stably parallelizable manifold \( M \) such that \( 4 \leq \dim M \leq 2r \text{cat}(M) - 4 \) satisfies the Ganea conjecture and this also permits to see that for the symplectic group \( \text{Sp}(3) \) we have \( \text{cat}(\text{Sp}(3) \times S^k) = \text{cat} \text{Sp}(3) + 1 \) for any \( k \geq 1 \).

The category of a space can be characterized by means of the Ganea fibrations and the invariant \( Q \text{cat} \) is obtained by applying fibrewise a base point free of the functor \( \Omega^\infty \Sigma^\infty \) to these fibrations. The precise definition of \( Q \text{cat} \) is based on a fibrewise construction due to Dror Farjoun. The first section is devoted to recall these notions and also to prove an unicity result concerning Dror Farjoun’s construction. This unicity result together with an explicit fibrewise construction for a base-point-free version of the functor \( \Omega^i \Sigma^i \) \((i \geq 1)\) leads in the second section to a characterization of the invariant \( Q \text{cat} \). This characterization is expressed in terms of iterated fibrewise suspensions of Ganea fibrations and constitutes the main tool of our proof of the formula \( Q \text{cat}(X \times S^k) = Q \text{cat}(X) + 1 \) that we give in Section 3. The applications of this product formula to the Ganea conjecture are also given in Section 3.

**Notation.** In this text a **space** is a compactly generated space having the homotopy type of a CW-complex. The category of these spaces is denoted by \( \mathcal{T}op \). Products and functional spaces in \( \mathcal{T}op \) are topologized in such a way that the product of two proclusions is also a proclusion and that the evaluation map \( Y^X \times X \to Y \) is continuous. The symbols \( \leftrightarrow, \to, \) and \( \sim \) denote, respectively, the cofibrations, fibrations, and weak equivalences. The restriction of a map \( f : X \to Y \) to a subspace \( A \) is denoted by \( f_{|A} : A \to Y \).

For \( X \in \mathcal{T}op \), we consider also the category \( \mathcal{T}op_X \) whose objects (called fibrewise spaces) are the morphisms of \( \mathcal{T}op \) of the form \( E \to X \) and whose morphisms (called fibrewise maps) between two objects \( E \to X \) and \( E' \to X \) are the morphisms \( E \to E' \) of \( \mathcal{T}op \) such that \( p \circ f = p' \). If no confusion is possible, we use the letter \( E \) for the object \( E \to X \) of \( \mathcal{T}op_X \). In particular, the trivial object \( \text{pr}_X : X \times Y \to X \) will be simply denoted by \( X \times Y \).

Finally, we denote by \( I^k \) the \( k \)-fold product of the interval \([0,1]\) and by \( \partial I^k \) the boundary of this space. This spaces are well-pointed with respect to the point \( e^k := (1,1,\ldots,1) \). The sphere \( S^k \) is defined to be the quotient \( I^k/\partial I^k \) and we denote by \( s \) the class of \( e^k \) in \( I^k/\partial I^k \).
2. Definition of $Q_{\text{cat}}$

In [2], Ganea characterized the L.-S. category of a well-pointed space $X$ by means of a sequence of fibrations $F_n(X) \xrightarrow{i_n(X)} G_n(X) \xrightarrow{g_n(X)} X$ which is defined inductively as follows. Start with the path fibration on $X: F_0(X) = \Omega X, G_0(X) = PX$ and $g_0(X)$ is the evaluation map. In order to define the $(n + 1)$th fibration from the $n$th one, consider the map $G_n(X) \cup CF_n(X) \to X$ obtained from $g_n(X)$ by mapping the cone on the base point. The standard decomposition of this map into a homotopy equivalence and a fibration gives: $G_n(X) \cup CF_n(X) \xrightarrow{\sim} G_{n+1}(X) \xrightarrow{g_{n+1}(X)} X$. Denote by $F_{n+1}(X)$ the fibre of $g_{n+1}(X)$ over the base point of $X$.

![Diagram]

By Ganea [2], the fibre $F_{n+1}(X)$ is homotopy equivalent to the join $F_n(X) \ast \Omega X$ and we thus have $F_k(X) \simeq \ast_{k+1} \Omega X$ where $\ast_{k+1}$ denotes the $(k + 1)$-fold join.

The sequence of fibrations $g_n(X): G_n(X) \to X$ gives a characterization of the L.S.-category:

**Theorem 1** (Ganea [2]). Let $X$ be a well-pointed pathwise connected space. Then $\text{cat}(X)$ is the least integer $n \in \mathbb{N}$ for which the fibration $g_n(X)$ admits a section.

In order to define the invariant $Q_{\text{cat}}$, in [11] Scheerer et al. apply to the Ganea fibrations a fibrewise version due to Dror Farjoun [1] of the functor $\Omega^\infty \Sigma^\infty$. This construction of Dror Farjoun is valid for any regular coaugmented functor $\lambda: \mathcal{F}op \to \mathcal{F}op$, that is a functor which sends contractible spaces to contractible spaces, preserves weak equivalences, and which is equipped with a natural transformation $\lambda: i_\ast \to \lambda$. Dror Farjoun shows that, for such a functor $\lambda$, there exists a functor $\tilde{\lambda}$ from the category of spaces over a space to itself such that for any map $p: E \to B$ we have a diagram

![Diagram]

in which:

(i) all the maps are natural;
(ii) the left square is commutative and the right square is homotopy commutative by a natural homotopy;
(iii) \( m_\lambda \circ j_\lambda = i_\lambda \);

(iv) the map induced by \( j_\lambda(E): E \to \tilde{\lambda}(E) \) between the homotopy fibres of \( p \) and \( p_\lambda \) over a point \( b \in B \) is naturally equivalent to the coaugmentation \( i_\lambda \).

We will say that such a functor \( \tilde{\lambda} \) is a \textit{functorial fibrewise extension} of the functor \( \lambda \). The construction given by Dror Farjoun that we recall at the end of this section is essentially unique:

**Proposition 2.** Let \( \lambda: \mathcal{F}op \to \mathcal{F}op \) be a regular coaugmented functor and let \( B \in \mathcal{F}op \). Dror Farjoun’s functorial fibrewise extension \( \tilde{\lambda}: \mathcal{F}op_B \to \mathcal{F}op_B \) is unique up to natural weak equivalence (over \( B \)).

We postpone also the proof of this proposition to the end of the section. Here, we point out that Dror Farjoun’s construction can only be applied to functors, the definition of which does not require base point. This led Scheerer et al. \([11]\) to define a \textit{base-point-free version} of the functor \( \Omega^k \Sigma^k \) that we recall now.

For \( Z \in \mathcal{F}op \) denote by \( \Sigma Z \) the unreduced suspension of \( Z \). That is \( \Sigma Z = Z \times I/\sim \) with \((z,0) \sim (z',0) \) and \((z,1) \sim (z',1) \) for \( z,z' \in Z \). For \( k \geq 2 \), set \( \Sigma^k Z := \Sigma(\Sigma^{k-1} Z) \) (where \( \Sigma^1 Z = \Sigma Z \)). The \( k \)-fold suspension can be described as a quotient of \( Z \times I^k \). As this will be useful in the next section, we precise here this quotient space structure. Set \( J^0 = \partial I \) and \( \rho^0 = \text{id} : \partial I \to J^0 \). For \( k \geq 2 \), write \( \partial I^k = I \times \partial I^{k-1} \cup \partial I \times I^{k-1} \) and consider the quotient space \( J^{k-1} = \partial I^k / \sim \) where the relation \( \sim \) is given by \((t,u) \sim (t',u)\) for any \( t,t' \in I \) and \( u \in \partial I^{k-1} \). Denote by \( \rho^{k-1} \) the identification map \( \partial I^k \to J^{k-1} \).

It is easy to see that the \( k \)-fold suspension of \( Z \) coincides with the pushout of the diagram

\[
J^{k-1} \buildrel {\rho^{k-1} \circ pr_{I^k}} \over \longrightarrow Z \times \partial I^k \cong Z \times I^k.
\]

Let us denote by \([z,t_1,\ldots,t_k]\) the class in \( \Sigma^k Z \) of \((z,t_1,\ldots,t_k) \in Z \times I^k \) and by \( j^k: \partial I^k \to \Sigma^k Z \) the map defined by \( j^k(t_1,\ldots,t_k) = [z,t_1,\ldots,t_k] \) where \( z \) is any arbitrary element of \( Z \). In \([11]\) the functor \( \hat{Q}^k \) is defined by \( \hat{Q}^k(Z) = \{ \omega: I^k \to \Sigma^k Z | \omega |_{\partial I^k} = j^k \} \) for \( k \geq 1 \) and by \( \hat{Q}^0 = \text{id} \) for \( k = 0 \). The functor \( \hat{Q}^k \) is a base-point-free version of \( \Omega^k \Sigma^k \). That means that we do not need the existence of a base point of \( Z \) to construct the space \( \hat{Q}^k(Z) \), but, nevertheless, if \( Z \) has a base point, we can consider the space \( \Omega^k \Sigma^k Z \) and we thus have a natural homotopy equivalence \( \hat{Q}^k(Z) \to \Omega^k \Sigma^k Z \). The functor \( \hat{Q}^k \) is naturally equipped with a coaugmentation \( i_{\hat{Q}^k}(Z) : Z \to \hat{Q}^k(Z) \) defined by \( i_{\hat{Q}^k}(z)(t_1,\ldots,t_k) = [z,t_1,\ldots,t_k] \) (naturally equivalent to the coaugmentation \( i_{\Omega^k \Sigma^k}(Z) : \Omega^k \Sigma^k(Z) \)) and is clearly a regular coaugmented functor. By applying Dror Farjoun’s construction we obtain thus for any \( n \geq 0 \) and any \( k \geq 0 \)

\[
\begin{align*}
F_n(X) \to \hat{Q}^k(F_n(X)) \quad \downarrow \quad G_n(X) \to \hat{Q}^k(G_n(X)) \\
g_n(X) \downarrow \quad (g_n(X))_{\hat{Q}^k} \downarrow \quad X \quad \downarrow \\
X
\end{align*}
\]

Following \([11]\), we consider the following approximations of the L.S.-category:
Definition 3. Let $X$ be a well-pointed space. For $k \in \mathbb{N}$, $Q^k \text{cat}(X)$ is the least integer $n$ such that the map $g_n(X)_{Q^k} : Q^k(G_n(X)) \to X$ has a homotopy section. If no such $n$ exists, one sets $Q^k \text{cat}(X) = \infty$.

For any $k \geq 0$ there exists a natural map $b_k : Q^k(Z) \to Q^{k+1}(Z)$ defined by $b_k(\omega)(t_1, \ldots, t_{k+1}) = [\omega(t_1, \ldots, t_k), t_{k+1}]$ which is compatible with the coaugmentations. Moreover, the map $b_k$ is naturally equivalent to the map $Q^k \Sigma^k Z \xrightarrow{\partial \Sigma^k Z} \Sigma^k Z \to \Sigma^k \Sigma^k Z$ where $\partial \Sigma^k Z : \Sigma^k Z \to \Omega \Sigma^k Z$ is the coaugmentation. In [11] Scheerer et al. set $Q := \lim_{k \to \infty} Q^k$ and define $Q \text{cat}(X)$ to be the least integer $n$ such that the map $g_n(X)_{Q^k} : Q(G_n(X)) \to X$ has a homotopy section. In this text we will work with an alternative definition of the limit invariant. More precisely, from the following commutative diagram induced by the map $b_k$:

\[
\begin{array}{ccc}
\overline{Q^k}(G_n(X)) & \xrightarrow{g_n(X)} & Q^{k+1}(G_n(X)) \\
\downarrow & & \downarrow \\
X & & X \\
\end{array}
\]

we deduce the inequalities

$$Q \text{cat}(X) \leq \cdots \leq Q^{k+1} \text{cat}(X) \leq Q^k \text{cat}(X) \leq \cdots \leq Q^1 \text{cat}(X) \leq Q^0 \text{cat}(X) = \text{cat}(X)$$

and we set

Definition 4. $Q^\infty \text{cat}(X) := \lim_{k \to \infty} Q^k \text{cat}(X)$.

The two limit invariants $Q \text{cat}$ and $Q^\infty \text{cat}$ agree for a large class of spaces.

Proposition 5. Let $X$ be a $(q-1)$-connected well-pointed space with $q \geq 1$. If $\text{dim} X < \infty$ then $Q \text{cat}(X) = Q^\infty \text{cat}(X)$.

Proof. From the inequalities above, we clearly have $Q \text{cat}(X) \leq Q^\infty \text{cat}(X)$. Conversely, suppose that the map $g_n(X)_{Q^k} : Q(G_n(X)) \to X$ admits a homotopy section $\sigma$. As $F_n(X) \simeq n+1 \Omega X$ is $(q(n+1)-2)$-connected, the space $\Sigma^k F_n(X)$ is $(q(n+1)-2+k)$-connected (hence at least 1-connected for $k \geq 2$) and the map $\Omega^k \Sigma^k F_n(X) \xrightarrow{\partial \Sigma^k F_n(X)} \Omega^k \Sigma^k F_n(X)$ is a $(2q(n+1)-3+k)$-equivalence for any $k \geq 2$. It follows that the map $\overline{Q^k}(F_n(X)) \to Q(F_n(X))$ is a $(2q(n+1)-3+k)$-equivalence for any $k \geq 2$ and so is the map $\overline{Q^k}(F_n(X)) \to \bar{Q}(F_n(X))$. As $\text{dim}(X) < \infty$, we can find an integer $k \geq 2$ such that $\text{dim} X < 2q(n+1) - 3+k$ and deduce from $\sigma$ a homotopy section $\sigma'$ of $g_n(X)_{Q^k} : Q(G_n(X)) \to X$. This implies $Q^\infty \text{cat}(X) \leq Q \text{cat}(X)$.

We come now to the construction of Dror Farjoun given in [1] and to the proof of Proposition 2.
2.1. Dror Farjou's construction of a functorial fibrewise extension

Let \( \lambda : \mathcal{F} \to \mathcal{F} \) be a regular coaugmented functor. The main part of the construction of the fibrewise extension \( \lambda \) is actually made in the category \( \mathcal{S} \) of simplicial sets. Using the adjoint functors \( \left| - \right| : \mathcal{S} \rightleftharpoons \mathcal{F} : \text{Sing} \), we associate with a functor \( \lambda : \mathcal{F} \to \mathcal{F} \) a functor \( \lambda_{\mathcal{S}} := \text{Sing} \circ \lambda \circ \left| - \right| : \mathcal{S} \to \mathcal{S} \). As \( \lambda \) is a regular coaugmented functor so is \( \lambda_{\mathcal{S}} \). We next explain the construction of a functorial fibrewise extension \( \lambda_{\mathcal{S}} \) for the regular coaugmented functor \( \lambda_{\mathcal{S}} : \mathcal{S} \to \mathcal{S} \).

Let \( B \in \mathcal{S} \) and \( p : E \to B \) a map in \( \mathcal{S} \) that we may suppose to be a fibration. Here, we always use the functorial decompositions of a map in a weak equivalence followed by a fibration or in a cofibration followed by a weak equivalence. We consider the simplex category \( \Delta_B \) of \( B \) in which the objects are the simplices \( \sigma : \Delta[n] \to B \) of \( B \) and the morphisms \( \tau : \Delta[m] \to B \) are the order preserving maps \( \tau : \sigma \in \Delta[n] \to \sigma \in \Delta[m] \) such that \( \tau \circ \sigma = \sigma \). Let \( B : \Delta_B \to \mathcal{S} \) the functor defined by \( B(\sigma : \Delta[n] \to \Delta[m]) = \sigma(\Delta[n]) \) where \( \sigma(\Delta[n]) \) denotes the pullback of \( p : E \to B \) by \( \sigma \). We have then for any \( \sigma : \Delta[n] \to B \) the following commutative diagram:

\[
\begin{array}{ccc}
\lambda_{\mathcal{S}}(\sigma(\Delta[n])) & \xrightarrow{\lambda_{\mathcal{S}}(\sigma)} & \sigma(\Delta[n]) \\
& \downarrow & \downarrow \sigma \\
\lambda_{\mathcal{S}}(\Delta[n]) & \xrightarrow{\lambda_{\mathcal{S}}} & \Delta[n] \\
& \downarrow & \\
& B \\
\end{array}
\]

and this diagram induces a commutative diagram between the homotopy colimits of the functors \( \sim E, \lambda_{\mathcal{S}} \circ E, \lambda_{\mathcal{S}} \circ B : \Delta_B \to \mathcal{S} \):

\[
\begin{array}{ccc}
hocolim \lambda_{\mathcal{S}} \circ E & \xrightarrow{\sim} & E \\
& \downarrow & \downarrow \sim \circ E \\
hocolim \lambda_{\mathcal{S}} \circ B & \xrightarrow{\sim} & B \\
& \downarrow & \\
& \sim \circ \lambda_{\mathcal{S}} \circ B \\
\end{array}
\]

As \( \lambda_{\mathcal{S}}(\Delta[n]) \) is contractible, the homotopy fibre of \( \text{hocolim} \lambda_{\mathcal{S}} \circ E \to \text{hocolim} \lambda_{\mathcal{S}} \circ B \) over a component is homotopy equivalent to \( \lambda_{\mathcal{S}}(F) \) where \( F \) is the fibre of \( p : E \to B \) over the corresponding component. In order to construct \( \lambda_{\mathcal{S}}(E) \), we form the pullback \( P \) of the associated fibration of the map \( \text{hocolim} \lambda_{\mathcal{S}} \circ E \to \text{hocolim} \lambda_{\mathcal{S}} \circ B \) by the map \( \text{hocolim} B \to \text{hocolim} \lambda_{\mathcal{S}} \circ B \) and we consider the induced map \( \text{hocolim} E \to P \). The space \( \lambda_{\mathcal{S}}(E) \) is obtained as the pushout of the associated cofibration of this map with the map \( \text{hocolim} E \to E \) and the map \( \lambda_{\mathcal{S}}(E) \to B \) follows from the universal property of pushouts.

\[
\begin{array}{ccc}
\text{hocolim} \lambda_{\mathcal{S}} \circ E & \xrightarrow{\sim} & \text{hocolim} E \\
& \downarrow & \downarrow \sim \circ E \\
& \bullet & \rightarrow \lambda_{\mathcal{S}}(E) \\
& \downarrow & \\
\text{hocolim} \lambda_{\mathcal{S}} \circ B & \xrightarrow{\sim} & \text{hocolim} B \\
\end{array}
\]
We finally obtain a diagram

\[
\begin{array}{ccc}
E & \xrightarrow{j_{\mathcal{F}}(E)} & \lambda_{\mathcal{F}}(E) \\
\downarrow p & & \downarrow \lambda_{\mathcal{F}}(p) \\
B & \xrightarrow{\iota_{\mathcal{F}}(B)} & \lambda_{\mathcal{F}}(B)
\end{array}
\]

satisfying all properties of a functorial fibrewise extension (in \(\mathcal{F}\)). Furthermore, notice that the map \(j_{\mathcal{F}}(E)\) is a cofibration.

We come back now to the topological situation. Let \(p : E \to B\) be a map that we may suppose to be a fibration (in \(\text{Top}\) also, we always use the functorial decompositions of a map in a weak equivalence followed by a fibration or in a cofibration followed by a weak equivalence). The realization of the construction above applied to the fibration \(\text{Sing}(E) \to \text{Sing}(B)\) gives us a natural diagram

\[
\begin{array}{ccc}
|\text{Sing}(E)| & \xrightarrow{|\lambda_{\mathcal{F}}(\text{Sing}(E))|} & |\lambda_{\mathcal{F}}(\text{Sing}(E))| \sim \lambda(E) \\
\downarrow & & \downarrow \\
|\text{Sing}(B)| & \xrightarrow{|\lambda_{\mathcal{F}}(\text{Sing}(B))|} & |\lambda_{\mathcal{F}}(\text{Sing}(B))| \sim \lambda(B)
\end{array}
\]

We form the pushout of the map \(|\text{Sing}(E)| \to |\lambda_{\mathcal{F}}(\text{Sing}(E))|\) (resp. \(|\text{Sing}(B)| \to |\text{Sing}(B)|\)) with the adjunction morphism \(|\text{Sing}(E)| \to E\) (resp. \(|\text{Sing}(B)| \to B\)). We define \(p_{\mu} : \lambda(E) \to B\) to be the induced map between the two pushouts. The others ingredients of a functorial fibrewise extension follow from the universal property of pushouts and the required conditions are easily checked.

**Proof of Proposition 2.** We deduce this result from the similar statement in the category \(\mathcal{F}\) that we prove now. We use here the notations of the construction of \(\lambda_{\mathcal{F}}\). Let \(p : E \to B\) be a fibration in \(\mathcal{F}\) and \(\mu\) be another functorial fibrewise extension of \(\lambda\). Without loss of generality, we may assume that \(p_{\mu} : \mu(E) \to B\) is also a fibration. From the properties (i)–(iv) of the functorial fibrewise extension \(\mu\), we deduce the existence for any simplex \(\sigma : \Delta[n] \to B\) of a diagram

\[
\begin{array}{ccc}
\lambda_{\mathcal{F}}\sigma^*(E) & \xrightarrow{i_{\mathcal{F}}\sigma^*} & \sigma^*(E) \\
\downarrow m_{\mu} & & \downarrow j_{\mu} \\
\mu\sigma^*(E) & \xrightarrow{\sim} & \mu(E) \\
\downarrow \sim & & \downarrow \\
(a_{\sigma}) & \xrightarrow{\sim} & \mu(E) \\
\lambda_{\mathcal{F}}(\Delta[n]) & \xrightarrow{\sim} & \Delta[n] \\
\downarrow p_{\mu} & & \downarrow \sigma \\
\lambda_{\mathcal{F}}(\Delta[n]) & \xrightarrow{\sim} & \Delta[n] \\
\end{array}
\]

in which the diagram \((a_{\sigma})\) is homotopy commutative by a natural homotopy, the other parts are exactly commutative, and the two right diagrams are pullbacks. Notice also that the diagram \((a_{\sigma})\)
is a homotopy pullback. We consider the homotopy colimits of the functors $\tilde{\lambda}_\mathcal{G} \circ \tilde{E}$, $\tilde{\lambda}_\mathcal{G} \circ \tilde{B}$, $\mu \circ \tilde{E} : \Delta_B \to \mathcal{G}$ and we obtain the following diagram:

Here, all subdiagrams are commutative except the diagram (b) which is homotopy commutative by a natural homotopy induced by all the natural homotopies of diagrams (a). Moreover, as the diagrams (a) are homotopy pullbacks so is the diagram (b). Also the upper right square is a homotopy pushout. This identifies, up to a chain of natural weak equivalences over $B$, the spaces $\mu(E)$ and $\lambda_\mathcal{G}(E)$. □

3. A characterization of $Q^k \text{cat}$

In this section, we explicate a functorial fibrewise extension of the functor $Q^k$ by means of iterated (unreduced) fibrewise suspensions of a fibration. By the unicity result that we proved in the previous section (Proposition 2) we will then obtain a characterization of the invariant $Q^k \text{cat}$.

We first recollect the material we will use about the fibrewise suspension and refer to [5] for complements on homotopy fibrewise theory.

3.1. Fibrewise suspension

Let $p : E \to B$ be a map. The (unreduced) fibrewise suspension of $E \in \mathcal{F}op_B$ is the fibrewise space $\tilde{p} : \Sigma_B E \to B$ obtained by the following pushout in $\mathcal{F}op_B$:

$$
E \times \{0,1\} \quad \longrightarrow \quad E \times I \\
p \times \text{id} \quad \downarrow \quad \downarrow \\
B \times \{0,1\} \quad \longrightarrow \quad \Sigma_B E.
$$

Denote by $s^1$ the map $B \times \{0,1\} \to \Sigma_B E$ in the diagram above. Let $[e,t]$ be the class in $\Sigma_B E$ of $(e,t) \in E \times I$. If $t \in \partial I$ and $e$ and $e'$ are two elements of $E$ such that $p(e) = p(e')$ then $[e,t] = [e',t]$. When $p$ is surjective, the map $s^1$ can thus be described by $s^1(b,t) = [e,t]$ where $e$ is any point of $E$ such that $p(e) = b$. 
Lemma 6. If the map \( p : E \to B \) is a fibration with fibre \( F \) over a point \( b \in B \) then the map \( \hat{p} : \Sigma_B E \to B \) is a fibration with fibre \( \Sigma F \) over \( b \).

Proof. We have the following pushout:

\[
\begin{align*}
B^l \times_B (E \times \{0,1\}) & \longrightarrow B^l \times_B (E \times I) \\
\downarrow & \\
B^l \times_B B \times \{0,1\} & \longrightarrow B^l \times_B \Sigma_B E.
\end{align*}
\]

From a lifting function \( \hat{\lambda} : B^l \times_B E \to E^l \) for the fibration \( p \), we can thus construct a lifting function \( \hat{\lambda} : B^l \times_B \Sigma_B E \to (\Sigma_B E)^l \) for the map \( \hat{p} \).

For \( k \geq 2 \), set \( \Sigma_B^k E := \Sigma_B (\Sigma_B^{k-1} E) \) and \( \hat{p}^k := \hat{p}^{k-1} \) (with \( \Sigma_B^1 E = \Sigma_B E \) and \( \hat{p}^1 := \hat{p} \)). The fibrewise space \( \Sigma_B^k E \) is naturally equipped with a map \( s^k : B \times \partial I^k \to \Sigma_B^k E \) which is defined inductively from \( s^1 : B \times \{0,1\} \to \Sigma_B E \) by means of the following diagram where the back face is a pushout:

\[
\begin{align*}
B \times \partial I^{k-1} \times \{0,1\} & \longrightarrow B \times \partial I^{k-1} \times I \\
\downarrow & \\
\Sigma_B^{k-1} E \times \{0,1\} & \longrightarrow \Sigma_B^{k-1} E \times I \\
\downarrow & \\
B \times I^{k-1} \times \{0,1\} & \longrightarrow B \times \partial I^k \\
\downarrow & \\
B \times \{0,1\} & \longrightarrow \Sigma_B^k E.
\end{align*}
\]

By construction, the map \( s^k \) makes commutative the diagram

\[
\begin{align*}
B \times \partial I^k & \xrightarrow{s^k} \Sigma_B^k E \\
\downarrow & \\
B \times I^k & \xrightarrow{p^k} B
\end{align*}
\]

The map \( s^k \) will play an important role in the fibrewise construction for the functor \( Q^k \) which we describe in the next section. Let us explicate it when \( p : E \to B \) is surjective. Recall that the \( k \)-fold suspension of a space \( F \) coincides with the pushout of the diagram

\[
\begin{align*}
J^{k-1} \xrightarrow{\rho^{k-1} \circ \partial} F \times \partial I^k & \to F \times I^k,
\end{align*}
\]

where \( \rho^0 : \partial I \to J^0 = \partial I \) is the identity map and, for \( k \geq 2 \), \( \rho^{k-1} : \partial I^k \to J^{k-1} = I \times \partial I^{k-1} \cup \partial I \times I^{k-1} / \sim \) is the identification map for the relation \( \sim \) given by \( (t,u) \sim (t',u) \) for any \( t,t' \in I \) and \( u \in \partial I^{k-1} \).
Similarly, we can see that the diagram

\[
\begin{array}{ccc}
E \times \partial I^k & \xrightarrow{p \times \rho^{k-1}} & E \times I^k \\
\downarrow & & \downarrow \\
B \times J^{k-1} & \to & \Sigma_B^k E
\end{array}
\]

is a pushout in \(\mathcal{Top}_B\). Therefore, if \(p : E \to B\) is surjective, we have, for \(b \in B\) and \((t_1, \cdots, t_k) \in \partial I^k\), \(s^k(b, t_1, \ldots, t_k) = [e, t_1, \ldots, t_k]\) where \(e \in E\) is any point such that \(p(e) = b\).

With the aid of the pushout above it is easy to prove the following lemma which will be useful to establish the product formula for the invariant \(Q^\infty\) cat.

**Lemma 7.** Let \(p : E \to B\) be a map and \(X\) be a space. Consider the map \(p \times \text{id} : E \times X \to B \times X\). Then, for any \(k \geq 1\), \(\Sigma_{B \times X}^k (E \times X) = \Sigma_B^k E \times X\).

**Proof.** The spaces \(\Sigma_{B \times X}^k (E \times X)\) and \(\Sigma_B^k E \times X\) agree both with the pushout of the diagram

\[
\begin{array}{ccc}
B \times X \times J^{k-1} & \xrightarrow{p \times \text{id} \times \rho^{k-1}} & E \times X \times \partial I^k \\
\downarrow & & \downarrow \\
\Sigma_B^k E \times X & \to & E \times X \times I^k \quad \square
\end{array}
\]

For \(k \geq 1\), the space \(\Sigma_B^k E\) is a well-pointed fibrewise space, that means that the map \(\hat{p}^k : \Sigma_B^k E \to B\) admits a section which is a cofibration. Indeed, the map \(e^k : B \to \Sigma_B^k E\) defined by \(e^k(b) = s^k(b, e^k)\) where \(e^k = (1, 1, \ldots, 1) \in I^k\) is the base point of \(\partial I^k\) both is a cofibration and a section of \(\hat{p}^k\). As for a well-pointed space in \(\mathcal{Top}\), we can, for a well-pointed object of \(\mathcal{Top}_B\), consider the reduced fibrewise suspension. Let us recall what we need about this notion.

Let \(E'\) be a fibrewise space such that the map \(p' : E' \to B\) admits a section \(\sigma : B \to E'\) which is a cofibration. The \(k\)-fold reduced fibrewise suspension of \(E\) is the fibrewise space given by the quotient space \(\Sigma_B^k E' = \Sigma_{B \times X}^k (E' \times X)\) where \([\sigma(b), u] \sim [\sigma(b), u']\) for all \(b \in B\) and \(u, u' \in I^k\) and by the map \(\Sigma_B^k E' \to B\) induced by \(\hat{p}'\). This fibrewise space is clearly well pointed by the map \(b \mapsto [\sigma(b), u]\) where \(u\) is any element of \(I^k\). Notice that the identification map \(\Sigma_B^k E' \to \Sigma_B^k E'\) is a homotopy equivalence. In the same way as the usual \(k\)-fold reduced suspension of \(X\) is homeomorphic to the smash product \(X \wedge S^k\), the \(k\)-fold reduced fibrewise suspension \(\Sigma_B^k E'\) \(\to B\) is homeomorphic to the fibrewise smash product \(E' \wedge_B (B \times S^k)\), the construction of which we recall now. First consider the fibrewise wedge \(E' \vee_B (B \times S^k)\) of \(E'\) and \(B \times S^k\) which is the pushout in \(\mathcal{Top}_B\) of the map \(\sigma : B \to E'\) and of the section \(B \to B \times S^k\) of the projection \(pr_B : B \times S^k \to B\) given by \(b \mapsto (b, s)\) where \(s = [e^k] \in S^k = I^k/\partial I^k\). The fibrewise smash-product \(E' \wedge_B (B \times S^k)\) of \(E'\) and \(B \times S^k\) is thus the pushout in \(\mathcal{Top}_B\) of the induced map \(E' \vee_B (B \times S^k) \to B\) and the obvious inclusion \(E' \vee_B (B \times S^k) \to E' \times_B (B \times S^k)\). We denote by \((p' \wedge_B pr_B)\) the induced map \(E' \wedge_B (B \times S^k) \to B \wedge_B B = B\) and by \(e' \wedge_B (b, u)\) the class in \(E' \wedge_B (B \times S^k)\) of an element \((e', (b, u)) \in E' \times_B (B \times S^k)\). The map \((p' \wedge_B pr_B)\) is equipped with the section \(B \to E' \wedge_B (B \times S^k)\) defined by \(b \mapsto e' \wedge_B (b, s) = \sigma(b) \wedge_B (b, u)\) (where \(e'\) is any point of \(E'\) such that \(p'(e') = b\) and \(u\) is any point of \(S^k\)). We have finally the following
where the two horizontal maps are fibrewise pointed, what means that they preserve the sections described above.

In particular, for \( i \geq 1 \), the space \( \Sigma^i_BE \) is a well-pointed fibrewise space through the map \( \iota^i : B \rightarrow \Sigma^i_BE \). We have thus

\[
\Sigma^{i+k}_BE \xrightarrow{\sim} \Sigma^i_B(\Sigma^i_BE) \xrightarrow{\sim} \Sigma^i_BE \wedge_B (B \times S^k)
\]

The spaces \( \Sigma^i_B(\Sigma^i_BE) \) and \( \Sigma^i_BE \wedge_B (B \times S^k) \) are, respectively, equipped with the maps \( B \times \Sigma^k \partial I^i = \Sigma_B(B \times \partial I^i) \xrightarrow{\Sigma^i_B(s)} \Sigma^i_BE \) and \( B \times (\partial I^i \wedge S^k) = (B \times \partial I^i) \wedge_B (B \times S^k) \xrightarrow{s^i \wedge \text{id}} \Sigma^i_BE \wedge_B (B \times S^k) \).

We have the following compatibility (over \( B \)) between these maps:

\[
B \times \partial I^{i+k} \xrightarrow{s^{i+k}} B \times \Sigma^k \partial I^i \xrightarrow{\Sigma^i_B(s)} B \times (\partial I^i \wedge S^k) \xrightarrow{s^i \wedge \text{id}} B \times S^k
\]

### 3.2. Fibrewise construction for \( Q^k \) and characterization of \( Q^k \) cat

Let \( F \) be a space and \( k \geq 1 \) be an integer. Recall that \( j^k : \partial I^k \rightarrow \Sigma^k F \) denotes the map defined by \( j^k(t_1, \ldots, t_k) = [f(t_1, \ldots, t_k)\] where \( f \) is any element of \( F \) and that \( Q^k(F) = \{ \omega : I^k \rightarrow \Sigma^k F | \omega_{|I^k} = j^k \} \).

For \( k \geq 1 \) consider the subspace \( Q^k_B(E) \) of the mapping space \( (\Sigma^k_B E)^k \) defined as follows:

\[
Q^k_B(E) = \{ \omega : I^k \rightarrow \Sigma^k_B E | \exists b \in B, \forall u \in I^k, \hat{p}^k \omega(u) = b \text{ and } \omega_{|I^k} = s^k(b, -) \}.
\]

We obtain thus a map \( \hat{p}^k : Q^k_B(E) \rightarrow B \) which associates with \( \omega \in Q^k_B(E) \) the value \( b = \hat{p}^k \omega(u) \) (where \( u \in I^k \) can be taken arbitrarily).

**Lemma 8.** If the map \( p : E \rightarrow B \) is a fibration with fibre \( F \) over \( b \in B \) then the map \( \hat{p}^k : Q^k_B(E) \rightarrow B \) is a fibration with fibre the space \( Q^k(F) \) over \( b \).

**Proof.** Let \( \Lambda : B^I \times_B \Sigma^k_B E \rightarrow (\Sigma^k_B E)^I \) be the lifting function constructed in Lemma 6. Define a map \( \hat{\Lambda} : B^I \times_B Q^k_B(E) \rightarrow (Q^k_B(E))^I \) by the following formula: for \( (z, \omega) \in B^I \times_B Q^k_B(E) \), set \( \hat{\Lambda}(z, \omega)(t)(u) = \Lambda(z, \omega(u))(t) \) where \( t \in I \) and \( u \in I^k \). The map \( \hat{\Lambda} \) is a lifting function for \( \hat{p}^k \). \( \square \)
Consider the map \( l_{Q^k_B} : E \to Q^k_B(E) \) defined by \( l_{Q^k_B}(e)(u) = [e, u] \) for \( e \in E \) and \( u \in I^k \) and denote by \( n_{Q^k_B} \) the map \( Q^k_B(E) \to Q^k(E) \) induced by the obvious map \( \Sigma^k_B E \to \Sigma_B E \). For any \( k \geq 1 \), we have thus the following commutative diagram:

\[
\begin{array}{ccc}
F & \longrightarrow & Q^k(F) \\
\downarrow & & \downarrow \\
E & \overset{l_{Q^k_B}}{\longrightarrow} & Q^k_B(E) \\
\downarrow & & \downarrow \\
B & \longrightarrow & Q^k(B)
\end{array}
\]

in which the composition \( E \to Q^k_B(E) \to Q^k(E) \) is exactly the coaugmentation. In other words, the functor \( Q^k_B : \mathcal{F}op_B \to \mathcal{F}op_B \) turns out to be a functorial fibrewise extension of the functor \( Q^k \). We deduce then the following result from Proposition 2:

**Proposition 9.** Let \( p : E \to B \) be a map. Then the map \( p_{\overline{Q^k}} : \overline{Q^k}(E) \to B \) and the map \( \tilde{p}^k : Q^k_B E \to B \) are naturally weakly equivalent over \( B \).

From this proposition and from the exponential law we then obtain the following characterizations of \( Q^k_{\text{cat}} \):

**Proposition 10.** Let \( X \) be a well-pointed space. The following conditions are equivalent:

(i) \( Q^k_{\text{cat}}(X) \leq n \).

(ii) The fibration \( g_n(X)^k : Q^k_X G_n(X) \to X \) admits a section.

(iii) The fibration \( g_n(X)^k : Q^k_X G_n(X) \to X \) admits a homotopy section.

(iv) There exists a map \( \phi : X \times I^k \to \Sigma^k_X G_n(X) \) such that the diagram

\[
\begin{array}{ccc}
X \times \partial I^k & \overset{s^k}{\longrightarrow} & \Sigma^k_X G_n(X) \\
\downarrow & & \downarrow \\
X & \longrightarrow & X.
\end{array}
\]

is commutative.

(v) There exists a map \( \phi : X \times I^k \to \Sigma^k_X G_n(X) \) such that in the diagram

\[
\begin{array}{ccc}
X \times \partial I^k & \overset{s^k}{\longrightarrow} & \Sigma^k_X G_n(X) \\
\downarrow & & \downarrow \\
X \times I^k & \longrightarrow & X.
\end{array}
\]

\( \phi|_{X \times \partial I^k} = s^k \) and \( (g_n(X)^k) \phi \) is homotopic to the projection \( X \times I^k \to X \) by a homotopy relative to \( X \times \partial I^k \).
With this characterization we can see that $Q^1 \text{cat} \ X$ is the least integer $n$ for which the fibrewise space $\Sigma^1_n \ G_\eta(X)$ is polarized in the sense of [6].

4. Product formula and application to the Ganea conjecture

In this section, we study the behaviour of the invariants $Q^i \text{cat}$, $Q^\infty \text{cat}$, and $Q \text{cat}$ with respect to the product of a space with a sphere and, from the formulae we obtain, we deduce new results concerning the Ganea conjecture.

4.1. Statement and applications

In Section 3.2, we will prove the following theorem:

**Theorem 11.** For any well-pointed space $X$, $i \geq 1$, and $k \geq 1$, the following inequality holds:

$$Q^{i+k} \text{cat}(X) + 1 \leq Q^i \text{cat}(X \times S^k).$$

On the other hand we know from [11] that $Q^i \text{cat}(X \times S^k) \leq Q^i \text{cat}(X) + 1$ for any $i \geq 1$ and $k \geq 1$. We thus obtain:

**Corollary 12.** Let $X$ be a well-pointed space. For any $k \geq 1$, $Q^\infty \text{cat}(X \times S^k) = Q^\infty \text{cat}(X) + 1$.

As $Q^\infty \text{cat}(X) + 1 = Q^\infty \text{cat}(X \times S^k) \leq \text{cat}(X \times S^k)$ we obtain the following sufficient condition for a space $X$ to satisfy the Ganea conjecture, i.e. for which one has $\text{cat}(X \times S^k) = \text{cat}(X) + 1$ for any $k \geq 1$.

**Corollary 13.** Let $X$ be a well-pointed space. If $Q^\infty \text{cat}(X) = \text{cat}(X)$ then $\text{cat}(X \times S^k) = \text{cat}(X) + 1$ for any $k \geq 1$.

As we mentioned in the introduction, there exist spaces $X$ for which the equality $\text{cat}(X \times S^k) = \text{cat}(X) + 1$ does not hold and the first of these spaces has been constructed by Iwase [4]. In [11], Scheerer et al. conjecture that the invariant $Q \text{cat}$ could permit to characterize the spaces satisfying the Ganea conjecture. More precisely, they conjecture that, for a finite CW-complex $X$, one has $\text{cat}(X \times S^k) = \text{cat}(X) + 1$ for any $k \geq 1$ if and only if $Q \text{cat}(X) = \text{cat}(X)$. Corollary 13 together with Proposition 5 enables us to establish one direction of this equivalence when $X$ is a finite-dimensional path-connected space.

**Corollary 14.** Let $X$ be a well-pointed path-connected space with $\dim(X) < \infty$. Then $X$ satisfies the Ganea conjecture when $Q \text{cat}(X) = \text{cat}(X)$. 
In addition, Corollary 13 permits to exhibit classes of spaces satisfying the Ganea conjecture. Certain of these classes are well known. For instance, we know from [11,9] that \( Q \text{cat}(X) = \text{cat}(X) \) when \( X \) is a rational space. We therefore have \( Q^\infty \text{cat}(X) = \text{cat}(X) \) for rational spaces and we rediscover the result of Hess and Jessup ([3,7]) that any rational space satisfies the Ganea conjecture. Notice however that the result of K. Hess which identifies the rational category \( (\text{cat}_0) \) with the module category \( (M\text{cat}_0) \) is used in the proof of the equality \( Q \text{cat}(X) = \text{cat}(X) \) for rational spaces. In [14], a sufficient condition for a space to satisfy the Ganea conjecture is obtained with the aid of an invariant denoted by \( \sigma \text{cat} \) and defined from iterated suspensions of the Ganea fibrations. More precisely, \( X \) satisfies the Ganea conjecture when \( \sigma \text{cat}(X) = \text{cat}(X) \) and this condition is used to exhibit spaces for which the equality \( \text{cat}(X \times S^k) = \text{cat}(X) + 1 \) holds. Since \( \sigma \text{cat}(X) \leq Q \text{cat}(X) \), the results of [14] on the Ganea conjecture can be attained through Corollary 13. In particular Corollary 13 permits to rediscover that any \((r - 1)\)-connected CW-complex \( X \) with \( r \geq 2 \) and \( \dim(X) \leq r\text{cat}(X) + 1 \) satisfies the Ganea conjecture. This result was first proved by Strom [13] and is improved when \( \text{cat}(X) \geq 2 \) by the following theorem:

**Theorem 15.** Let \( X \) be a \((r - 1)\)-connected CW-complex with \( r \text{cat}(X) \geq 3 \). If \( \dim(X) \leq 2r \text{cat}(X) - 3 \) then \( Q \text{cat}(X) = Q^\infty \text{cat}(X) = \text{cat}(X) \). In particular \( X \) satisfies the Ganea conjecture.

This theorem is established by an obvious induction from Proposition 16 below. We point here out that this result enables us to exhibit new spaces satisfying the Ganea conjecture, for instance:

**Example.** The symplectic group \( Sp(3) \) satisfies the Ganea conjecture. Indeed, \( Sp(3) \) is 2-connected, of dimension 21 and we know from [12] that \( \text{cat}(Sp(3)) \geq 4 \).

Notice also that Rudyak gave in [8] a condition for closed stably parallelizable manifolds to satisfy the Ganea conjecture which is very close to the condition of Theorem 15. More precisely, Rudyak proved that if \( M \) is a \((r - 1)\)-connected closed stably parallelizable manifold such that \( r \geq 1 \) and \( 4 \leq \dim(M) \leq 2r \text{cat}(M) - 4 \) then \( M \) satisfies the Ganea conjecture. As this condition implies that \( r \text{cat}(X) \geq 3 \) the result of Rudyak is entirely contained in Theorem 15. Since it is not known today whether \( \text{cat}(Sp(3)) = 4 \) or \( \text{cat}(Sp(3)) > 4 \), let us observe that the result on \( Sp(3) \) given above cannot be deduced from Rudyak’s result.

Let us finally remark that the bound \( \dim(X) \leq 2r \text{cat}(X) - 3 \) given in Theorem 15 is the best possible. In [4], Iwase shows that the space \( X = S^8 \cup_{2z} e^{30} \), where \( z \) is the generator of the direct summand \( Z/4Z \) of \( \pi_29(S^8) \cong Z/4Z \oplus (Z/2Z)^3 \), is a counterexample to the Ganea conjecture. For this space, we have \( r = 8 \), \( \text{cat}(X) = 2 \) and thus \( \dim(X) = 2r \text{cat}(X) - 2 \).

**Proposition 16.** Let \( X \) be a \((r - 1)\)-connected CW-complex and \( n \) be an integer such that \( rn \geq 3 \). If for \( i \geq 0 \) one has \( Q^i \text{cat}(X) = n \) and \( \dim(X) \leq 2rn - 3 + i \) then \( Q^{i+1} \text{cat}(X) = n \).

**Proof.** As \( Q^{i+1} \text{cat}(X) \leq Q^i \text{cat}(X) = n \), we only have to prove that \( Q^{i+1} \text{cat}(X) \geq n \). Suppose that this is not true. We thus have \( Q^{i+1} \text{cat}(X) \leq n - 1 \) and there exists a map \( \sigma : X \to \overline{Q^{i+1}}(G_{n-1}(X)) \).
such that \((g_{n-1}(X))_{Q^i} \circ \sigma \simeq id\). Consider the following diagram:

\[
\begin{array}{ccc}
Q^i(F_{n-1}(X)) & \rightarrow & Q^{i+1}(F_{n-1}(X)) \\
\downarrow & & \downarrow \\
\overline{Q}^i(G_{n-1}(X)) & \rightarrow & \overline{Q}^{i+1}(G_{n-1}(X)) \\
\downarrow \quad g_{n-1}(X)_{Q^i} & & \downarrow \quad g_{n-1}(X)_{Q^{i+1}} \\
X & & X
\end{array}
\]

As \(X\) is \((r-1)\)-connected, the space \(F_{n-1}(X)\) is \((rn-2)\)-connected. The condition \(rn \geq 3\) ensures that \(F_{n-1}(X)\) is at least 1-connected. Therefore, the map \(Q^i(F_{n-1}(X)) \rightarrow Q^{i+1}(F_{n-1}(X))\) is a \((2rn-3+i)\)-equivalence for all \(i \geq 0\) and so is the map \(\overline{Q}^i(G_{n-1}(X)) \rightarrow \overline{Q}^{i+1}(G_{n-1}(X))\). Since \(\text{dim}(X) \leq 2rn+i-3\), there exists a map \(\sigma' : X \rightarrow Q^i(G_{n-1}(X))\) such that \(\overline{Q}^i \circ \sigma' \simeq \sigma\). The map \(\sigma'\) turns out to be a homotopy section of \(g_{n-1}(X)_{\overline{Q}^i}\) which contradicts the hypothesis \(Q^i \text{cat}(X) = n\).

4.2. Proof of Theorem 11

The first ingredient of the proof consists of a characterization of the number \(Q^i \text{cat}(X \times S^k)\). Before we give it, we recall the similar characterization of \(\text{cat}(X \times S^k)\). Consider the map

\[
\gamma_{n+1} : \Gamma_{n+1} = G_n(X) \times S^k \cup_{F_n(X) \times s} CF_n(X) \times s \rightarrow X \times S^k
\]

which extends the map \(g_n(X) \times id : G_n(X) \times S^k \rightarrow X \times S^k\) by mapping \(CF_n(X) \times s\) onto the point \(* \times s\) where \(*\) and \(s\) are the respective base points of \(X\) and \(S^k\). This map is known (see [4]) to be a \((n+1)\)-LS map in the sense of [10]. That means that there exist two maps \(\xi : G_{n+1}(X \times S^k) \rightarrow \Gamma_{n+1}\) and \(\psi : \Gamma_{n+1} \rightarrow G_{n+1}(X \times S^k)\) such that \(\gamma_{n+1} \circ \xi \simeq g_{n+1}(X \times S^k)\) and \(g_{n+1}(X \times S^k) \circ \psi \simeq \gamma_{n+1}\). More precisely, we can construct a commutative diagram as follows:

\[
\begin{array}{ccc}
G_{n+1}(X \times S^k) & \rightarrow & \Gamma_{n+1} \\
\downarrow \quad g_{n+1}(X \times S^k) & & \downarrow \quad \gamma_{n+1} \\
X \times S^k & \rightarrow & G_{n+1}(X \times S^k)
\end{array}
\]

\[
\begin{array}{ccc}
G_{n+1}(X \times S^k) & \leftarrow & \Gamma_{n+1} \\
\downarrow \quad \Gamma_{n+1} & & \downarrow \quad g_{n+1}(X \times S^k) \\
X \times S^k & \leftarrow & X \times S^k
\end{array}
\]

(1)

It follows immediately from this that \(\text{cat}(X \times S^k) \leq n+1\) if and only if \(\gamma_{n+1}\) admits a homotopy section. For \(Q\text{'}^i \text{cat}(X \times S^k)\), we consider the space

\[
\Gamma'_{n+1} = \Sigma^i X G_n(X) \times S^k \cup_{\Sigma^i F_n(X) \times s} \Sigma^i F_n(X) \times s
\]

and the map

\[
\gamma'_{n+1} : \Gamma'_{n+1} \rightarrow X \times S^k
\]
which extends the map $g_n(X)^i \times \text{id} : \Sigma^n G_n(X) \times S^k \to X \times S^k$ by mapping $C\Sigma^n F_n(X) \times s$ onto the point $* \times s$. Denote also by $s^i \times \text{id}_{S^k}$ the composite $X \times \partial I^i \times S^k \to \Sigma^n X G_n(X) \times S^k \to I^i_{n+1}$. We thus have:

**Proposition 17.** $Q^i \text{cat}(X \times S^k) \leq n + 1$ if and only if there exists a map $\phi : X \times S^k \times I^i \to I^i_{n+1}$ such that in the diagram

$$
\begin{array}{ccc}
X \times \partial I^i \times S^k & \xrightarrow{s^i \times \text{id}_{S^k}} & I^i_{n+1} \\
\downarrow & & \downarrow \\
X \times I^i \times S^k & \xrightarrow{\phi} & X \times S^k.
\end{array}
$$

$\phi|_{X \times \partial I^i \times S^k} = s^i \times \text{id}_{S^k}$ and $\gamma_{n+1}^i \phi$ is homotopic to the projection $X \times I^i \times S^k \to X \times S^k$ by a homotopy relative to $X \times \partial I^i \times S^k$.

**Proof.** As the fibrewise suspension preserves weak equivalences, we can construct from diagram 1 the following commutative diagram:

$$
\begin{array}{ccc}
\Sigma^n X \times S^k G_{n+1}(X \times S^k) & \cong & \Sigma^n X \times S^k (G_n(X \times S^k) \cup CF_n(X \times S^k)) \\
\downarrow & & \downarrow \\
X \times S^k \times \partial I^i & \xrightarrow{s^i} & \Sigma^n X \times S^k \Gamma_{n+1} \\
\downarrow & & \downarrow \\
X \times S^k \times I^i & \xrightarrow{\phi} & X \times S^k \\
\downarrow & & \downarrow \\
X \times S^k & \xrightarrow{\gamma_{n+1}^i} & X \times S^k.
\end{array}
$$

Using Proposition 10 and the lifting lemma, we thus deduce from this diagram that $Q^i \text{cat}(X \times S^k) \leq n + 1$ if and only if there exists a map $\phi : X \times S^k \times I^i \to \Sigma^n X \times S^k \Gamma_{n+1}$ such that in the diagram

$$
\begin{array}{ccc}
X \times S^k \times \partial I^i & \xrightarrow{s^i} & \Sigma^n X \times S^k \Gamma_{n+1} \\
\downarrow & & \downarrow \\
X \times S^k \times I^i & \xrightarrow{\phi} & X \times S^k \\
\downarrow & & \downarrow \\
X \times S^k & \xrightarrow{\gamma_{n+1}^i} & X \times S^k.
\end{array}
$$

$\phi|_{X \times \partial I^i \times S^k} = s^i$ and $\gamma_{n+1}^i \phi$ is homotopic to the projection $X \times I^i \times S^k \to X \times S^k$ by a homotopy relative to $X \times S^k \times \partial I^i$.

In order to obtain the statement we next prove that the map $\gamma_{n+1}^i$ is exactly the map $\gamma_{n+1}^i$. For that consider the following commutative diagram in which $J^{i-1}$ denotes the quotient of $\partial I^i$ introduced
in the first section:

\[
\begin{align*}
\ast \times s \times J^{i-1} & \hookrightarrow CF_n(X) \times s \times \partial I^i \twoheadrightarrow CF_n(X) \times s \times I^i \\
\ast \times s \times J^{i-1} & \hookrightarrow F_n(X) \times s \times \partial I^i \twoheadrightarrow F_n(X) \times s \times I^i \\
X \times S^k \times J^{i-1} & \hookrightarrow G_n(X) \times S^k \times \partial I^i \twoheadrightarrow G_n(X) \times S^k \times I^i
\end{align*}
\]

By taking the pushouts of the lines, we obtain the following diagram:

\[
\Sigma^l_{X \times S^k}(G_n(X) \times S^k) \hookrightarrow \Sigma^l F_n(X) \times s \twoheadrightarrow \Sigma^l CF_n(X) \times s
\]

and, on the other side, the pushouts of the columns give arise to the following diagram:

\[
X \times S^k \times J^{i-1} \hookrightarrow \Gamma_{n+1} \times \partial I^i \twoheadrightarrow \Gamma_{n+1} \times I^i.
\]

As the colimit of a diagram is unique, this identifies the spaces \(\Sigma^l_{X \times S^k} \Gamma_{n+1} \) and \(\Sigma^l_{X \times S^k}(G_n(X) \times S^k) \cup \Sigma^l CF_n(X) \times s\) and clearly this identification holds over \(X \times S^k\). Without changing the result, we can replace in diagram 2 above the cofibration \(\Sigma^l F_n(X) \times s \twoheadrightarrow \Sigma^l CF_n(X) \times s\) by the cofibration \(\Sigma^l F_n(X) \times s \twoheadrightarrow CF_n(X) \times s\) and the space \(\Sigma^l_{X \times S^k}(G_n(X) \times S^k) \) by \(\Sigma^l_{X \times S^k}(G_n(X) \times S^k)\) (see Lemma 7). This achieves to identify the maps \(\gamma^{l+1}_{n+1}\) and \(\gamma^{l+1}\). It is finally easy to check that, under this identification, the map \(s^i : X \times S^k \times \partial I^i \to \Sigma^l_{X \times S^k} \Gamma_{n+1}\) corresponds to the map \(s^i \times id_{S^k} : X \times S^k \times \partial I^i \to \Gamma_{n+1}\). This achieves the proof. \(\square\)

We come now to the heart of the proof of Theorem 11. Suppose that \(Q^l \text{cat}(X \times S^k) \leq n + 1\). Recall from the end of Section 2.1 that we have a commutative diagram as follows:

\[
\begin{align*}
X \times \partial I^{i+k} \xrightarrow{s^{i+k}} \Sigma^{i+k} G_n(X) & \\
\sim & \\
X \times (\partial I^i \land S^k) \xrightarrow{s^i \land X \text{id}_{X \times S^k}} \Sigma^i G_n(X) \land_X X \times S^k & \\
\sim & \\
X \times I^{i+k} \xrightarrow{g_n(X)^i \land_X pr_X} X & \\
\sim & \\
X \times (I^i \land S^k) \xrightarrow{X} X
\end{align*}
\]

In order to prove that \(Q^{i+k} \text{cat}(X) \leq n\) it suffices to construct a map \(\theta : X \times (I^i \land S^k) \to \Sigma^i G_n(X) \land_X X \times S^k\) such that \(\theta|_{X \times (\partial I^i \land S^k)} = s^i \land_X \text{id}_{X \times S^k}\) and \((g_n(X)^i \land_X pr_X) \circ \theta\) is homotopic to the projection \(X \times (I^i \land S^k) \to X\) relatively to \(X \times (\partial I^i \land S^k)\).
First we construct a map $\mu : \Gamma'_{n+1} = \Sigma^i X G_n(X) \times S^k \cup_{\Sigma^i F_n(X) \times s} C \Sigma^i F_n(X) \times s \to \Sigma^i X G_n(X) \land_X X \times S^k$ and a commutative diagram:

\[
\begin{array}{cccc}
X \times S^k \times \partial I^i & \xrightarrow{\phi} & \Gamma'_{n+1} & \xrightarrow{\mu} & \Sigma^i X G_n(X) \land_X X \times S^k \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\
X \times S^k \times I^i & \to & X \times S^k & \to & X.
\end{array}
\]

For that, notice that the composite $\Sigma^i F_n(X) \times s \to \Sigma^i X G_n(X) \times S^k \to \Sigma^i X G_n(X) \land_X (X \times S^k) \to (\Sigma^i X G_n(X) \lor_X (X \times S^k))$ factors through $\Sigma^i X G_n(X) \land_X (X \times S^k)$. The map $\Sigma^i F_n(X) \times s \to \Sigma^i X G_n(X) \land_X (X \times S^k)$ so obtained makes commutative the following diagram:

\[
\begin{array}{cccc}
C \Sigma^i F_n(X) \times s & \xrightarrow{\ast} & \Sigma^i F_n(X) \times s & \xrightarrow{\ast} & \Sigma^i X G_n(X) \times S^k \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\
X & \leftarrow & \Sigma^i X G_n(X) \land_X (X \times S^k) & \to & \Sigma^i X G_n(X) \land_X (X \times S^k).
\end{array}
\]

The map $\mu$ is obtained by the universal property between the pushout of the upper line and the pushout of the lower line. It is easy to see that $\mu$ is over the projection $pr_X : X \times S^k \to X$ and that

the composite $\mu \circ (s^i \times id_{S^k})$ coincides with the composite $X \times \partial I^i \times S^k \xrightarrow{s^i \times id_{S^k} \times s^i} \Sigma^i X G_n(X) \times_X X \times S^k \to \Sigma^i X G_n(X) \land_X X \times S^k$.

From the hypothesis $Q' \text{cat}(X \times S^k) \leq n + 1$ we know that there exists a map $\phi : X \times S^k \times I^i \to \Gamma'_{n+1}$ such that $\phi|_{X \times S^k \times \partial I^i} = s^i \times id_{S^k}$ and $\gamma'_{n+1} \phi$ is homotopic to the projection $X \times I^i \times S^k \to X \times S^k$ by a homotopy relative to $X \times \partial I^i \times S^k$.

Denote by $\tilde{\phi}$ the composite $\mu \circ \phi$. As the restrictions of the maps $\tilde{\phi}$ and $\mu \circ (s^i \times id_{S^k})$ to the space $X \times S^k \times e^i$ factor through the canonical section $X \to \Sigma^i X G_n(X) \land_X X \times S^k$ we obtain the following diagram:

\[
\begin{array}{cccc}
X \times (S^k \times \partial I^i) & \to & \Sigma^i X G_n(X) \land_X X \times S^k \\
\downarrow & & \downarrow & & \downarrow \\
X \times (S^k \times I^i) & \to & X.
\end{array}
\]

in which the symbol $\simeq$ denotes the semi-smash product (i.e., $S^k \simeq Z = S^k \times Z/S^k$), the upper triangle commutes exactly and the lower triangle commutes up to a homotopy relative to $X \times (S^k \simeq \partial I^i)$. Recall now that the projection $S^k \simeq \partial I^i \to S^k \land \partial I^i$ admits a homotopy section and, since $S^k \simeq I^i$
is contractible, we can construct a commutative diagram

\[
\begin{array}{ccc}
S^k \wedge \partial I^i & \longrightarrow & S^k \times \partial I^i \\
\downarrow & & \downarrow \\
S^k \wedge I^i & \longrightarrow & S^k \times I^i
\end{array}
\]

This leads finally to a diagram

\[
\begin{array}{ccc}
X \times (S^k \wedge \partial I^i) & \longrightarrow & \Sigma^i_X G_n(X) \wedge_X X \times S^k \\
\downarrow & & \downarrow \\
X \times (S^k \wedge I^i) & \longrightarrow & X.
\end{array}
\]

in which the lower triangle is homotopy commutative by a homotopy relative to \(X \times (S^k \wedge \partial I^i)\) and the upper triangle is homotopy commutative by a homotopy \(h\) verifying \((\Sigma^i_X G_n(X) \wedge_X X \times S^k) \circ h(x,u,t) = x\) for all \((x,u,t) \in X \times (S^k \wedge \partial I^i) \times I\). As the map \(X \times (S^k \wedge \partial I^i) \to X \times (S^k \wedge I^i)\) is a cofibration, the relative lifting lemma enables us to exhibit a map \(\theta : X \times (I^i \wedge S^k) \to X \times (S^k \wedge \partial I^i)\) such that \(\theta |_{X \times (\partial I^i \wedge S^k)} = s^i \wedge_X id_X \wedge S^k\) and \((\Sigma^i_X G_n(X) \wedge_X X \times S^k) \circ \theta \) is homotopic to the projection \(X \times (I^i \wedge S^k) \to X\) relatively to \(X \times (\partial I^i \wedge S^k)\). This achieves the proof. \(\square\)

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References


