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Localization of Triangulated Categories and Derived Categories

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INTRODUCTION

The notion of quotient and localization of abelian categories by dense subcategories (i.e., Serre classes) was introduced by Gabriel, and plays an important role in ring theory [6, 13]. The notion of triangulated categories was introduced by Grothendieck and developed by Verdier [9, 16] and is recently useful in representation theory [8, 4, 14]. The quotient of triangulated categories by épaisse subcategories is constructed in [16]. Both quotients were indicated by Grothendieck, and they resemble each other. In this paper, we will consider triangulated categories and derived categories from the point of view of localization of abelian categories. Verdier gave a condition which is equivalent to the one that a quotient functor has a right adjoint, and considered a relation between épaisse subcategories [16]. We show that localization of triangulated categories is similarly defined, and have a relation between localizations and épaisse subcategories. Beilinson, Bernstein, and Deligne introduced the notion of t-structure which is similar to torsion theory in abelian categories [2]. We, in particular, consider a stable t-structure, which is an épaisse subcategory, and deal with a correspondence between localizations of triangulated categories and stable t-structures. And then recollement, in the sense of [2], is equivalent to bilocalization. Next, we show that quotient and localization of abelian categories induce quotient and localization of its derived categories.

In Section 1, we recall standard notations and terminologies of quotient and localization of abelian categories. In Section 2, we define localization of triangulated categories, and consider a relation between localizations and stable t-structures (Theorem 2.6). In Section 3, we show that if $A \to A/C$ is a quotient of abelian categories, then $D^*(A) \to D^*(A/C)$ is a quotient of triangulated categories, where $* = +, -$, or b (Theorem 3.2). Moreover,

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under some conditions, if $A \to A/C$ is a localization of triangulated categories.

In Section 4, we study quotient and localization of derived categories of modules by using similar module to tilting modules with finite projective dimension [4, 11] (Propositions 4.2 and 4.3). In Section 5, we apply Section 4 to the situation of module categories of finite dimensional algebras over a fixed field k and by calculating the Grothendieck groups we give a bound to the number of non-isomorphic indecomposable modules which are direct summands of such a module (Propositions 5.1, 5.2, Corollaries 5.5, 5.7, and 5.8). And we consider relations between localizations and ring epimorphisms (Proposition 5.3 and Theorem 5.4). Finally, in connection with recollement of derived categories of modules which was introduced by Cline, Parshall, and Scott [5, 12], we consider relations between localizations and idempotent ideals (Proposition 5.9, Theorem 5.10, and Corollary 5.11).

Throughout this note, we assume that all categories are skeletally small.

1. Preliminaries

Let A be an abelian category. A full subcategory C of A is called dense if for every exact sequence $0 \to X \to Y \to Z \to 0$ in A, the following condition holds: $X, Z \in C$ if and only if $Y \in C$. We denote by $\phi(C)$ the set of morphisms f such that $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are in C. Then $\phi(C)$ is a multiplicative system. And then C is an abelian category and the quotient category A/C is defined. In this case, we will say that $0 \to C \to A \xrightarrow{Q} A/C \to 0$ is exact. The right adjoint of Q is called a section functor. If there exists a section functor S, $\{A/C; Q, S\}$ is called a localization of A. In this case, C is called a localizing subcategory of A. Then S is fully faithful. On the other hand, if $T: A \to B$ is an exact functor between abelian categories which has a fully faithful right adjoint $S: B \to A$, then $\operatorname{Ker} T$ is a localizing subcategory of A, and T induces an equivalence between $A/\operatorname{Ker} T$ and B. Colocalization of C is also defined, and similar results hold [6, 13]. We apply these ideas to triangulated categories in the next section.

2. Localization of Triangulated Categories

Given two triangulated categories D and D', a grade functor from D to D' is an additive functor $F: D \to D'$ and an isomorphism $\Phi: FT \to T'F$, where T and T' are the translation functors of D and D', respectively. A grade functor (F, Φ) is called a ∂-functor if for every distinguished triangle

(X, Y, Z, u, v, w) in D, $(X, Y, Z, Fu, Fv, \Phi_X \circ Fw)$ is distinguished in D' (we often simply write F unless it confounds us) [9, I, Sect. 1; 16, I, Sect. 1, No. 1]. If F has a right or left adjoint G, then G is a ∂-functor, also [10, 1.6 Proposition].

A subcategory U of D is called épaisse if U is a full triangulated subcategory and if U satisfies the following condition: For any $f\colon X \to Y$, which factors through an object in U and which has a mapping cone in C, X and Y are objects in U. We denote by $\phi(U)$ the set of morphisms f which is contained in a distinguished triangle (X, Y, Z, f, g, h) where Z is an object of U. Then $\phi(U)$ is a multiplicative system which satisfies the following conditions:

(FR-1) $s \in \phi(U)$ if and only if $Ts \in \phi(U)$, where T is the translation functor.

(FR-2) Given distinguished triangles (X, Y, Z, u, v, w), (X', Y', Z', u', v', w'), if f and g are morphisms in $\phi(U)$ such that $u' \circ f = g \circ u$, then there exists a morphism h in $\phi(U)$ such that (f, g, h) is a morphism of distinguished triangles (see [16, I, Sect. 2, No. 1] for details).

And the quotient category D/U is defined. In this case, we will say that $0 \to U \xrightarrow{K} D \xrightarrow{Q} D/U \to 0$ is exact (see [2, 1.4.4; 16, I, Sect. 2, No. 3] for details.

Lemma 2.1 (cf. [16, 5-3 Proposition]). *Let D be a triangulated category, U a épaisse subcategory of D, and $Q\colon D \to D/U$ a natural quotient. For $M \in D$, the following are equivalent.*

(a) *For every* $f\colon X \to Y \in \phi(U)$, $\mathrm{Hom}(f, M)\colon \mathrm{Hom}_D(Y, M) \to \mathrm{Hom}_D(X, M)$ *is bijective.*

(b) $\mathrm{Hom}_D(U, M) = 0$.

(c) *For every* $X \in D$, $Q(X, M)\colon \mathrm{Hom}_D(X, M) \to \mathrm{Hom}_{D/U}(QX, QM)$ *is bijective.*

Proof. (a) $\Rightarrow$ (b). For every object $U \in U$, $0 \to U \xrightarrow{1} U \to$ is a distinguished triangle. Then $0 = \mathrm{Hom}_D(0, M) \simeq \mathrm{Hom}_D(U, M)$.

(b) $\Rightarrow$ (c). Every morphism of $\mathrm{Hom}_{D/U}(QX, QM)$ is represented by a diagram

$$X \xleftarrow{\ s\ } X' \xrightarrow{\ f\ } M$$

where s contained in a distinguished triangle $U \to X' \to X \to$ with $U \in \mathcal{U}$. Then there exists $f': X \to M$ in $\mathcal{A}$ such that $f = f' \circ s$, because $\mathrm{Hom}_{\mathcal{D}}(U, M) = 0$. Hence $Q(X, M)$ is surjective. If a morphism $g: X \to M$ satisfies that there exist a morphism $s: X' \to X$, where s is contained in a distinguished triangle $U \xrightarrow{r} X' \xrightarrow{s} X \xrightarrow{t}$ with $U \in \mathcal{U}$, such that $g \circ s = 0$, then there exist $u: U[1] \to M$ such that $g = u \circ t$. Therefore $g = 0$, because $u \in \mathrm{Hom}_{\mathcal{D}}(U, M) = 0$. Hence $Q(X, M)$ is injective.

(c) $\Rightarrow$ (a). Let $f: X \to Y$ be a morphism in $\phi(\mathcal{U})$. Then we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(Y, M) & \xrightarrow{\mathrm{Hom}(f, M)} & \mathrm{Hom}_{\mathcal{D}}(X, M) \\ {\scriptstyle Q(Y, M)}\downarrow & & \downarrow{\scriptstyle Q(X, M)} \\ \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QY, QM) & \xrightarrow[\mathrm{Hom}(Qf, QM)]{} & \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QX, QM). \end{array}$$

According to (c), $Q(X, M)$ and $Q(Y, M)$ are bijective. Because $QU = 0$, $\mathrm{Hom}(Qf, QM)$ is bijective. Hence $\mathrm{Hom}(f, M)$ is bijective.

An object M is called $\mathcal{U}$-closed if it satisfies the equivalent conditions of Lemma 2.1 [16, 5-4]. Let $0 \to \mathcal{U} \xrightarrow{K} \mathcal{D} \xrightarrow{Q} \mathcal{D}/\mathcal{U} \to 0$ be an exact sequence of triangulated categories. The right adjoint of Q is called a section functor. If there exists a section functor S, then $\{\mathcal{D}/\mathcal{U}; Q, S\}$ is called a localization of $\mathcal{D}$, and $0 \to \mathcal{U} \xrightarrow{K} \mathcal{D} \xrightarrow{Q} \mathcal{D}/\mathcal{U} \to 0$ is called localization exact.

LEMMA 2.2. *For every object $V \in \mathcal{D}/\mathcal{U}$, SV is $\mathcal{U}$-closed.*

Proof. For every $f: X \to Y \in \phi(\mathcal{U})$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(Y, SV) & \xrightarrow{\mathrm{Hom}(f, SV)} & \mathrm{Hom}_{\mathcal{D}}(X, SV) \\ \wr & & \wr \\ \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QY, V) & \xrightarrow[\mathrm{Hom}(Qf, V)]{\sim} & \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QX, V). \end{array}$$

Therefore $\mathrm{Hom}(f, SV)$ is an isomorphism. By Lemma 2.1, SV is $\mathcal{U}$-closed.

Let $\Phi: QS \to 1_{\mathcal{D}/\mathcal{U}}$ and $\Psi: 1_{\mathcal{D}} \to SQ$ be adjunction arrows.

PROPOSITION 2.3. *Let $\{\mathcal{D}/\mathcal{U}; Q, S\}$ be a localization of $\mathcal{D}$.*

(a) *Φ is an isomorphism (i.e., S is fully faithful).*

(b) *For every object $X \in \mathcal{D}$, the distinguished triangle $U \to X \xrightarrow{\Psi_X} SQX \to$ satisfies that U is in $\mathcal{U}$.*

Proof. (a) For every $X \in D$ and $Y \in D/U$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_D(X, SY) & = & \mathrm{Hom}_D(X, SY) \\ {\scriptstyle Q(X,SY)}\downarrow & & \wr\wr \\ \mathrm{Hom}_{D/U}(QX, QSY) & \xrightarrow[\mathrm{Hom}(QX, \Phi_Y)]{} & \mathrm{Hom}_{D/U}(QX, Y) \end{array}$$

By Lemmas 2.1 and 2.2, $Q(X, SY)$ is an isomorphism. Then $\mathrm{Hom}(QX, \Phi_Y)$ is an isomorphism. For any $Z \in D/U$, there exists $X \in D$ such that $Z \simeq QX$. Hence Φ is an isomorphism.

(b) It suffices to show that for any $X \in D$, $Q\Psi_X$ is an isomorphism. By $QX \xrightarrow{Q\Psi_X} QSQX \xrightarrow[\sim]{\Phi_{QX}} QX = 1_{QX}$, $Q\Psi_X$ is an isomorphism.

COROLLARY 2.4. *Let $M \in D$. Then M is U-closed if and only if $M \simeq SQM$.*

THEOREM 2.5. *Let D and E be triangulated categories, $T: D \to E$ a ∂-functor which has a fully faithful right adjoint $S: E \to D$. Then T induces an equivalence between $D/\mathrm{Ker}\ T$ and E. In particular, $0 \to \mathrm{Ker}\ T \to D \xrightarrow{T} E \to 0$ is localization exact.*

Proof. Consider $Q: D \to D/\mathrm{Ker}\ T$, then by the universal property of Q we have a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{T} & E \\ & {\scriptstyle Q}\searrow \quad \nearrow{\scriptstyle R} & \\ & D/\mathrm{Ker}\ T & \end{array}$$

If $f: X \to Y$ is a morphism in D, then Tf is an isomorphism if and only if Qf is an isomorphism. For every object $M \in D$, $TM \to TSTM$ is an isomorphism, then $QM \to QSTM$ is an isomorphism. Therefore $Q \to QST$ is an isomorphism. By the universal property of Q and $QST = QSRQ$, $1_{D/\mathrm{Ker}\ T} \simeq QSR$. Next, $RQS = TS \simeq 1_E$. Hence R is an equivalence.

Let U and V be full subcategories of D such that: (a) U and V are stable for translations; (b) $\mathrm{Hom}_D(U, V) = 0$; (c) For every $X \in D$, there exists a distinguished triangle (U, X, V) with $U \in U$ and $V \in V$. Then U and V are épaisse subcategories of D, and (U, V) is t-structure in the sense of Beilinson, Bernstein, and Deligne [2, 1.3]. We will call (U, V) a stable t-structure. Moreover, there exist exact sequences $0 \to U \xrightarrow{K} D \xrightarrow{Q} V \to 0$ and $0 \to V \xrightarrow{R} D \xrightarrow{Q'} U \to 0$ such that Q is the left adjoint of R and that Q'

is the right adjoint of K, where K and R are natural inclusions (see [2, 1.4.4] for details). Namely, $\{V; Q, R\}$ is a localization of D, and $\{U; K, Q'\}$ is a colocalization of D. By Theorem 2.5 and [16, 6-6 Proposition], and their duals, D/U is a localization of D if and only if U is a colocalization of D, and D/U is a colocalization of D if and only if U is a localization of D. By [2, 1.4.8], we know that recollement, in the sense of [2, 1.4.3], is equivalent to bilocalization.

PROPOSITION 2.6. *Let D be a triangulated category. If $\{V; Q, R\}$ is a localization of D, then R is fully faithful, and (KU, RV) is a stable t-structure, where $U = \operatorname{Ker} Q$ and K is a natural inclusion. Conversely, if (U, V) is a stable t-structure in D, then a natural inclusion $R\colon V \to D$ has a left adjoint Q such that $\{V; Q, R\}$ is a localization.*

Proof. Let $\{V; Q, R\}$ be a localization of D. Then, by $\operatorname{Hom}_D(KU, RV) \simeq \operatorname{Hom}_D(QKU, V) = 0$ and Proposition 2.3, it is clear that R is fully faithful and (KU, RV) is a stable t-structure. The converse version is true by the above.

We have the same result of Cline, Parshall, and Scott [5, Sect. 1, Theorem 1.1] under the weak conditions.

PROPOSITION 2.7. *Let $F\colon D \to E$ be a ∂-functor of triangulated categories. Assume that F has a fully faithful right (resp., left) adjoint $G\colon E \to D$. If F has a left (resp., right) adjoint $H\colon E \to D$, then H is a fully faithful ∂-functor. In this case,* $(\operatorname{Ker} F, D, E)$ *is a recollement.*

Proof. According to Theorem 2.5, Proposition 2.3, Proposition 2.6, and their duals, it is clear.

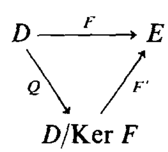
3. LOCALIZATION OF DERIVED CATEGORIES

Let A be an additive category, $K(A)$ a homotopy category of A, and $K^+(A)$, $K^-(A)$ and $K^b(A)$ full subcategories of $K(A)$ generated by the bounded below complexes, the bounded above complexes, and the bounded complexes, respectively. For an abelian category A, a derived category $D(A)$ (resp., $D^+(A)$, $D^-(A)$, and $D^b(A)$) of A is a quotient of $K(A)$ (resp., $K^+(A)$, $K^-(A)$, and $K^b(A)$) by a multiplicative set of quasi-isomorphisms. Then $K^*(A)$ and $D^*(A)$ are triangulated categories, where $*$ = nothing, $+$, $-$, or b [9, 16]. In general, we denote by $K^*(A)$ a localizing subcategory of $K(A)$ (i.e., $K^*(A)$ is a full triangulated subcategory of $K(A)$ and $D^*(A) \to D(A)$ is a fully faithful ∂-functor, where $D^*(A)$ is a quotient of $K^*(A)$ by a multiplicative set of quasi-isomorphisms) [9, I, Sect. 5; 16, II, Sect. 1, No. 1]. For a thick abelian subcategory C of A (i.e., C is extension closed in A), we denote by $D^*_C(A)$ a full subcategory of $D^*(A)$ generated by complexes of which all homologies are in C [9, I, Sect. 4].

Let $\partial(D^*(A), D(B))$ be a category of ∂-functors from $D^*(A)$ to $D(B)$ and $\mathrm{Hom}_\partial(F, G)$ the set of morphisms from F to G for $F, G \in \partial(D^*(A), D(B))$. Given a ∂-functor $F\colon K^*(A) \to K(B)$, we obtain a right derived functor $R^*F\colon D^*(A) \to D(B)$ when there exists an object R^*F in $\partial(D^*(A), D(B))$ such that $\mathrm{Hom}_\partial(R^*F, ?) \simeq \mathrm{Hom}_\partial(Q^*_A \circ F, ? \circ Q_B)$, where $Q^*_A\colon K^*(A) \to D^*(A)$, $Q_B\colon K(B) \to D(B)$ are natural quotients, [9, I, Sect. 5; 16, I, Sect. 2]. When $R^+F\colon D^+(A) \to D(B)$ exists, we say F has right homological dimension $\leqq n$ on A if $R^iF(X) = 0$ for all $X \in A$ and for all $i > n$ [9, I, Sect. 5; 16, I, Sect. 2, No. 2]. And an object X in A is called a right F-acyclic object if $R^iF(X) = 0$ for all $i > 0$. We also denote by R^*F a right derived functor of an induced ∂-functor from $F\colon A \to B$.

Let $F\colon A \to B$ be a left exact additive functor between abelian categories. If A has enough injectives, and F has finite right homological dimension on A, then RF, R^-F, and R^bF exist, and $RF|_{D^*(A)} \simeq R^*F$, and moreover, R^*F has image in $D^*(B)$, where $* = +, -$, or b [9, I, Sect. 5]. We often denote by $R^{*,\#}F$, $R^*F|_{D^\#(A)}$ when $D^\#(A)$ is a full subcategory of $D^*(A)$. On the other hand, if A and B have enough injectives and projectives, respectively, and if the derived functor $R^{+,b}F\colon D^b(A) \to D(B)$ has image in $D^b(B)$ and $R^{+,b}F\colon D^b(A) \to D^b(B)$ has a left adjoint, then F has a left adjoint $G\colon B \to A$ and the derived functor $L^{-,b}G\colon D^b(B) \to D(A)$ has image in $D^b(A)$, and which is the left adjoint of $R^{+,b}F$ [4, (3.1) Lemma].

LEMMA 3.1. *Let D and E be triangulated categories and $F\colon D \to E$ a ∂-functor. Consider the commutative diagram*



If F' is full dense, then F' is an equivalence.

THEOREM 3.2. *Let $0 \to C \to A \xrightarrow{Q} A/C \to 0$ be an exact sequence of abelian categories. Then $0 \to D^*_C(A) \to D^*(A) \xrightarrow{Q^*} D^*(A/C) \to 0$ is an exact sequence of triangulated categories, where $* = +, -$, or b.*

Proof. According to $\mathrm{Ker}\, Q^* = D^*_C(A)$ and Lemma 3.1, it suffices to show that the induced ∂-functor $Q'^*\colon D^*(A)/D^*_C(A) \to D^*(A/C)$ is full dense.

(I). *The case of* $*=b$. (i) Q'^b *is dense.* Let $X^{\cdot}: \cdots \to 0 \to X^{-n} \xrightarrow{\partial_{-n}} X^{-n+1} \xrightarrow{\partial_{-n+1}} \cdots \xrightarrow{\partial_{-1}} X^0 \to 0 \to \cdots$ be a complex in $D^b(A/C)$. Then $X^{\cdot}$ is represented by a diagram in A:

$$\begin{array}{ccccccccc}
 & & X'^{-n} & & & X'^{-n+1} & & X'^{-1} & \\
 & {}^{s_{-n}}\swarrow & & \searrow{}^{d_{-n}} & {}^{s_{-n+1}}\swarrow & & \searrow{}^{d_{-n+1}} \quad {}^{s_{-1}}\swarrow & & \searrow{}^{d_{-1}} \\
\cdots \to 0 \to X^{-n} & & & & X^{-n+1} & & \cdots & & X^0 \to 0 \to \cdots,
\end{array}$$

where $s_i \in \phi(C)$ for all i. By induction on i, we have the following commutative diagram in A:

$$\begin{array}{ccc}
X''^{-i} & \xrightarrow{d'_{-i}} & X''^{-i+1} \\
{\scriptstyle t_{-i}}\downarrow & & \downarrow{\scriptstyle s'_{-i+1}} \\
X'^{-i} & \xrightarrow[d_{-i}]{} & X^{-i+1},
\end{array}$$

and we have $s'_{-i} = s_{-i} \circ t_{-i} \in \phi(C)$ and $d'_{-i+1} \circ d'_{-i} = 0$ for all i. Indeed, it is clear in case of $i \geqq 1$ by taking $X''^{-i+1} := X'^{-i+1}$, $X''^{-i} := X^{-i}$, $s_{-i+1} := 1_{X^{-i+1}}$, $t_{-i} := 1_{X^{-i}}$, and $d'_{-i} := d_{-i}$. Next, by the property of a multiplicative system, we have the following commutative diagram in A:

$$\begin{array}{ccc}
X''''^{-i-1} & \xrightarrow{d''_{-i-1}} & X''^{-i} \\
{\scriptstyle s''_{-i-1}}\downarrow & & \downarrow{\scriptstyle s'_{-i}} \\
X'^{-i-1} & \xrightarrow[d_{-i-1}]{} & X^{-i},
\end{array}$$

where $s''_{-i-1} \in \phi(C)$. Since $\partial_{-i} \circ \partial_{-i-1} = 0$, there exists $t'_{-i-1}: X'''^{-i-1} \to X''''^{-i-1}$ such that $s_{-i-1} \circ s''_{-i-1} \circ t'_{-i-1} \in \phi(C)$ and $s'_{-i+1} \circ d'_{-i} \circ d''_{-i-1} \circ t'_{-i-1} = 0$. Then there exists $t''_{-i-1}: X''^{-i-1} \to X'''^{-i-1} \in \phi(C)$ such that $d'_{-i} \circ d'_{-i-1} \circ t'_{-i-1} \circ t''_{-i-1} = 0$. Let $t_{-i-1} := s''_{-i-1} \circ t'_{-i-1} \circ t''_{-i-1}$, $d'_{-i-1} := d''_{-i-1} \circ t'_{-i-1} \circ t''_{-i-1}$, $s'_{-i-1} := s_{-i-1} \circ t_{-i-1}$. Then we have the following commutative diagram in A:

$$\begin{array}{ccc}
X''^{-i-1} & \xrightarrow{d'_{-i-1}} & X''^{-i} \\
{\scriptstyle t_{-i-1}}\downarrow & & \downarrow{\scriptstyle s'_{-i}} \\
X'^{-i-1} & \xrightarrow[d_{-i-1}]{} & X^{-i},
\end{array}$$

and we have $s'_{-i-1} = s_{-i-1} \circ t_{-i-1} \in \phi(C)$ and $d'_{-i} \circ d'_{-i-1} = 0$. It is easy to see that $X''^{\cdot} = (X''^i, d'_i)$ is a complex in $D^b(A)$ such that $QX''^{\cdot} \simeq X^{\cdot}$, and $Q'^b X''^{\cdot} \simeq X^{\cdot}$.

(ii) Q'^b *is full.* (a) We first show that for every morphism $f: X^\cdot \to Y^\cdot$ of complexes in $K^b(A/C)$, there exist a complex $X''^\cdot$ and morphisms $s': X''^\cdot \to Y^\cdot$ and $f': X''^\cdot \to Y^\cdot$ of complexes in $K^b(A)$ such that $f \circ Qs' = Qf'$ and Qs' is an isomorphism in $K^b(A/C)$. By (i), $f = (f_i): X^\cdot \to Y^\cdot$ is represented by the following diagram in A:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow 0 \longrightarrow & X^{-n} & \xrightarrow{\partial_{-n}} & X^{-n+1} & \xrightarrow{\partial_{-n+1}} & \cdots & \xrightarrow{\partial_{-1}} & X^0 & \longrightarrow 0 \\
& \uparrow{\scriptstyle s_{-n}} & & \uparrow{\scriptstyle s_{-n+1}} & & & & \uparrow{\scriptstyle s_0} & \\
& X'^{-n} & & X'^{-n+1} & & \cdots & & X'^0 & \\
& \downarrow{\scriptstyle f'_{-n}} & & \downarrow{\scriptstyle f'_{-n+1}} & & & & \downarrow{\scriptstyle f'_0} & \\
\cdots \longrightarrow 0 \longrightarrow & Y^{-n} & \xrightarrow[d_{-n}]{} & Y^{-n+1} & \xrightarrow[d_{-n+1}]{} & \cdots & \xrightarrow[d_{-1}]{} & Y^0 & \longrightarrow 0,
\end{array}$$

where $s_i \in \phi(C)$ for all i. By induction on i, we have the following commutative diagram in A:

$$\begin{array}{ccc}
X^{-i} & \xrightarrow{\partial_{-i}} & X^{-i+1} \\
{\scriptstyle s'_{-i}}\uparrow & & \uparrow{\scriptstyle s'_{-i+1}} \\
X''^{-i} & \xrightarrow{\partial'_{-i}} & X''^{-i+1} \\
{\scriptstyle f''_{-i}}\downarrow & & \downarrow{\scriptstyle f''_{-i+1}} \\
Y^{-i} & \xrightarrow[d_{-i}]{} & Y^{-i+1},
\end{array}$$

where $s'_{-i} \in \phi(C)$, and we have $f_{-i} \circ Qs''_{-i} = Qf''_{-i}$ and $\partial'_{-i+1} \circ \partial'_{-i} = 0$ for all i. Indeed, it is clear in case of $i \geqq 0$ by taking $s'_{-i} := s_{-i}$, $f''_{-i} := f'_{-i}$ and $\partial'_{-i} := \partial_{-i}$. Next, by the property of a multiplicative system, we have the following commutative diagram in A:

$$\begin{array}{ccc}
X^{-i-1} & \xrightarrow{\partial_{-i-1}} & X^{-i} \\
{\scriptstyle s'''_{-i-1}}\uparrow & & \uparrow{\scriptstyle s'_{-i}} \\
X''''^{-i-1} & \xrightarrow[\partial'''_{-i-1}]{} & X''^{-i}
\end{array}$$

where $s'''_{-i-1} \in \phi(C)$. Since f is a morphism of complexes in $C^b(A/C)$, there exist $t'_{-i-1}: X''''^{-i-1} \to X''''^{-i-1}$ and $t_{-i-1}: X''''^{-i-1} \to X'^{-i-1}$ such that $s'''_{-i-1} \circ t'_{-i-1} = s_{-i-1} \circ t_{-i-1} \in \phi(C)$ and such that $d_{-i-1} \circ f'_{-i-1} \circ t_{-i-1} = f''_{-i} \circ \partial'''_{-i-1} \circ t'_{-i-1}$. Let $\partial''_{-i-1} := \partial'''_{-i-1} \circ t'_{-i-1}$, $f'''_{-i-1} := f'_{-i-1} \circ t_{-i-1}$,

$s''_{-i-1} := s'''_{-i-1} \circ t'_{-i-1} = s_{-i-1} \circ t_{-i-1}$. Then we have the following commutative diagram in A:

$$\begin{array}{ccc}
X^{-i-1} & \xrightarrow{\partial_{-i-1}} & X^{-i} \\
{\scriptstyle s''_{-i-1}}\uparrow & & \uparrow{\scriptstyle s'_{-i}} \\
X'''^{-i-1} & \xrightarrow{\partial''_{-i-1}} & X''^{-i} \\
{\scriptstyle f'''_{-i-1}}\downarrow & & \downarrow{\scriptstyle f''_{-i}} \\
Y^{-i-1} & \xrightarrow[d_{-i-1}]{} & Y^{-i},
\end{array}$$

where $s''_{-i-1} \in \phi(C)$ and we have $f_{-i-1} \circ Qs''_{-i-1} = Qf'''_{-i-1}$. Since $s'_{-i+1} \circ \partial'_{-i} \circ \partial''_{-i-1} = \partial_{-i} \circ \partial_{-i-1} \circ s''_{-i-1} = 0$, there exists $t''_{-i-1}: X''^{-i-1} \to X'''^{-i-1} \in \phi(C)$ such that $\partial'_{-i} \circ \partial''_{-i-1} \circ t''_{-i-1} = 0$. Let $\partial'_{-i-1} := \partial''_{-i-1} \circ t''_{-i-1}$, $f''_{-i-1} := f'''_{-i-1} \circ t''_{-i-1}$, $s'_{-i-1} := s''_{-i-1} \circ t''_{-i-1}$. Then we have the following commutative diagram in A:

$$\begin{array}{ccc}
X^{-i-1} & \xrightarrow{\partial_{-i-1}} & X^{-i} \\
{\scriptstyle s'_{-i-1}}\uparrow & & \uparrow{\scriptstyle s'_{-i}} \\
X''^{-i-1} & \xrightarrow{\partial'_{-i-1}} & X''^{-i} \\
{\scriptstyle f''_{-i-1}}\downarrow & & \downarrow{\scriptstyle f''_{-i}} \\
Y^{-i-1} & \xrightarrow[d_{-i-1}]{} & Y^{-i},
\end{array}$$

where $s'_{-i-1} \in \phi(C)$, and we have $f_{-i-1} \circ Qs'_{-i-1} = Qf''_{-i-1}$ and $\partial'_{-i} \circ \partial'_{-i-1} = 0$. It is easy to see that $X''^{\cdot} = (X''^{i}, \partial'_{i})$ is a complex in $K^b(A)$ and $s' = (s'_i)$, $f'' = (f''_i)$ are morphism in $K^b(A)$ such that $f \circ Qs' = Qf''$.

(b) Any morphism $f: X^{\cdot} \to Y^{\cdot}$ in $D^b(A/C)$ is represented by the following diagram in $K^b(A/C)$:

$$\begin{array}{ccccc}
 & & X_1^{\cdot} & & \\
 & {\scriptstyle t}\swarrow & & \searrow{\scriptstyle f'} & \\
X^{\cdot} & & & & Y^{\cdot},
\end{array}$$

where t is a quasi-isomorphism. According to (a), it is easy to see that there exist morphisms $t': X_2^{\cdot} \to X^{\cdot}$ and $s: X_2^{\cdot} \to X_1^{\cdot}$ in $K^b(A)$ such that $t \circ Qs = Qt'$, Qs is an isomorphism in $K^b(A/C)$, and Qt' is a quasi-isomorphism, and that there exist morphisms $f'': X_3^{\cdot} \to Y^{\cdot}$ and $s': X_3^{\cdot} \to X_1^{\cdot}$ in $K^b(A)$ such that $f' \circ Qs' = Qf''$ and Qs' is an isomorphism in $K^b(A/C)$. We have the following morphism of distinguished triangles in $K^b(A)$:

$$\begin{array}{ccccccc} X_4^{\cdot} & \xrightarrow{r} & X_3^{\cdot} & \longrightarrow & Z^{\cdot} & \longrightarrow & \\ {\scriptstyle t''}\downarrow & & \downarrow{\scriptstyle s'} & & \| & & \\ X_2^{\cdot} & \xrightarrow[s]{} & X_1^{\cdot} & \longrightarrow & Z^{\cdot} & \longrightarrow & . \end{array}$$

By $QZ^{\cdot}=0$ in $K^b(A/C)$, r and t'' are isomorphisms in $K^b(A/C)$. Then $Q(t'\circ t'')$ is a quasi-isomorphism in $K^b(A/C)$. Taking by the same symbols induced morphisms from $K^b(A)$ into $D^b(A)$, $f\circ r$ and $t'\circ t''$ are morphisms in $D^b(A)$ such that $t'\circ t''\in\phi(D_C^*(A))$ and $f\circ Q^b(t'\circ t'')=f\circ Rt\circ Q^b(s\circ t'')=Rf'\circ Q^b(s'\circ r)=Q^b(f''\circ r)$, where $R\colon K^b(A/C)\to D^b(A/C)$ is a natural quotient. Hence Q'^b is full.

(II). *The Case of* $*=-$. Let $X^{\cdot}\colon\cdots\to X^{-n}\to X^{-n+1}\to\cdots\to X^0\to 0\to\cdots$ be a complex in $D^-(A/C)$, and $X_i^{\cdot}\colon\cdots\to 0\to X^{-i}\to X^{-i+1}\to\cdots\to X^0\to 0\to\cdots$ a truncated complex in $D^b(A/C)$. Then, by (I), there exists a complex $X_i'^{\cdot}$ in $D^b(A)$ such that $s_i\colon Q'^bX_i'^{\cdot}\simeq X_i^{\cdot}$. Moreover, for a natural inclusion $X_i^{\cdot}\subsetneq X_{i+1}^{\cdot}$, we have a commutative diagram

$$\begin{array}{ccc} Q'^bX_i'^{\cdot} & \hookrightarrow & Q'^bX_{i+1}'^{\cdot} \\ {\scriptstyle s_i}\wr\| & & \|\wr{\scriptstyle s_{i+1}} \\ X_i^{\cdot} & \hookrightarrow & X_{i+1}^{\cdot}. \end{array}$$

Hence $Q'^-X'^{\cdot}=\varinjlim Q'^bX_i'^{\cdot}\simeq X^{\cdot}$, where $X'^{\cdot}=\varinjlim X_i'^{\cdot}$. For any morphism $f\colon X^{\cdot}\to Y^{\cdot}$ in $K^-(A/C)$, we have a commutative diagram

$$\begin{array}{ccc} X_i^{\cdot} & \xrightarrow{f_i} & Y_i^{\cdot} \\ \downarrow & & \downarrow \\ X^{\cdot} & \xrightarrow[f]{} & Y^{\cdot}. \end{array}$$

According to (I), there exist a complex $Z_i^{\cdot}$ and morphisms $s_i'\colon Z_i^{\cdot}\to X_i^{\cdot}$ and $f_i'\colon X'^{\cdot}\to Y'^{\cdot}$ of complexes in $K^b(A)$ such that $f_i\circ Qs_i'=Qf_i'$ and Qs_i' is an isomorphism in $K^b(A/C)$. Moreover, for all i, we have the following commutative diagram:

$$\begin{array}{ccccc} X_i^{\cdot} & \xleftarrow{s_i'} & Z_i^{\cdot} & \xrightarrow{f_i'} & Y_i^{\cdot} \\ \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\ X_{i+1}^{\cdot} & \xleftarrow[s_{i+1}']{} & Z_{i+1}^{\cdot} & \xrightarrow[f_{i+1}']{} & Y_{i+1}^{\cdot}. \end{array}$$

Then we have $f \circ Qs' = Qf'$, where $Qs' = \varinjlim Qs'_i$ is an isomorphism in $K^b(A/C)$ and $Qf' = \varinjlim Qf'_i$. As well as (I), Q'^- is also full.

(III). *The Case of* $* = +$. By (I) with the arrows reversed and the dual of (II), it is trivial.

Remark. By the proof of Theorem 3.2, $K^*(A) \xrightarrow{Q^*} K^*(A/C)$ is also a quotient functor, where $* = +, -$, or b.

COROLLARY 3.3. *Let* $0 \to C \to A \to A/C \to 0$ *be a localization* $\{A/C; Q, T\}$ *of* A. *Assume that* A/C *has enough injectives. Then* $0 \to D^+_C(A) \to D^+(A) \to D^+(A/C) \to 0$ *is localization exact, that is,* $\{D^+(A); Q^+, R^+T\}$ *is a localization of* $D^+(A)$.

Proof. For any $Y^{\cdot} \in D^+(A/C)$, there exists a complex $I^{\cdot} = (I^i, d_i) \in K^+(A/C)$ where all I^i are injective such that $Y^{\cdot} \simeq I^{\cdot}$ in $D^+(A/C)$. Then, given any $X^{\cdot} \in D^+(A)$ and $Y^{\cdot} \in D^+(A/C)$, we have $\mathrm{Hom}_{D^+(A/C)}(Q^+X^{\cdot}, Y^{\cdot}) \simeq \mathrm{Hom}_{K^+(A/C)}(Q^+X^{\cdot}, I^{\cdot}) \simeq \mathrm{Hom}_{K^+(A)}(X^{\cdot}, TI^{\cdot})$. Since S is exact and T is the right adjoint of S, $TI^{\cdot} = (TI^i, Td_i)$ is a complex in $K^+(A)$, where all TI^i are injective. Then we have $\mathrm{Hom}_{K^+(A)}(X^{\cdot}, TI^{\cdot}) \simeq \mathrm{Hom}_{D^+(A)}(X^{\cdot}, Q^+Y^{\cdot})$. Then R^+T is the right adjoint of Q^+. According to Theorem 3.2 and Proposition 2.6, we are done.

4. LOCALIZATION OF DERIVED CATEGORIES OF MODULES

Equivalences of derived categories of modules were considered in [8, 4, 14]. For a ring A, we denote by Mod A (resp., mod A) the category of right A-modules (resp. finitely presented right A-modules). According to Theorem 3.2 for a finitely generated projective A-module P, we have $0 \to D^*_{\mathrm{Ker}\,Q}(\mathrm{Mod}\,A) \to D^*(\mathrm{Mod}\,A) \xrightarrow{Q^*} D^*(\mathrm{Mod}\,B) \to 0$ is exact, where $B = \mathrm{End}_A(P)$, $Q = \mathrm{Hom}_A(P, ?)$, and $* = +, -$, or b. Moreover, Q^+ (resp., Q^-) is a localization (resp., a colocalization) cf. [5, Proposition 2.1]. Let T be a right A-module such that: (a) $0 \to P_n \to \cdots \to P_1 \to P_0 \to T \to 0$ is exact, where all P_i are finitely generated projective; (b) $\mathrm{Ext}^i_A(T, T) = 0$ for any $i > 0$. Cline, Pasrshall, and Scott showed that $\{D^-(\mathrm{Mod}\,B); L^-(? \otimes_B T), R^-\mathrm{Hom}_A(T, ?)\}$ is a colocalization of $D^-(\mathrm{Mod}\,A)$, where $B := \mathrm{End}_A(T)$, and that if $\mathrm{pdim}_B T < \infty$, then $\{D^b(\mathrm{Mod}\,B); L^b(? \otimes_B T), R^b\mathrm{Hom}_A(T, ?)\}$ is a colocalization of $D^b(\mathrm{Mod}\,A)$ (see [4, (4.2)]). In this section, we consider quotient and localization of derived categories of modules categories for rings.

For a complex $X^{\cdot} := (X^i, d_i)$, we define the following truncations [9, I, Sect. 7]:

$$\sigma_{>n}(X^{\cdot}): \cdots \to 0 \to \mathrm{Im}\, d_n \to X^{n+1} \to X^{n+2} \to \cdots,$$

$$\sigma_{\leqq n}(X^{\cdot}): \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Ker}\, d_n \to 0 \to \cdots.$$

For $m \leqq n$, we denote by $D^{[m,n]}(\text{Mod } A)$ a full subcategory of $D^b(\text{Mod } A)$ generated by complexes of which homology $H^i = 0$ ($i < m$ or $n < i$).

LEMMA 4.1. *Let $F: C \to D$ be a ∂-functor between triangulated categories. If there exists a family T of objects in C satisfying the following conditions:*

(a) *For every $X \in C$, there exists an object $T_X \in T$ and a morphism $s_X: T_X \to X$ such that Z belongs to* Ker F, *where $T_X \to X \to Z \to$ is a distinguished triangle;*

(b) *For X and $Y \in C$, there exists a morphism $f': T_X \to Y$ in C such that $f \circ Fs_Y = Ff'$ for any $f \in \text{Hom}_D(FX, FY)$;*

(c) *For every $Y \in D$, there exists an object $X \in T$ such that $Y \simeq FX$;*

then $0 \to \text{Ker } F \to C \to D \to 0$ *is exact.*

Proof. It is clear by Lemma 3.1.

PROPOSITION 4.2. *Let T be a right A-module with a finitely generated projective resolution, $B := \text{End}_A(T)$, and $F := \text{Hom}_A(T, ?)$: $\text{Mod } A \to \text{Mod } B$. Assume that T satisfies the following conditions:*

(a) $\text{Ext}^i_A(T, T) = 0$ ($i \geqq 1$),

(b) $\text{pdim } T_A \leqq 1$.

Then $0 \to \text{Ker } R^bF \to D^b(\text{Mod } A) \xrightarrow{R^bF} D^b(\text{Mod } B) \to 0$ *is exact.*

Proof. Let T be a family of complexes $X^\cdot: \cdots \to 0 \to X^m \to \cdots \to X^{n-1} \to X^n \to 0 \to \cdots$ (for all $m \leqq n$) $\in K^b(\text{Mod } A)$, where $X^m \in$ F-rac A and X^i is direct sums of T ($m < i \leqq n$). It suffices to show that T satisfies the conditions of Lemma 4.1. Since $\text{Ext}^i_A(T, T) = 0$ ($i > 0$) and pdim $T \leqq 1$, if X is generated by T, then $\text{Ext}^i_A(T, X) = 0$. The condition (b) implies the existence of R^bF. Since $D^b(\text{Mod } B) \simeq K^{-,b}(\text{Free } B)$, given $Y^\cdot \in D^{[m,n]}(\text{Mod } B)$, there exists $T^\cdot \in K^-(\text{Sum } T)$ such that $FT^\cdot \xrightarrow{\sim} Y^\cdot$ in $D^b(\text{Mod } B)$. Furthermore, $F(\sigma_{>t} T^\cdot) \xrightarrow{\sim} \sigma_{>t}(FT^\cdot) \xrightarrow{\sim} FT^\cdot$ ($t < m-1$). Since Im d_t is generated by T, $\sigma_{>t}(T^\cdot)$ is in T. Then (c) of Lemma 4.1 is satisfied. Given $X^\cdot \in D^{[m,n]}(\text{Mod } A)$, there exists $I^\cdot \in K^{[m,n+1]}$(F-rac A) such that $X^\cdot \simeq I^\cdot$ in $D^b(\text{Mod } A)$. For $FI^\cdot$, there exist $T^\cdot \in K^-(\text{Sum } T)$ and $f: T^\cdot \to I^\cdot$ in $K^-(\text{Mod } A)$ such that Ff is an isomorphism in $D^-(\text{Mod } B)$. Similarly, $f = T^\cdot \to \sigma_{>t}(T^\cdot) \xrightarrow{g} I^\cdot$ and Fg is an isomorphism in $D^b(\text{Mod } B)$ ($t < m-1$). Then (a) of Lemma 4.1 is satisfied. Moreover, for every $X^\cdot$ and $Y^\cdot \in D^{[m,n]}(\text{Mod } A)$, there exist $T^\cdot$ and $T'^\cdot \in K^-(\text{Sum } T)$ such that $FT^\cdot \simeq X^\cdot$ and $FT'^\cdot \simeq Y^\cdot$, and then

$$\text{Hom}_{D^b(\text{Mod } B)}(R^bF(X^\cdot), R^bF(Y^\cdot)) \simeq \text{Hom}_{D^b(\text{Mod } B)}(FT^\cdot, FT'^\cdot).$$

Since $FT^{\cdot}$ and $FT'^{\cdot} \in D^{[m,n+1]}(\text{Mod } B)$, for $t < m-1$, we have $\text{Hom}_{D^-(\text{Mod } B)}(FT^{\cdot}, \sigma_{\leq t}(FT'^{\cdot})[i]) = 0$ (for all i), then

$$\begin{aligned}\text{Hom}_{D^-(\text{Mod } B)}(FT^{\cdot}, FT'^{\cdot}) &\simeq \text{Hom}_{D^-(\text{Mod } B)}(FT^{\cdot}, \sigma_{>t}(FT'^{\cdot})) \\ &\simeq \text{Hom}_{K^-(\text{Mod } B)}(FT^{\cdot}, \sigma_{>t}(FT'^{\cdot}))\end{aligned}$$

and

$$\text{Hom}_{K^-(\text{Mod } B)}(FT^{\cdot}, \sigma_{>t}(FT'^{\cdot})) \simeq \text{Hom}_{K^b(\text{Mod } B)}(\sigma_{>t-1}(FT^{\cdot}), \sigma_{>t}(FT'^{\cdot}))$$

Hence we have

$$\simeq \text{Hom}_{K^b(\text{Mod } A)}(\sigma_{>t-1} T^{\cdot}, \sigma_{>t} T'^{\cdot}).$$

The condition (b) of Lemma 4.1 is satisfied.

PROPOSITION 4.3. *Let A and B be semiprimary rings and F: Mod $A \to$ Mod B a left exact additive functor. Assume that $R^{+,b}F$ has image in $D^b(\text{Mod } B)$ and that $R^{+,b}F$: $D^b(\text{Mod } A) \to D^b(\text{Mod } B)$ is a colocalization. Then there exists a right B-A-bimodule T such that*: (a) $F \simeq \text{Hom}_A(T, ?)$; (b) $B \simeq \text{End}_A(T)$; (c) $\text{Ext}^i_A(T, T) = 0$ $(i \geqq 1)$; (d) pdim T_A, $\text{pdim}_B T < \infty$. *Furthermore, $R^{+,b}F \simeq R^bF$ and L^bG is the left adjoint of R^bF, where $G = ? \otimes_B T$.*

Proof. There exists a left adjoint G of F such that $L^{-,b}G$: $D^b(\text{Mod } B) \to D^b(\text{Mod } A)$ is the left adjoint of $R^{+,b}F$, by [4, (3, 1) Lemma]. Let $T := GB$, then T is a B-A-bimodule such that $F \simeq \text{Hom}_A(T, ?)$ and $G \simeq ? \otimes_B T$. Let J_A and J_B be jacobson radicals of A and B, respectively. Since $R^{+,b}F$ has image in $D^b(\text{Mod } B)$, $R^{+,b}F(A/J_A)$ is in $D^b(\text{Mod } A)$, and then there exists an integer n such that $R^iF(A/J_A) \simeq \text{Ext}^i_A(T, A/J_A) = 0$ $(i > n)$. By [1, Proposition 7], pdim $T_A < \infty$. Similarly, $\text{pdim}_B T < \infty$. Then $R^{+,b}F \simeq R^bF$ and $L^{-,b}G \simeq L^bG$. Next, since R^bF is a colocalization and since $T \simeq L^bG(B)$, $B \simeq R^bF \circ L^bG(B) \simeq R^bF(T)$, and T is Ker R^bF-coclosed, by the dual of Lemma 2.2. Hence $R^iF(T) \simeq \text{Ext}^i_A(T, T) = 0$ $(i \neq 0)$, and $B \simeq \text{End}_B(B) \simeq \text{End}_A(T)$ as rings, by the dual of Lemma 2.1.

COROLLARY 4.4. *Under the condition of Proposition* 4.5, *we have* gl dim $B \leqq$ gl dim $A + \text{pdim}_B T$.

Proof. Since L^bG: $D^b(\text{Mod } B) \to D^b(\text{Mod } A)$ is fully faithful, for any B-modules M, N, we have

$$\begin{aligned}\text{Ext}^i_B(M, N) &\simeq \text{Hom}_{D^b(\text{Mod } B)}(M, N[i]) \\ &\simeq \text{Hom}_{D^b(\text{Mod } A)}(L^bG(M), L^bG(N)[i]).\end{aligned}$$

Let $\text{pdim}_B T := n$, then $L^bG(M)$ and $L^bG(N)$ are in $D^{[-n,0]}(\text{Mod B})$. Hence we have $\text{Hom}_{D^b(\text{Mod } A)}(L^bG(M), L^bG(N)[i]) = 0$ for $i > \text{gl dim } A + n$.

5. Application to Finite Dimensional Algebras

In this section, we consider only the case of finite dimensional algebras over a fixed field k. Let $D := \text{Hom}_k(?, k)$, then D is a duality between mod A and A-mod, where A-mod is the category of finitely generated left A-modules. Then D induces the duality, which we use the symbol D, between $D^*(\text{mod } A)$ and $D^\#(A\text{-mod})$, where $(*, \#) = (+, -), (-, +)$, or (b, b), by $(DX^\cdot)^i = DX^{-i}$, where $X^\cdot = (X^i, d_i)$. For a finite dimensional algebra A, we know that the Grothendieck group of mod A is isomorphic to a free abelian group which has the complete set of non-isomorphic indecomposable projective A-modules as a basis. We denote by Grot($\mathcal{A}$) the Grothendieck group of $\mathcal{A}$, where $\mathcal{A}$ is an abelian category or a triangulated category. Here, we use $\text{Grot}(\text{mod } A) \simeq \text{Grot}(D^b(\text{mod } A))$ and Proposition of Grothendieck (see [7] for details).

Proposition 5.1. *Let T be a finitely generated right A-module such that*: (a) $\text{Ext}^1_A(T, T) = 0$; (b) $\text{pdim } T_A \leqq 1$. Then $0 \to \text{Ker } R^bF \to D^b(\text{mod } A) \xrightarrow{R^bF} D^b(\text{mod } B) \to 0$ *is exact.*

Proof. It is trivial by Proposition 4.2.

Remark. According to Bongartz's lemma [3, 2.1 Lemma] and an equivalence of derived categories ([4, (2.1) Theorem] or [14, Theorem 4.1.1]), we get another proof of Proposition 5.1 by Theorem 3.2.

Proposition 5.2. *Let A and B be finite dimensional algebras, F: mod $A \to$ mod B a left exact additive functor. Then $R^{+,b}F$ has image in $D^b(\text{mod } B)$ and $R^{+,b}F\colon D^b(\text{mod } A) \to D^b(\text{mod } B)$ is a colocalization if and only if there exists a finitely generated B-A-bimodule T such that*:

(a) $F \simeq \text{Hom}_A(T, ?)$,

(b) $B \simeq \text{End}_A(T)$,

(c) $\text{Ext}^i_A(T, T) = 0$ *for all* $i > 0$,

(d) $\text{pdim } T_A$, $\text{pdim}_B T < \infty$.

Proof. By Proposition 4.3 and [4, (4.2)], it is clear.

PROPOSITION 5.3. *Let* $\pi: A \to C$ *be a ring homomorphism between finite dimensional algebras,* $E := ?\otimes_C C_A$: mod $C \to$ mod A, *and* $E' := {}_AC\otimes_C ?$: C-mod $\to A$-mod. *Then the following are equivalent.*

(a) $E: D^-(\text{mod } C) \to D^-(\text{mod } A)$ *has a left adjoint* $\hat{G}$ *such that* $\{D^-(\text{mod } C); \hat{G}, E\}$ *is a localization of* $D^-(\text{mod } A)$.

(b) $E: D^+(\text{mod } C) \to D^+(\text{mod } A)$ *has a right adjoint* $\hat{F}$ *such that* $\{D^+(\text{mod } C); E, \hat{F}\}$ *is a colocalization of* $D^+(\text{mod } A)$.

(c) $E': D^-(C\text{-mod}) \to D^-(A\text{-mod})$ *has a left adjoint* $\hat{G}'$ *such that* $\{D^-(C\text{-mod}); \hat{G}', E'\}$ *is a localization of* $D^-(A\text{-mod})$.

(d) $E': D^+(C\text{-mod}) \to D^+(A\text{-mod})$ *has a right adjoint* $\hat{F}'$ *such that* $\{D^+(C\text{-mod}); E', \hat{F}'\}$ *is a colocalization of* $D^+(A\text{-mod})$.

(e) π *is a ring epimorphism, and* $\text{Tor}_i^A(C, C) = 0$ *for all* $i > 0$. *Moreover, in this case,* $\text{Ext}_A^i(C_A, C_A) = \text{Ext}_A^i({}_AC, {}_AC) = 0$ *for all* $i > 0$.

Proof. It is well known that π is a ring epimorphism if and only if the natural morphism $C\otimes_A C \to C$ is an isomorphism as an C-C-bimodule morphism. If π is a ring epimorphism, then the natural ring morphism $C \to \text{End}(C_A)$ is an isomorphism (see [15]).

(e)$\Rightarrow$(a). Let $G := ?\otimes_A C_C$, then G is the left adjoint of E. For $X^\cdot \in D^-(\text{mod } A)$, there exists a complex $P^\cdot \in K^-(\text{free } A)$ such that $X^\cdot \simeq P^\cdot$ in $D^-(\text{mod } A)$. Given $Y^\cdot \in D^-(\text{mod } C)$, we have

$$\begin{aligned} \text{Hom}_{D^-(\text{mod } A)}(X^\cdot, EY^\cdot) &\simeq \text{Hom}_{K^-(\text{mod } A)}(P^\cdot, EY^\cdot) \\ &\simeq \text{Hom}_{K^-(\text{mod } C)}(GP^\cdot, Y^\cdot). \end{aligned}$$

Since E is exact and G is the left adjoint of E, $GP^\cdot$ is in $K^-(\text{proj } C)$, where proj C is a category of finitely generated projective C-modules. Then we have

$$\text{Hom}_{D^-(\text{mod } A)}(X^\cdot, EY^\cdot) \simeq \text{Hom}_{D^-(\text{mod } C)}(L^-G(X^\cdot), Y^\cdot).$$

And for $Y^\cdot \in D^-(\text{mod } C)$, there exists a complex $Q^\cdot \in K^-(\text{free } C)$ such that $Y^\cdot \simeq Q^\cdot$ in $D^-(\text{mod } C)$. Since C_A is left G-acyclic and $C\otimes_A C \simeq C$, we have $E\circ L^-G(Y)^\cdot) \simeq E\circ G(Q^\cdot) \simeq Q^\cdot$. Hence $E\circ L^-G \simeq 1_{D^-(\text{mod } C)}$.

(a)$\Rightarrow$(e). By the above, we have $\hat{G} \simeq L^-G$, where $G = ?\otimes_A C_C$. Then $L^-G\circ E(C) \xrightarrow{\sim} C$ in $D^-(\text{mod } C)$, and hence the natural morphism $C\otimes_A C \to C$ is an isomorphism and $\text{Tor}_i^A(C, C) = 0$ for all $i > 0$.

(c)$\Leftrightarrow$(e). This is similar to (a)$\Leftrightarrow$(e).

(a)$\Leftrightarrow$(d) and (b)$\Leftrightarrow$(c). Since $DED \simeq E'$ and $DE'D \simeq E$, they are trivial by the duality. Since $C \xrightarrow{\sim} R^+\,\text{Hom}_A(C_A, ?)\circ E(C)$ and $C \xrightarrow{\sim} R^+\,\text{Hom}_A({}_AC, ?)\circ E'(C)$ in $D^+(\text{mod } C)$ and in $D^+(C\text{-mod})$ by (b) and (d), respectively, we have $\text{Ext}_A^i(C_A, C_A) = \text{Ext}_A^i({}_AC, {}_AC) = 0$ for all $i > 0$.

THEOREM 5.4. *Let* $\pi: A \to C$ *be a ring homomorphism between finite dimensional algebras,* $E := ? \otimes_C C_A$: mod $C \to$ mod A, *and* $E' := {}_A C \otimes_C ?$: C-mod $\to$ A-mod. *Then the following are equivalent.*

(a) $E: D^b(\text{mod } C) \to D^b(\text{mod } A)$ *has a right adjoint* $\hat{F}$ *such that* $\{D^b(\text{mod } C); E, \hat{F}\}$ *is a colocalization of* $D^b(\text{mod } A)$.

(b) $E': D^b(C\text{-mod}) \to D^b(A\text{-mod})$ *has a left adjoint* $\hat{G}'$ *such that* $\{D^b(C\text{-mod}); \hat{G}', E'\}$ *is a localization of* $D^b(A\text{-mod})$.

(c) (i) *The natural morphism* $C \to \text{End}(C_A)$ *is an isomorphism as a ring*; (ii) pdim $C_A < \infty$; *and* (iii) $\text{Ext}^i_A(C_A, C_A) = 0$ *for all* $i > 0$.

(d) (i) π *is a ring epimorphism*; (ii) pdim $C_A < \infty$; *and* (iii) $\text{Tor}^A_i(C, C) = 0$ *for all* $i > 0$.

Proof. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(d) and (a)$\Rightarrow$(c). They are trivial by Proposition 5.3.

(c)$\Rightarrow$(a). This is trivial by Proposition 5.2.

For a finitely generated right A-module T_A, let add T_A be the full subcategory of mod A generated by direct summands of finite direct summs of T_A.

COROLLARY 5.5. *Let* A *be a finite dimensional algebra,* T *a finitely generated right* A*-module,* $B := \text{End}(T_A)$, *and* $C := \text{End}({}_B T)^{op}$. *Assume that* T *satisfies the following conditions*:

(a) $\text{Ext}^i_A(T, T) = 0$ $(i \geqq 1)$,

(b) pdim $T_A < \infty$,

(c) *there exists an exact sequence* $0 \to C \to T_0 \to T_1 \to \cdots \to T_n \to 0$ *in* mod A, *where all* T_i *are in* add T_A.

Then $R^{+,b}F$ *has image in* $D^b(\text{mod } B)$ *and* $R^{+,b}F$: $D^b(\text{mod } A) \to D^b(\text{mod } B)$ *is a colocalization. In this case,* ${}_B T$ *is a left tilting* B*-module with finite projective dimension, in the sense of* [11], *and* C *satisfies the conditions of Theorem* 5.4.

Proof. First, it suffices to show that pdim ${}_B T < \infty$. By the conditions (a) and (c), we have the following exact sequence in B-mod: $0 \to \text{Hom}_A(T_n, T) \to \cdots \to \text{Hom}_A(T_0, T) \to \text{Hom}_A(C, T) \to 0$. It is easy to see that ${}_B T$ is a direct summand of $\text{Hom}_A(C, T)$, and that all $\text{Hom}_A(T_i, T)$ are left projective B-modules. Then pdim ${}_B T \leqq n$. Next, applying $\text{Hom}_B(?, T)$ to the above sequence, we get

$$\begin{array}{ccccccc}
\text{Hom}_B(\text{Hom}_A(C, T), T) & \to & \text{Hom}_B(\text{Hom}_A(T_0, T), T) & \to & \cdots & \to & \text{Hom}_B(\text{Hom}_A(T_n, T), T) \\
\wr\| & & \wr\| & & & & \wr\| \\
0 \to C & \longrightarrow & T_0 & \longrightarrow & \cdots & \longrightarrow & T_n \longrightarrow 0.
\end{array}$$

Then $\mathrm{Ext}^i_B(\mathrm{Hom}_A(C, T), T) = 0$, and $\mathrm{Ext}^i_B(T, T) = 0$ for all $i > 0$. By the condition (b), we have a projective resolution of T in $\mathrm{mod}\, A$: $0 \to P_m \to \cdots \to P_1 \to P_0 \to T \to 0$. Applying $\mathrm{Hom}_A(?, T)$ to it, we get an exact sequence in B-mod:

$$0 \to \mathrm{Hom}_A(T, T) \to \mathrm{Hom}_A(P_0, T) \to \mathrm{Hom}_A(P_1, T) \to \cdots \to \mathrm{Hom}_A(P_m, T) \to 0,$$

where $B = \mathrm{Hom}_A(T, T)$ and all $\mathrm{Hom}_A(P_i, T)$ are in add T_A. Hence ${}_BT$ is a left tilting B-module with finite projective dimension. Then it is easy to see that $\{D^b(\mathrm{mod}\, C); ? \otimes_A C_C, R^b \mathrm{Hom}_A(C_A, ?)\}$ is a colocalization of $D^b(\mathrm{mod}\, A)$.

COROLLARY 5.6. *Under the condition of Proposition* 5.2, *we have* $\mathrm{gl\,dim}\, B \leqq \mathrm{gl\,dim}\, A + \mathrm{pdim}_B\, T$.

Proof. It is trivial by Corollary 4.4.

For a finitely generated A-module M, let $n(M)$ be a number of non-isomorphic indecomposable modules which are direct summands of M.

COROLLARY 5.7. *Let T be a finitely generated right A-module such that*: (a) $\mathrm{Ext}^i_A(T, T) = 0$ $(i \geqq 1)$; (b) $\mathrm{pdim}\, T_A$, $\mathrm{pdim}\, {}_BT < \infty$, *where* $B = \mathrm{End}_A(T)$. *Then we have* $n(T) \leqq n(A)$.

Proof. According to Proposition 5.2, $0 \to \mathrm{Ker}\, R^bF \to D^b(\mathrm{mod}\, A) \to D^b(\mathrm{mod}\, B) \to 0$ is a colocalization. Then, by [7, Sect. 3], we have the following split exact sequence:

$$0 \to \mathrm{Grot}(\mathrm{Ker}\, R^bF) \to \mathrm{Grot}(D^b(\mathrm{mod}\, A)) \to \mathrm{Grot}(D^b(\mathrm{mod}\, B)) \to 0.$$

Since $\mathrm{Grot}(D^b(\mathrm{mod}\, A)) \simeq \mathrm{Grot}(\mathrm{mod}\, A) \simeq Z^{n(A)}$ and $\mathrm{Grot}(D^b(\mathrm{mod}\, B)) \simeq \mathrm{Grot}(\mathrm{mod}\, B) \simeq Z^{n(T)}$, we have $n(T) \leqq n(A)$.

COROLLARY 5.8. *Let T be a finitely generated right A-module such that*: (a) $\mathrm{Ext}^i_A(T, T) = 0$ $(i \geqq 1)$; (b) $\mathrm{pdim}\, T_A < \infty$; (c) *there exists an exact sequence* $0 \to C \to T_0 \to T_1 \to \cdots \to T_n \to 0$ *in* $\mathrm{mod}\, A$, *with* $T_i \in \mathrm{add}\, T$ *for all* i, *where* $C = \mathrm{Biend}(T_A)^{op}$. *Then we have* $n(C) = n({}_BT) = n(T_A) \leqq n(A)$.

Cline, Parshall, and Scott studied relations between a derived category of modules of a ring and the one of its residue ring [12, Theorem 2.7; 5, Sects. 1, 2]. We consider relations between derived categories and idempotent ideals.

PROPOSITION 5.9. *Let A be a finite dimensional algebra, e an idempotent of A, I*: $\mathrm{mod}\, A/AeA \to \mathrm{mod}\, A$ *a natural inclusion, and Q*: $\mathrm{mod}\, A \to \mathrm{mod}\, eAe$ *a natural quotient. Then the following are equivalent.*

(a) $0 \to D^b(\text{mod } A/AeA) \xrightarrow{I^b} D^b(\text{mod } A) \xrightarrow{Q^b} D^b(\text{mod } eAe) \to 0$ *is exact.*

(b) $\text{Tor}_i^A(A/AeA, A/AeA) = 0$ *for all* $i > 0$.

(c) $\text{Tor}_i^A(AeA, AeA) = 0$ *for all* $i > 0$, (ii) $AeA \otimes_A AeA \simeq AeA$.

Proof. (a) ⇒ (b). According to (a), I^b is fully faithful. Then

$$\begin{aligned}\text{Tor}_i^A(A/AeA, A/AeA) &\simeq \text{DExt}_A^i(A/AeA_A, D(A/AeA)_A) \\ &\simeq \text{DHom}_{D^b(\text{mod } A/AeA)}(A/AeA, D(A/AeA)[i]) \\ &= 0 \qquad \text{for all} \quad i > 0.\end{aligned}$$

(b) ⇒ (a). By Proposition 5.3, I^b is fully faithful. According to [5, (1.3)], $D^b(\text{mod } A/AeA) \simeq D^b_{\text{mod } A/AeA}(\text{mod } A)$. Then we are done by Theorem 3.2.

(b) ⇔ (c). This is easy.

Remark. As well as [12, Theorem 2.7], it is easy to see that $0 \to D^-(\text{mod } A/AeA) \xrightarrow{I^-} D^-(\text{mod } A) \xrightarrow{Q^-} D^-(\text{mod } eAe) \to 0$ is colocalization exact if and only if (i) $\text{Tor}_i^{eAe}(Ae, eA) = 0$ for all $i > 0$, and (ii) $Ae \otimes_{eAe} eA \simeq AeA$. In this case, we have $\text{Tor}_i^A(A/AeA, A/AeA) = 0$ for all $i > 0$.

THEOREM 5.10. *The following are equivalent.*

(a) $0 \to D^b(\text{mod } A/AeA) \xrightarrow{I^b} D^b(\text{mod } A) \xrightarrow{Q^b} D^b(\text{mod } eAe) \to 0$ *is colocalization exact.*

(b) (i) $\text{Tor}_i^A(A/AeA, A/AeA) = 0$ *for all* $i > 0$, (ii) $\text{pdim}\, _A AeA < \infty$.

(c) (i) $\text{Ext}_A^i({}_A A/AeA, {}_A A/AeA) = 0$ *for all* $i > 0$, (ii) $\text{pdim}\, _A AeA < \infty$.

(d) (i) $\text{Tor}_i^A(AeA, AeA) = 0$ *for all* $i > 0$, (ii) $AeA \otimes_A AeA \simeq AeA$, (iii) $\text{pdim}\, _A AeA < \infty$.

(e) (i) $\text{Tor}_i^{eAe}(Ae, eA) = 0$ *for all* $i > 0$, (ii) $Ae \otimes_{eAe} eA \simeq AeA$, (iii) $\text{pdim}\, _{eAe} eA < \infty$.

Proof. (a) ⇔ (b) ⇔ (c). According to Section 2, this is trivial by Theorem 5.4 and Proposition 5.9.

(b) ⇔ (d). This is easy.

(a) ⇔ (e). See [12, Theorem 2.7].

COROLLARY 5.11. *The following are equivalent.*

(a) $0 \to D^b(\text{mod } A/AeA) \xrightarrow{I^b} D^b(\text{mod } A) \xrightarrow{Q^b} D^b(\text{mod } eAe) \to 0$ *is bilocalization exact (i.e.,* $\{D^b(\text{mod } A/AeA), D^b(\text{mod } A), D^b(\text{mod } eAe)\}$ *is recollement.)*

(b) (i) $\mathrm{Tor}_i^A(A/AeA, A/AeA)=0$ *for all* $i>0$, (ii) $\mathrm{pdim}\,{}_A AeA$, $\mathrm{pdim}\, AeA_A < \infty$.

(c) (i) $\mathrm{Ext}^i_A(A/AeA_A, A/AeA_A)=0$ *for all* $i>0$, (ii) $\mathrm{pdim}\,{}_A AeA$, $\mathrm{pdim}\, AeA_A < \infty$.

(c′) (i) $\mathrm{Ext}^i_A({}_A A/AeA, {}_A A/AeA)=0$ *for all* $i>0$, (ii) $\mathrm{pdim}\,{}_A AeA$, $\mathrm{pdim}\, AeA_A < \infty$.

(d) (i) $\mathrm{Tor}_i^A(AeA, AeA)=0$ *for all* $i>0$, (ii) $AeA \otimes_A AeA \simeq AeA$, (iii) $\mathrm{pdim}\,{}_A AeA$, $\mathrm{pdim}\, AeA_A < \infty$.

(e) (i) $\mathrm{Tor}_i^{eAe}(Ae, eA)=0$ *for all* $i>0$, (ii) $Ae \otimes_{eAe} eA \simeq AeA$, (iii) $\mathrm{pdim}_{eAe}\, eA$, $\mathrm{pdim}\, Ae_{eAe} < \infty$.

Proof. By Theorem 5.10 and its dual, this is clear.

Remark. (1) Under the condition of Proposition 5.2, global dimensions of A or B are not necessarily finite. Indeed, let A be a finite dimensional algebra over a field k with the following quiver with relations

$$a \xleftarrow{\ \delta\ } \overset{\overset{\alpha}{\circlearrowleft}}{b} \xleftarrow{\ \beta\ } c \xleftarrow{\ \gamma\ } d$$

with $\delta \circ \alpha = \alpha^2 = \delta \circ \beta = \beta \circ \gamma = 0$. Then $\mathrm{gl\,dim}\, A = \infty$. Let $T := I(c) \oplus (I(c)/S(c))$, where $S(c)$ is a simple right A-module corresponding with a vertex c, and $I(c)$ is an injective hull of $S(c)$. Then $\mathrm{pdim}\, T_A = 2$ and $\mathrm{Ext}^i_A(T, T)=0$ for all $i>0$. Next, $B := \mathrm{End}_A(T)$ has a quiver with a relation: $e \to f \circlearrowleft \zeta$ with $\zeta^2 = 0$. Then we have $\mathrm{gl\,dim}\, B = \infty$ and $\mathrm{pdim}\,{}_B T = 1$. Hence $R^b\,\mathrm{Hom}_A(T, ?)\colon D^b(\mathrm{mod}\, A) \to D^b(\mathrm{mod}\, B)$ is a colocalization functor which has $L^b(? \otimes_B T)$ as a cosection functor.

(2) The above example satisfies the conditions of Corollary 5.5.
(3) Under the conditions of Proposition 5.2, when we know if $\mathrm{Ker}\, R^bF$ is not zero, then $\mathrm{Grot}(\mathrm{Ker}\, R^bF)$ is not zero (for example, A is hereditary), $D^b(\mathrm{mod}\, A)$ is equivalent to $D^b(\mathrm{mod}\, B)$ if and only if $n(T) = n(A)$.

References

1. M. Auslander, On the dimension of modules and algebras, III, *Nagoya Math J.* **9** (1955), 67–77.
2. A. A. Beilinson, P. Bernstein, and P. Deligne, Faisceaux pervers, *Astérisque* **100** (1982).
3. K. Bongartz, "Tilted Algebras," pp. 26–38, Lecture Notes in Math., Vol. 903, Springer-Verlag, Berlin, 1982.
4. E. Cline, B. Parshall, and L. Scott, Derived categories and Morita theory, *J. Algebra* **104** (1986), 397–409.
5. E. Cline, B. Parshall, and L. Scott, Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* **391** (1988), 85–99.

6. P. GABRIEL, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.
7. A. GROTHENDIECK, "Groupes de classes des catégories abéliennes et triangulees," pp. 351–371, Lecture Notes in Math., Vol. 589, Springer-Verlag, Berlin, 1977.
8. D. HAPPEL, On the derived category of a finite-dimensional algebra, *Comment. Math. Helv.* **62** (1987), 339–389.
9. R. HARTSHORNE, "Residue and Duality," Lecture Notes in Math., Vol. 20, Springer-Verlag, Berlin, 1966.
10. B. KELLER AND D. VOSSIECK, Sous les catégories dérivées, *C. R. Acad. Sci. Paris* **305** (1987), 225–228.
11. Y. MIYASHITA, Tilting modules of finite projective dimension, *Math. Z.* **193** (1986), 113–146.
12. B. PARSHALL AND L. SCOTT, Derived categories, quasi-hereditary algebras, and algebraic groups, preprint, Lect. at Ottawa-Moonsonee Workshop in Algebra, 1987.
13. N. POPESCU, "Abelian Categories with Applications to Rings and Modules," Academic Press, London/New York, 1973.
14. J. RICKARD, Equivalences of derived categories of modules, preprint.
15. B. STENSTRÖM, "Rings of Quotients: An Introduction to Methods of Ring Theory," Springer-Verlag, Berlin, 1975.
16. J. VERDIER, "Catégories dérivées, état 0," pp. 262–311, Lecture Notes in Math., Vol. 569, Springer-Verlag, Berlin, 1977.