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A complete classification of quintic space curves with rational rotation-minimizing frames

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ABSTRACT

An adapted orthonormal frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ on a space curve $\mathbf{r}(t)$, where $\mathbf{f}_1 = \mathbf{r}'/|\mathbf{r}'|$ is the curve tangent, is *rotation-minimizing* if its angular velocity satisfies $\boldsymbol{\omega} \cdot \mathbf{f}_1 \equiv 0$, i.e., the normal-plane vectors $\mathbf{f}_2, \mathbf{f}_3$ exhibit no instantaneous rotation about $\mathbf{f}_1$. The simplest space curves with *rational* rotation-minimizing frames (RRMF curves) form a subset of the quintic spatial *Pythagorean-hodograph* (PH) *curves*, identified by certain non-linear constraints on the curve coefficients. Such curves are useful in motion planning, swept surface constructions, computer animation, robotics, and related fields. The condition that identifies the RRMF quintics as a subset of the spatial PH quintics requires a rational expression in four quadratic polynomials $u(t), v(t), p(t), q(t)$ and their derivatives to be reducible to an analogous expression in just two polynomials $a(t), b(t)$. This condition has been analyzed, thus far, in the case where $a(t), b(t)$ are also quadratic, the corresponding solutions being called *Class I* RRMF quintics. The present study extends these prior results to provide a complete categorization of all possible PH quintic solutions to the RRMF condition. A family of *Class II* RRMF quintics is thereby newly identified, that correspond to the case where $a(t), b(t)$ are linear. Modulo scaling/rotation transformations, Class II curves have five degrees of freedom, as with the Class I curves. Although Class II curves have rational RMFs that are only of degree 6 – as compared to degree 8 for Class I curves – their algebraic characterization is more involved than for the latter. Computed examples are used to illustrate the construction and properties of this new class of RRMF quintics. A novel approach

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for generating RRMF quintics, based on the sum-of-four-squares decomposition of positive real polynomials, is also introduced and briefly discussed.

1. Introduction

Let $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ be an *adapted* orthonormal frame on a space curve $\mathbf{r}(t)$, such that $\mathbf{f}_1$ coincides with the curve tangent $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ while $\mathbf{f}_2, \mathbf{f}_3$ span the normal plane at each curve point. The variation of such a frame may be specified by its angular velocity $\boldsymbol{\omega}$ through the differential equations

$$\mathbf{f}_1' = \boldsymbol{\omega} \times \mathbf{f}_1, \qquad \mathbf{f}_2' = \boldsymbol{\omega} \times \mathbf{f}_2, \qquad \mathbf{f}_3' = \boldsymbol{\omega} \times \mathbf{f}_3,$$

and the characteristic property of a *rotation-minimizing frame* (RMF) is that its angular velocity satisfies $\boldsymbol{\omega} \cdot \mathbf{f}_1 \equiv 0$, i.e., $\mathbf{f}_2, \mathbf{f}_3$ have no instantaneous rotation about $\mathbf{f}_1$. Such frames are of great interest in applications concerned with controlling the orientation of a rigid body along a spatial trajectory; for example, in swept surface constructions, computer animation, and robot path planning (Farouki and Han, 2003; Farouki et al., in press; Jüttler and Mäurer, 1999; Klok, 1986; Sír and Jüttler, 2005; Wang and Joe, 1997; Wang et al., 2008).

Recent studies (Farouki, 2010; Farouki et al., 2009, in press; Farouki and Sakkalis, 2010; Farouki and Sakkalis , 2011) have established the possibility of constructing *rational* rotation-minimizing frames on a special class of polynomial space curves of minimum degree 5 — the so-called *RRMF curves*. Such curves are necessarily Pythagorean-hodograph (PH) curves (Farouki, 2008), since only PH curves admit rational unit tangents. The RRMF curves can thus be characterized through the identification of constraints on the coefficients of PH curves, that are sufficient and necessary for a rational RMF.

The focus of this paper is on classifying the lowest-degree (quintic) *polynomial* curves with rational RMFs. An approach for generating *rational* curves with rational RMFs has recently been proposed in Bartoň et al. (2010), based upon the observation that Möbius transformations in $\mathbb{R}^3$ preserve the PH property of (polynomial or rational) curves, and the rotation-minimizing nature of rational adapted frames defined on them. For example, degree 6 rational PH curves with rational RMFs can be generated from Möbius transformations of planar PH cubics (whose Frenet frames are rational and trivially rotation-minimizing). The polynomial RRMF curves analyzed herein are more fundamental, since the RRMF property is intrinsic to their algebraic structure, rather than a consequence of its maintenance when such curves are imaged by an RRMF-preserving map.

Also, the possibility that rational RMFs may exist on space curves that are not PH curves, but nevertheless have rational unit tangents, is not addressed here. Planar curves with this property are known to exist, the simplest example arising from a rational quadratic parameter transformation applied to the parabola, which results in an improper (doubly-traced) parameterization (Arrondo et al., 1997; Farouki and Sederberg, 1995; Lü, 1995). Because of the absence of a complete characterization of *space* curves with this property (analogous to that presented in Lü (1995) for the planar case), and the improper parameterizations, no further consideration of this possibility is made here.

A polynomial PH space curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ is characterized (Farouki, 2008) by the property that its derivative components satisfy, for some polynomial $\sigma(t)$, the Pythagorean condition

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t). \tag{1}$$

The *quaternion* and *Hopf map* forms (Choi et al., 2002) are two convenient models for the construction of spatial PH curves. The former generates a Pythagorean hodograph $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ from a quaternion polynomial[2]

$$\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}, \tag{2}$$

[2] Calligraphic characters such as $\mathcal{A}$ denote quaternions, their scalar and vector parts being indicated by $\mathrm{scal}(\mathcal{A})$ and $\mathrm{vect}(\mathcal{A})$. Bold symbols denote complex numbers or vectors in $\mathbb{R}^3$ — the meaning should be clear from the context.

and its conjugate $\mathcal{A}^*(t) = u(t) - v(t)\,\mathbf{i} - p(t)\,\mathbf{j} - q(t)\,\mathbf{k}$ through the product

$$\begin{aligned}\mathbf{r}'(t) &= \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) = [u^2(t) + v^2(t) - p^2(t) - q^2(t)]\,\mathbf{i} \\ &\quad + 2\,[u(t)q(t) + v(t)p(t)]\,\mathbf{j} + 2\,[v(t)q(t) - u(t)p(t)]\,\mathbf{k}.\end{aligned} \tag{3}$$

The latter employs two complex polynomials

$$\boldsymbol{\alpha}(t) = u(t) + \mathrm{i}\,v(t), \qquad \boldsymbol{\beta}(t) = q(t) + \mathrm{i}\,p(t) \tag{4}$$

to generate a Pythagorean hodograph through the expression

$$\mathbf{r}'(t) = (|\boldsymbol{\alpha}(t)|^2 - |\boldsymbol{\beta}(t)|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)), 2\,\mathrm{Im}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t))). \tag{5}$$

The equivalence of (3) and (5) is seen by setting $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t)$, and identifying the imaginary unit i with the quaternion element $\mathbf{i}$. See Farouki (2008) for a thorough treatment of these two representations. The parametric speed $\sigma(t) = |\mathbf{r}'(t)|$ of the PH curve $\mathbf{r}(t)$ defined by integrating $\mathbf{r}'(t)$ — i.e., the derivative of its arc length s with respect to the parameter t — is the polynomial

$$\sigma(t) = |\mathcal{A}(t)|^2 = |\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2 = u^2(t) + v^2(t) + p^2(t) + q^2(t). \tag{6}$$

The *Euler–Rodrigues frame* (ERF) is a rational adapted frame, defined (Choi and Han, 2002) on any spatial PH curve by

$$(\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)) = \frac{(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t), \mathcal{A}(t)\,\mathbf{j}\,\mathcal{A}^*(t), \mathcal{A}(t)\,\mathbf{k}\,\mathcal{A}^*(t))}{|\mathcal{A}(t)|^2}, \tag{7}$$

that is a useful reference for identifying rational RMFs. Here, $\mathbf{e}_1(t)$ is the curve tangent while $\mathbf{e}_2(t)$, $\mathbf{e}_3(t)$ span the normal plane. The ERF is given explicitly in terms of the polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$ as

$$\begin{aligned}\mathbf{e}_1 &= \frac{(u^2 + v^2 - p^2 - q^2)\,\mathbf{i} + 2(uq + vp)\,\mathbf{j} + 2(vq - up)\,\mathbf{k}}{u^2 + v^2 + p^2 + q^2}, \\ \mathbf{e}_2 &= \frac{2(vp - uq)\,\mathbf{i} + (u^2 - v^2 + p^2 - q^2)\,\mathbf{j} + 2(uv + pq)\,\mathbf{k}}{u^2 + v^2 + p^2 + q^2}, \\ \mathbf{e}_3 &= \frac{2(up + vq)\,\mathbf{i} + 2(pq - uv)\,\mathbf{j} + (u^2 - v^2 - p^2 + q^2)\,\mathbf{k}}{u^2 + v^2 + p^2 + q^2}.\end{aligned} \tag{8}$$

Now if the PH curve defined by (3) or (5) admits a rational RMF $(\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t))$ then $\mathbf{e}_1 = \mathbf{f}_1$ is the curve tangent, and the normal-plane vectors $\mathbf{f}_2(t)$, $\mathbf{f}_3(t)$ must be obtainable from the ERF normal-plane vectors $\mathbf{e}_2(t)$, $\mathbf{e}_3(t)$ by a rational rotation — i.e., for relatively prime polynomials $a(t)$, $b(t)$ we must have

$$\begin{bmatrix}\mathbf{f}_2(t)\\ \mathbf{f}_3(t)\end{bmatrix} = \frac{1}{a^2(t) + b^2(t)}\begin{bmatrix}a^2(t) - b^2(t) & -2\,a(t)b(t)\\ 2\,a(t)b(t) & a^2(t) - b^2(t)\end{bmatrix}\begin{bmatrix}\mathbf{e}_2(t)\\ \mathbf{e}_3(t)\end{bmatrix}. \tag{9}$$

This is equivalent (Han, 2008) to the requirement that

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{ab' - a'b}{a^2 + b^2} \tag{10}$$

for relatively prime polynomials $a(t)$, $b(t)$. The expression on the left is just the component $\omega_1 = \boldsymbol{\omega}\cdot\mathbf{t}$ of the ERF angular velocity $\boldsymbol{\omega}$ in the direction of $\mathbf{e}_1 = \mathbf{f}_1$, while that on the right is the angular velocity of the normal-plane rotation (9) that maps $\mathbf{e}_2$, $\mathbf{e}_3$ onto $\mathbf{f}_2$, $\mathbf{f}_3$. Thus, the condition (10) requires the existence of a rational normal-plane rotation that exactly cancels the ω_1 component of the ERF angular velocity. In terms of the Hopf map representation, condition (10) is equivalent to requiring the existence of a complex polynomial $\mathbf{w}(t) = a(t) + \mathrm{i}\,b(t)$, with $\gcd(a(t), b(t)) = 1$, such that

$$\frac{\mathrm{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}')}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2} = \frac{\mathrm{Im}(\overline{\mathbf{w}}\mathbf{w}')}{|\mathbf{w}|^2}. \tag{11}$$

The simplest non-planar curves with rational RMFs are quintics (Farouki, 2010; Farouki et al., 2009; Han, 2008). To define a PH quintic through (3) or (5), the polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$ must be quadratic. Satisfaction of the RRMF condition (10) by PH quintics has thus far been considered only in the case where $a(t), b(t)$ are assumed to also be quadratic. In this case, simple constraints on the coefficients of the quaternion polynomial (2) or the complex polynomials (4) have been identified (Farouki, 2010) that are sufficient and necessary for a rational RMF. These were also shown (Farouki and Sakkalis, 2010) to be equivalent to a certain polynomial divisibility condition. The simplicity of the constraints identifying these RRMF quintics, together with (modulo scaling/rotation transformations) their five residual degrees of freedom, facilitates development of algorithms (Farouki et al., in press) for the design of rational rotation-minimizing rigid-body motions by the interpolation of initial/final positions and orientations.

This study extends and completes these prior results by enumerating a complete categorization of all solutions to (10) by PH quintics — i.e., when $u(t), v(t), p(t), q(t)$ are quadratic but $a(t), b(t)$ are of *unrestricted* degree. The most important outcome of this classification is the identification of a novel non-trivial family of RRMF quintics that satisfy (10) with $a(t), b(t)$ linear rather than quadratic. Because the rational normal-plane rotation in (9) is quadratic rather than quartic, this new class of RRMF quintics admit rational RMFs of lower degree than those for which $a(t), b(t)$ are quadratic, and they also have five essential degrees of freedom. However, their algebraic characterization appears to be inherently more complicated.

The plan for this paper is as follows. Section 2 introduces a reduction to *normal form*, which is used to determine simple criteria that identify degenerate (linear or planar) RRMF curves, allowing the subsequent analysis to focus on proper RRMF curves, i.e., true space curves. Section 3 further exploits the normal form reduction to facilitate the enumeration of all possible proper PH quintic solutions to the RRMF condition. In addition to the known class of RRMF quintics satisfying (10) with $\deg(a^2 + b^2) = 4$, this enumeration reveals the existence of a novel class of proper RRMF quintics satisfying (10) with $\deg(a^2 + b^2) = 2$, having the same number of freedoms as the previously-known solutions. Finally, Section 4 considers the "inverse" problem of generating RRMF curves from quadruples $u(t), v(t), p(t), q(t)$ obtained from the (infinitely many) decompositions $f(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t)$ of any given strictly positive real polynomial $f(t)$, while Section 5 summarizes the results of this paper, and identifies open problems for further investigation.

2. Degenerate PH curves

Since every straight line and every planar PH curve is trivially an RRMF curve, and we are interested in true space curves, instances of (3) or (5) that define straight lines or planar curves will be called *degenerate* spatial PH curves. We present here new criteria to identify such degenerate curves, based on Lemma 1 below and the fact that two real polynomials $f(t), g(t)$ are linearly dependent if and only if they satisfy $fg' = f'g$. These criteria are independent of non-essential coefficients, and are easy to test in practice.

As in earlier studies (Farouki, 2010), the analysis can be greatly simplified by invoking a scaling/rotation transformation to eliminate non-essential freedoms that do not influence the intrinsic nature of a spatial PH curve. We call this transformation *reduction to normal form*.

Lemma 1. *Let $\boldsymbol{\alpha}(t) = u(t) + \mathrm{i}\, v(t)$, $\boldsymbol{\beta}(t) = q(t) + \mathrm{i}\, p(t)$ be complex polynomials, where $u(t), v(t), p(t), q(t)$ are real polynomials of degree $m \geq 1$. Then complex values $\boldsymbol{\mu}, \boldsymbol{\nu}$ can be chosen such that, under the transformation*

$$\begin{bmatrix} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{bmatrix} \rightarrow \begin{bmatrix} \boldsymbol{\mu} & -\overline{\boldsymbol{\nu}} \\ \boldsymbol{\nu} & \overline{\boldsymbol{\mu}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{bmatrix}, \tag{12}$$

the polynomials $v(t), p(t), q(t)$ are of degree $m - 1$ at most.

Proof. If we write $\boldsymbol{\alpha}(t) = \mathbf{a}_m t^m + \cdots + \mathbf{a}_1 t + \mathbf{a}_0$ and $\boldsymbol{\beta}(t) = \mathbf{b}_m t^m + \cdots + \mathbf{b}_1 t + \mathbf{b}_0$ (where $\mathbf{a}_k = u_k + \mathrm{i}\, v_k$ and $\mathbf{b}_k = q_k + \mathrm{i}\, p_k$ for $k = 0, \ldots, m$) the coefficients transform according to

$$\begin{bmatrix} \mathbf{a}_k \\ \mathbf{b}_k \end{bmatrix} \rightarrow \begin{bmatrix} \boldsymbol{\mu} & -\overline{\boldsymbol{\nu}} \\ \boldsymbol{\nu} & \overline{\boldsymbol{\mu}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_k \\ \mathbf{b}_k \end{bmatrix}$$

for $k = 0, \ldots, m$. In particular, with the choices $\boldsymbol{\mu} = \overline{\mathbf{a}}_m / (|\mathbf{a}_m|^2 + |\mathbf{b}_m|^2)$ and $\boldsymbol{\nu} = -\mathbf{b}_m / (|\mathbf{a}_m|^2 + |\mathbf{b}_m|^2)$ we obtain $(\mathbf{a}_m, \mathbf{b}_m) \rightarrow (1, 0)$. $\square$

When $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ determine a PH curve $\mathbf{r}(t)$ with hodograph $\mathbf{r}'(t)$ specified by (5), the map (12) defines (Farouki, 2010) a scaling/rotation of the hodograph in $\mathbb{R}^3$, that does not alter its intrinsic nature. From Lemma 1, we may henceforth assume, without loss of generality, that $u(t) = t^m + \cdots + u_1 t + u_0$ while $v(t), p(t), q(t)$ are of degree $m-1$ at most. We call a quadruple of polynomials $(u(t), v(t), p(t), q(t))$ of this form *normal*.

Now $\mathbf{r}(t)$ is planar if and only if $x'(t), y'(t), z'(t)$ are linearly dependent. Then we observe from (3) that, in normal form, $x'(t)$ is of degree $2m$, while $y'(t), z'(t)$ are of degree $2m-1$ at most. Therefore, $\mathbf{r}(t)$ is planar if and only if $y'(t)$ and $z'(t)$ are linearly dependent, i.e., $y'z'' = y''z'$, which is equivalent to

$$(p^2+q^2)(uv'-u'v) = (u^2+v^2)(qp'-q'p). \tag{13}$$

Furthermore, $\mathbf{r}(t)$ is a straight line if and only if $p(t) = q(t) = 0$. Indeed, when $\mathbf{r}(t)$ is a line $x'(t), y'(t)$ and $x'(t), z'(t)$ are linearly dependent, respectively. But since $x'(t)$ is of degree $2m$ and $y'(t), z'(t)$ are of degree $2m-1$ at most, we must have $y'(t) = z'(t) = 0$, and this implies that $p(t) = q(t) = 0$, since $u^2(t) + v^2(t) \neq 0$. The converse is trivial. These results may be summarized as follows.

Proposition 1. *Let $\mathbf{r}(t)$ be a PH curve with hodograph defined by the normal quadruple $(u(t), v(t), p(t), q(t))$ as above. Then*

1. *$\mathbf{r}(t)$ is a plane curve, other than a straight line, if and only if (13) is satisfied with $(p(t), q(t)) \neq (0,0)$.*
2. *On the other hand, $\mathbf{r}(t)$ is a straight line if and only if $(p(t), q(t)) = (0,0)$.*

In normal form, a *degenerate* RRMF curve is either a straight line or planar curve that satisfies (13) and has vanishing torsion, while a *proper* RRMF curve is a true space curve that does not satisfy (13) and thus has non-vanishing torsion.

3. Classification of RRMF quintics

In previous studies (Farouki, 2010; Farouki et al., 2009; Farouki and Sakkalis, 2010) the RRMF quintics have been studied under the assumption that (10) and (11) are satisfied with $u^2+v^2+p^2+q^2 = a^2+b^2$ and $|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2 = |\mathbf{w}|^2$, respectively. Also, it was shown in Farouki and Sakkalis (2010) that $\deg(a^2+b^2) \le \deg(u^2+v^2+p^2+q^2)$ is a necessary condition for the satisfaction of (10). Thus, for RRMF quintics with $\deg(u,v,p,q) = 2$, the possible solutions to (10) may have (i) $\deg(a,b) = 2$, (ii) $\deg(a,b) = 1$, or (iii) $\deg(a,b) = 0$. Henceforth, we refer to these solutions as follows.

Definition 1. A PH curve defined by (3) with $\deg(u,v,p,q) = 2$ is called a Class I, II, or III RRMF quintic according to whether it satisfies (10) with $\deg(a,b) = 2$, 1, or 0.

It will be shown below that the Class III RRMF quintics are planar curves, while the Class I quintics have been thoroughly analyzed before Farouki (2010), and Farouki and Sakkalis (2010). The significant new outcome of this analysis is the existence of the novel family of Class II RRMF quintics, which includes true space curves.

3.1. Class I RRMF quintics

The spatial PH quintic curves that belong to this class correspond to the case where $a(t), b(t)$ are assumed to be quadratic – i.e., case (i) above – and hence

$$uv'-u'v-pq'+p'q = \gamma(ab'-a'b) \quad \text{and} \quad u^2+v^2+p^2+q^2 = \gamma(a^2+b^2)$$

for some non-zero constant γ (one may, without loss of generality, set $\gamma = 1$). It was shown by Farouki (2010) that, if the quadratic quaternion polynomial (2) or complex polynomials (4) are specified in the Bernstein basis

$$b_k^m(t) = \binom{m}{k}(1-t)^{m-k}t^k, \quad k = 0, \ldots, m,$$

on $t \in [0,1]$ as

$$\mathcal{A}(t) = \mathcal{A}_0\, b_0^2(t) + \mathcal{A}_1\, b_1^2(t) + \mathcal{A}_2\, b_2^2(t), \tag{14}$$

or

$$\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}_0\, b_0^2(t) + \boldsymbol{\alpha}_1\, b_1^2(t) + \boldsymbol{\alpha}_2\, b_2^2(t), \qquad \boldsymbol{\beta}(t) = \boldsymbol{\beta}_0\, b_0^2(t) + \boldsymbol{\beta}_1\, b_1^2(t) + \boldsymbol{\beta}_2\, b_2^2(t), \tag{15}$$

then imposition of the coefficient constraints

$$\mathrm{vect}(\mathcal{A}_2\, \mathbf{i}\, \mathcal{A}_0^*) = \mathcal{A}_1\, \mathbf{i}\, \mathcal{A}_1^* \tag{16}$$

or

$$\mathrm{Re}(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2 - \boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2) = |\boldsymbol{\alpha}_1|^2 - |\boldsymbol{\beta}_1|^2 \quad \text{and} \quad \boldsymbol{\alpha}_0\overline{\boldsymbol{\beta}}_2 + \boldsymbol{\alpha}_2\overline{\boldsymbol{\beta}}_0 = 2\,\boldsymbol{\alpha}_1\overline{\boldsymbol{\beta}}_1 \tag{17}$$

is sufficient and necessary for a proper RRMF curve satisfying (10) with $a(t), b(t)$ quadratic, or equivalently (11) with $\mathbf{w}(t)$ quadratic. Since these Class I RRMF quintics have been thoroughly analyzed before, Farouki (2010) and Farouki and Sakkalis (2010), we shall not dwell further on them here. Instead, we focus henceforth on the Class II and III RRMF quintics.

3.2. Class II RRMF quintics

We begin with the observation that a scaling/rotation transformation does not influence the RRMF nature of a spatial PH curve.

Lemma 2. *If the RRMF condition (11) is satisfied by complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ and $\mathbf{w}(t)$, it is also satisfied upon replacing them by $\boldsymbol{\mu}\,\boldsymbol{\alpha}(t) - \overline{\boldsymbol{\nu}}\,\boldsymbol{\beta}(t)$, $\boldsymbol{\nu}\,\boldsymbol{\alpha}(t) + \overline{\boldsymbol{\mu}}\,\boldsymbol{\beta}(t)$ and $\boldsymbol{\eta}\,\mathbf{w}(t)$, for any complex numbers $(\boldsymbol{\mu}, \boldsymbol{\nu}) \neq (0, 0)$ and $\boldsymbol{\eta} \neq 0$.*

Proof. For complex numbers $(\boldsymbol{\mu}, \boldsymbol{\nu}) \neq (0, 0)$ the linear map (12) applied to the polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ yields

$$\begin{aligned} |\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2 &\to (|\boldsymbol{\mu}|^2 + |\boldsymbol{\nu}|^2)\,(|\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2), \\ \overline{\boldsymbol{\alpha}}(t)\boldsymbol{\alpha}'(t) + \overline{\boldsymbol{\beta}}(t)\boldsymbol{\beta}'(t) &\to (|\boldsymbol{\mu}|^2 + |\boldsymbol{\nu}|^2)\,(\overline{\boldsymbol{\alpha}}(t)\boldsymbol{\alpha}'(t) + \overline{\boldsymbol{\beta}}(t)\boldsymbol{\beta}'(t)), \end{aligned}$$

and hence the left-hand side of (11) is unaltered. Similarly, we have $\mathrm{Im}(\overline{\mathbf{w}}(t)\mathbf{w}'(t)) \to |\boldsymbol{\eta}|^2\, \mathrm{Im}(\overline{\mathbf{w}}(t)\, \mathbf{w}'(t))$ and $|\mathbf{w}(t)|^2 \to |\boldsymbol{\eta}|^2\, |\mathbf{w}(t)|^2$ when $\mathbf{w}(t) \to \boldsymbol{\eta}\,\mathbf{w}(t)$, and consequently the right-hand side of (11) is also unchanged. □

Now from Lemma 1 we may henceforth assume, without loss of generality, that

$$u(t) = t^2 + u_1 t + u_0, \qquad v(t) = v_1 t + v_0, \qquad p(t) = p_1 t + p_0, \qquad q(t) = q_1 t + q_0. \tag{18}$$

We are primarily concerned here with case (ii), but before addressing it we quickly dispense with case (iii).

Case (iii): $\deg(a^2 + b^2) = 0$. Since $ab' - a'b = 0$, we deduce from (10) and (18) that $uv' - u'v - pq' + p'q = -v_1 t^2 - 2v_0 t + u_0 v_1 - u_1 v_0 - p_0 q_1 + p_1 q_0 = 0$, so we must have $v_1 = v_0 = u_0 v_1 - u_1 v_0 - p_0 q_1 + p_1 q_0 = 0$, which imply that $uv' - u'v = 0$, since $v(t) = 0$, and $qp' - q'p = p_1 q_0 - p_0 q_1 = 0$. But then condition (13) is satisfied, i.e., the curve is planar and is thus a degenerate RRMF curve. In fact, satisfaction of (10) with $\deg(a^2 + b^2) = 0$ identifies curves on which the ERF is rotation-minimizing and, as we have just seen, all quintics with this property are planar. The simplest non-planar curves with rotation-minimizing ERFs are (Choi and Han, 2002) of degree 7.

We focus henceforth on case (ii), which yields non-degenerate RRMF curves — i.e., true spatial PH quintics with rational rotation-minimizing frames. It transpires that, in normal form, these Class II RRMF quintics incorporate five free parameters — as with the Class I RRMF quintics satisfying (10) with $\deg(a^2 + b^2) = 4$. In prior studies (Farouki, 2010; Farouki et al., 2009; Farouki and Sakkalis, 2010) these latter curves were called "generic" RRMF quintics, in the expectation that solutions to (10) with $\deg(u^2 + v^2 + p^2 + q^2) = 4$ and $\deg(a^2 + b^2) < 4$ would comprise a "lower-dimension subspace" of the complete set of non-degenerate RRMF quintics. Since this is not the case, the terminology of Definition 1 is henceforth adopted. We now proceed with the analysis of the Class II RRMF quintics.

Case (ii): $\deg(a^2 + b^2) = 2$. Since $a(t), b(t)$ are linear and relatively prime, Lemma 2 indicates that we may, without loss of generality, assume $a(t) = t - r$, $b(t) = s$ for $r, s \in \mathbb{R}$ with $s \neq 0$. Let $w = uv' - u'v - pq' + p'q$ and $\sigma = u^2 + v^2 + p^2 + q^2$. Then (10) implies that

$$[(t - r)^2 + s^2]\, w(t) = -s\, \sigma(t). \tag{19}$$

Now from (18) we have

$$\begin{aligned} w(t) &= -\,v_1t^2 - 2v_0t + u_0v_1 - u_1v_0 - p_0q_1 + p_1q_0, \\ \sigma(t) \;&=\; t^4 + 2u_1t^3 + (2u_0 + u_1^2 + v_1^2 + p_1^2 + q_1^2)t^2 \\ &\quad + 2(u_0u_1 + v_0v_1 + p_0p_1 + q_0q_1)t + u_0^2 + v_0^2 + p_0^2 + q_0^2, \end{aligned} \tag{20}$$

and comparing like powers of t on the left and right sides in (19) yields

$$\begin{aligned} & s - v_1 = 0, \\ & 2u_1s + 2v_1r - 2v_0 = 0, \\ & (2u_0 + u_1^2 + v_1^2 + p_1^2 + q_1^2)s + u_0v_1 - u_1v_0 - p_0q_1 + p_1q_0 - v_1(r^2 + s^2) + 4v_0r = 0, \\ & 2(u_0u_1 + v_0v_1 + p_0p_1 + q_0q_1)s - 2(u_0v_1 - u_1v_0 - p_0q_1 + p_1q_0)r - 2v_0(r^2 + s^2) = 0, \\ & (u_0^2 + v_0^2 + p_0^2 + q_0^2)s + (u_0v_1 - u_1v_0 - p_0q_1 + p_1q_0)(r^2 + s^2) = 0. \end{aligned} \tag{21}$$

This is a system of five equations in ten variables, and its solutions may be characterized as follows.

Proposition 2. *The (real) solutions of the system (21) can be parameterized in terms of the free variables r, u_1, v_1, p_1, q_1 with $v_1 \neq 0$, as $s = v_1$ and either*

$$u_0 = -(u_1 + r)r, \qquad v_0 = (u_1 + r)v_1, \qquad p_0 = v_1q_1 - p_1r, \qquad q_0 = -(v_1p_1 + q_1r), \tag{22}$$

or

$$\begin{aligned} u_0 &= -\,(u_1 + r)r - \frac{4v_1^2(p_1^2 + q_1^2)}{(u_1 + 2r)^2 + 9v_1^2 + p_1^2 + q_1^2}, \\ v_0 &= (u_1 + r)v_1, \\ p_0 &= v_1q_1 - p_1r + \frac{4v_1^2((u_1 + 2r)p_1 - 3v_1q_1)}{(u_1 + 2r)^2 + 9v_1^2 + p_1^2 + q_1^2}, \\ q_0 &= -(v_1p_1 + q_1r) + \frac{4v_1^2((u_1 + 2r)q_1 + 3v_1p_1)}{(u_1 + 2r)^2 + 9v_1^2 + p_1^2 + q_1^2}. \end{aligned} \tag{23}$$

Proof. We first substitute $s = v_1$ and $v_0 = (u_1 + r)v_1$ from the first two of Eqs. (21) into the remaining three. From the third and fourth equations, we then obtain

$$(p_1^2 + q_1^2)\, p_0 = f_0 \quad \text{and} \quad (p_1^2 + q_1^2)\, q_0 = g_0,$$

where f_0, g_0 are polynomials in $u_0, u_1, v_1, p_1, q_1, r$. Assume for now that $p_1^2 + q_1^2 \neq 0$. Substituting for p_0, q_0 from the above into the fifth of Eqs. (21) and solving for u_0, we obtain the first expression in either (22) or in (23).

Substituting for u_0 from (22) into the third and fourth of Eqs. (21) then yields $p_0 = v_1q_1 - p_1r$ and $q_0 = -v_1p_1 - q_1r$, thus completing the solution (22). On the other hand, substituting from (23) for u_0 we obtain (through a judicious re-arrangement of terms) the expressions for p_0, q_0 given in (23).

Suppose now that $p_1^2 + q_1^2 = 0$. In that case, the third equation gives $u_0 = -(u_1 + r)r$ and on substituting this and $s = v_1$, $v_0 = (u_1 + r)v_1$ into the fifth equation we obtain $(p_0^2 + q_0^2)v_1 = 0$. Since $v_1 \neq 0$, we see that $p_0 = q_0 = 0$ as required by solution (22). Finally, note that the denominator in the expressions for u_0, p_0, q_0 in (23) is never zero, since by assumption $v_1 \neq 0$. This concludes the proof. □

For curves defined by the solution (22), we have

$$\begin{aligned} x'(t) &= [(t - r)^2 + v_1^2]\,[(t + u_1 + r)^2 - p_1^2 - q_1^2], \\ y'(t) &= 2q_1\,[(t - r)^2 + v_1^2]\,(t + u_1 + r), \\ z'(t) &= -\,2p_1\,[(t - r)^2 + v_1^2]\,(t + u_1 + r). \end{aligned}$$

Such curves are evidently *planar* and *non-primitive*, since $x'(t), y'(t), z'(t)$ are linearly dependent and have a non-constant common factor. The curves defined by the solution (23), on the other hand, are primitive and are true space curves with $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' \neq 0$.

Defining the complex numbers $\mathbf{a}_k = u_k + \mathrm{i}\, v_k$ and $\mathbf{b}_k = q_k + \mathrm{i}\, p_k$ for $k = 0, 1, 2$ and writing $\boldsymbol{\gamma} = -r + v_1\mathrm{i}$, $\boldsymbol{\zeta} = \mathbf{a}_1 - 2\,\overline{\boldsymbol{\gamma}}$, $\boldsymbol{\eta} = \mathbf{b}_1$, the solutions (23) that define proper Class II RRMF quintics can be more compactly expressed as

$$\mathbf{a}_0 = (\mathbf{a}_1 - \boldsymbol{\gamma})\boldsymbol{\gamma} - \frac{4v_1^2|\boldsymbol{\eta}|^2}{|\boldsymbol{\zeta}|^2 + |\boldsymbol{\eta}|^2}, \qquad \mathbf{b}_0 = \boldsymbol{\gamma}\,\boldsymbol{\eta} + \frac{4v_1^2\overline{\boldsymbol{\zeta}}\boldsymbol{\eta}}{|\boldsymbol{\zeta}|^2 + |\boldsymbol{\eta}|^2}. \tag{24}$$

Since they provide a means of generating Class II RRMF quintics in terms of one real and two complex free parameters – r and $\mathbf{a}_1, \mathbf{b}_1$ – the relations (24) might be considered analogous to the generating formulas for Class I RRMF quintics specified in Proposition 1 of Farouki (2010). However, they are obviously more complicated than Eqs. (15) in Farouki (2010), and have thus far eluded a reduction to simple sufficient-and-necessary coefficient constraints, analogous to (16) and (17) for Class I curves. This problem deserves further attention, but at present it seems clear that Class I RRMF quintics have a simpler algebraic structure than Class II.

Remark 1. On a PH quintic, the ERF vectors (8) are quartic rational functions of the curve parameter. For Class I RRMF quintics, satisfying (10) with $\deg(a(t), b(t)) = 2$, the RMF normal-plane vectors defined by (9) are rational functions of degree 8. Since the solution (23) identifies RRMF quintics satisfying (10) with $\deg(a(t), b(t)) = 1$, the RMF vectors (9) on these Class II RRMF quintics are only of degree 6.

Example 1. Choosing the values $r = 1, u_1 = -1, v_1 = 2, p_1 = 0, q_1 = -2$ in (23) gives

$$s = 2, \qquad u_0 = -\frac{64}{41}, \qquad v_0 = 0, \qquad p_0 = \frac{28}{41}, \qquad q_0 = \frac{50}{41},$$

and hence we have

$$u(t) = t^2 - t - \frac{64}{41}, \qquad v(t) = 2\,t, \qquad p(t) = \frac{28}{41}, \qquad q(t) = -2\,t + \frac{50}{41}$$

and

$$a(t) = t - 1, \qquad b(t) = 2,$$

which satisfy

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{ab' - a'b}{a^2 + b^2} = \frac{-2}{t^2 - 2\,t + 5}.$$

The resulting hodograph components

$$x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) = t^4 - 2\,t^3 - \frac{87}{41}\,t^2 + 8\,t + \frac{812}{1681},$$
$$y'(t) = 2\,[u(t)q(t) + v(t)p(t)] = -4\,t^3 + \frac{264}{41}\,t^2 + \frac{268}{41}\,t - \frac{6400}{1681},$$
$$z'(t) = 2\,[v(t)q(t) - u(t)p(t)] = -\frac{384}{41}\,t^2 + \frac{256}{41}\,t + \frac{3584}{1681},$$

define a primitive curve with $\gcd(x'(t), y'(t), z'(t)) = 1$ and they satisfy $x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t)$, where

$$\sigma(t) = (t^2 - 2\,t + 5)\left(t^2 + \frac{36}{41}\right).$$

The hodograph defines a true space curve, as can be verified from the fact that condition (13) is not satisfied, and $\mathbf{r}(t)$ has the non-constant torsion

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{-7872}{(t^2 - 2\,t + 5)^2(1681\,t^2 - 738\,t + 2997)},$$

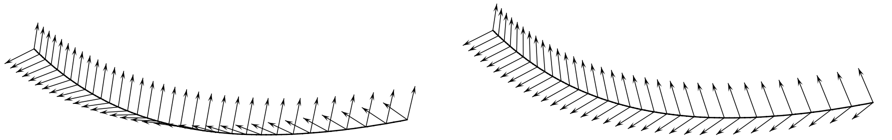

Fig. 1. The RRMF quintic $\mathbf{r}(t)$ of Example 1 for $t \in [1.5, 2.5]$, showing the variation of the principal normal and binormal vectors of the Frenet frame (left), and the rational RMF normal-plane vectors along the curve (right). The initial RMF orientation is chosen to agree with the Frenet frame at the left point.

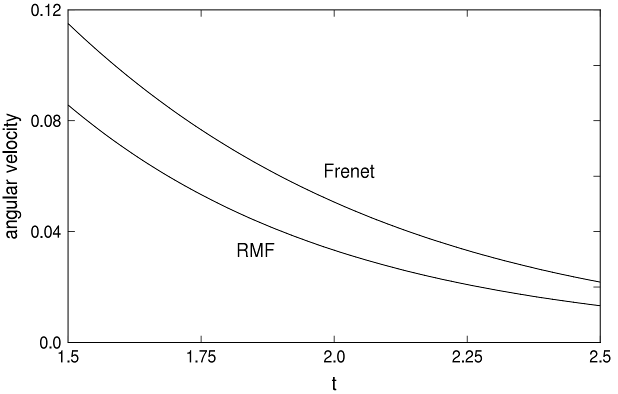

Fig. 2. Comparison of angular speed for the Frenet frame and the rational RMF along the RRMF quintic of Example 1, over the parameter interval $t \in [1.5, 2.5]$ illustrated in Fig. 1.

while the curvature is given by

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\,\mathbf{r}'\,|^3} = \frac{164\sqrt{(t^2 - 2\,t + 5)(1681\,t^2 - 738\,t + 2997)}}{(t^2 - 2\,t + 5)^2(41\,t^2 + 36)^2}.$$

The rational RMF normal-plane vectors, obtained from (8) and (9), are of degree 6 in the curve parameter t. Fig. 1 compares these vectors with the Frenet frame normal-plane vectors, over the interval $t \in [1.5, 2.5]$. The angular speed of the Frenet frame and the rational RMF are compared over the same interval in Fig. 2.

Example 2. Choosing the values $r = 2, u_1 = -2, v_1 = 1, p_1 = 2, q_1 = 1$ in (23) gives

$$s = 1, \qquad u_0 = -\frac{10}{9}, \qquad v_0 = 0, \qquad p_0 = -\frac{25}{9}, \qquad q_0 = -\frac{20}{9},$$

and thus we obtain

$$u(t) = t^2 - 2\,t - \frac{10}{9}, \qquad v(t) = t, \qquad p(t) = 2t - \frac{25}{9}, \qquad q(t) = t - \frac{20}{9}$$

and

$$a(t) = t - 2, \qquad b(t) = 1,$$

so that

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{ab' - a'b}{a^2 + b^2} = \frac{-1}{t^2 - 4\,t + 5}.$$

The resulting hodograph components

$$x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) = t^4 - 4\,t^3 - \frac{20}{9}\,t^2 + 20\,t + \frac{925}{81},$$

$$y'(t) = 2\,[u(t)q(t) + v(t)p(t)] = 2\,t^3 - \frac{40}{9}\,t^2 + \frac{10}{9}\,t + \frac{400}{81},$$

$$z'(t) = 2\,[v(t)q(t) - u(t)p(t)] = -\,4\,t^3 + \frac{140}{9}\,t^2 - \frac{100}{9}\,t - \frac{500}{81},$$

define a primitive curve with $\gcd(x'(t), y'(t), z'(t)) = 1$ and they satisfy $x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t)$, where

$$\sigma(t) = (t^2 - 4\,t + 5)\left(t^2 + \frac{25}{9}\right).$$

Again, this example defines a true space curve, with the curvature and torsion functions

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\,\mathbf{r}'\,|^3} = \frac{18\sqrt{5\,(t^2 - 4\,t + 5)(81\,t^2 - 180\,t + 325)}}{(t^2 - 4\,t + 5)^2(9\,t^2 + 25)^2},$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{-\,108}{(t^2 - 4\,t + 5)^2(81\,t^2 - 180\,t + 325)}.$$

Finally, although case (i) with $\deg(a^2 + b^2) = 4$ has been thoroughly treated by Farouki (2010) and Farouki et al. (2009), for completeness we briefly consider the special instance of this case where $\gcd(w, \sigma)$ is of degree 2, i.e., $w(t)$ is a factor of $\sigma(t)$.

Remark 2. Let $\mathbf{r}(t)$ be an RRMF quintic defined by the polynomials (18), such that (10) is satisfied with $\deg(a^2 + b^2) = 4$ and $\gcd(w, \sigma)$ is of degree 2. Then $p(t) = q(t) = 0$.

Proof. In this case, we must have $w = ab' - a'b$ and $\sigma = a^2 + b^2$, so that $\gcd(ab' - a'b, a^2 + b^2)$ is also of degree 2. Since $\gcd(ab' - a'b, a^2 + b^2) = \gcd(2(aa' + bb'), a^2 + b^2)$ by Lemma 4.1 of Farouki and Sakkalis (2010), $a^2 + b^2$ must have complex conjugate roots $\mathbf{z}, \bar{\mathbf{z}} = r \pm \mathrm{i}\, s$ (where $r, s \in \mathbb{R}$ and $s \neq 0$) of multiplicity 2 each. Therefore,

$$\sigma(t) = a^2(t) + b^2(t) = [(t - r)^2 + s^2]^2. \tag{25}$$

Also, $(ab' - a'b)/(a^2 + b^2)$ must have the form given in Eq. (12) of Farouki and Sakkalis (2010), namely

$$\frac{ab' - a'b}{a^2 + b^2} = \pm\,\frac{\mathrm{i}}{2}\left[\frac{2}{t - \mathbf{z}} - \frac{2}{t - \bar{\mathbf{z}}}\right].$$

Consequently, $w = ab' - a'b$ may be written as

$$w(t) = \pm\,2s\,[(t - r)^2 + s^2]. \tag{26}$$

Choosing the $-$ sign above and comparing (25)–(26) with (20) we obtain, after simplification, the equations

$$\begin{aligned}
&v_1 - 2s = 0,\\
&v_0 + 2rs = 0,\\
&u_1 + 2r = 0,\\
&2su_0 - p_0q_1 + p_1q_0 - 2r^2s + 2s^3 = 0,\\
&2u_0 + p_1^2 + q_1^2 - 2r^2 + 2s^2 = 0,\\
&2ru_0 - p_0p_1 - q_0q_1 - 2r^3 + 2rs^2 = 0,\\
&u_0^2 + p_0^2 + q_0^2 - r^4 + 2r^2s^2 - s^4 = 0.
\end{aligned}$$

We claim that $p_1^2 + q_1^2 = 0$. Assume the contrary. Then, solving the fourth and sixth equations for p_0, q_0 and taking into account the fifth equation, we obtain $p_0 = -(sq_1 + rp_1)$, $q_0 = sp_1 - rq_1$.

Substituting for p_0, q_0 into the seventh equation, and using the fifth equation again, gives the quadratic

$$u_0^2 - 2(r^2 + s^2)u_0 + r^4 + 2r^2s^2 - 3s^4 = 0$$

for u_0, with solutions $u_0 = r^2 - s^2$ and $u_0 = r^2 + 3s^2$. However, since both solutions contradict the fifth equation, we must have $p_1 = q_1 = 0$. In that case, the fifth and seventh equations give $p_0 = q_0 = 0$, so that $p(t) = q(t) = 0$, as claimed. Clearly, $\mathbf{r}(t)$ is then just a straight line (the x-axis). $\square$

Table 1
Classification and properties of low-degree RRMF curves.

n	Type	$\deg(u, v, p, q)$	$\deg(a, b)$	RMF degree
1	Straight line	0	0	0
3	Planar PH cubic	1	0	2
5	Class I RRMF quintic	2	2	8
5	Class II RRMF quintic	2	1	6
5	Planar PH quintic	2	0	4

In conclusion, Table 1 summarizes all RRMF curves of degree $n \leq 5$ generated by (3) or (5). Straight lines are obtained by choosing constants for $u(t), v(t), p(t), q(t)$ so that $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ is a constant vector, and any unit vectors $\mathbf{f}, \mathbf{g}$ such that $\mathbf{f} \times \mathbf{g} = \mathbf{t}$ then define a rational RMF. As shown in Han (2008), the only cubic RRMF curves – corresponding to linear polynomials $u(t), v(t), p(t), q(t)$ – are planar PH cubics. Finally, the case where $u(t), v(t), p(t), q(t)$ are quadratic generates the simplest non-planar RRMF curves, designated Class I or Class II quintics according to whether (10) is satisfied with $\deg(a, b) = 2$ or 1, respectively, while the case of quintics satisfying (10) with $\deg(a, b,) = 0$ corresponds (see Section 3.2) to planar PH quintics.

Note also that additional RRMF curves can be generated by multiplying (3) or (5) with a scalar polynomial $w(t)$. However, such curves have non-primitive hodographs (i.e., $\gcd(x'(t), y'(t), z'(t)) \neq 1$) and to obtain a *regular* curve satisfying $|\mathbf{r}'(t)| \neq 0$ for all t, one must ensure that $w(t)$ has no real roots. If $\deg(w) = l$ and $\deg(u, v, p, q) = m$, the resulting curves are of degree $l + 2m + 1$. For $m = 0$ this generates only non-uniformly parameterized straight lines, since multiplying by $w(t)$ cannot change the fixed direction of $\mathbf{r}'(t)$. Similarly, it generates only planar curves for $m = 1$, since multiplying $\mathbf{r}'(t)$ by $w(t)$ cannot alter the fact that $\mathbf{n} \cdot \mathbf{r}'(t) = 0$, where $\mathbf{n}$ is the normal to the plane in which the PH cubic $\mathbf{r}(t)$ resides. Hence, spatial RRMF curves generated by this approach must be of degree $n > 5$.

4. Inverse problem for RRMF quintics

Thus far, we have investigated the conditions on a quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ (assumed primitive, i.e., $\gcd(u(t), v(t), p(t), q(t)) = 1$) that ensure satisfaction of the RRMF condition (10). Note that, since $\mathcal{A}(t)$ is primitive, the polynomial $|\mathcal{A}(t)|^2 = u^2(t) + v^2(t) + p^2(t) + q^2(t)$ is a *positive* real polynomial of even degree.

Now any positive polynomial $f(t)$ of degree $2m$ can be expressed as a sum of squares of four polynomials,

$$f(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t), \tag{27}$$

in infinitely many ways. This follows from the well-known fact (No. 44, Part VI in Pólya and Szegö (1976)) that such polynomials can always be expressed as a sum of squares of two polynomials, $f(t) = g^2(t) + h^2(t)$ with $g(t)h(t) \neq 0$, since by defining the quaternion polynomial $\mathcal{A}(t) = g(t) + (\lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k})\,h(t)$ we see that $f(t) = |\mathcal{A}(t)|^2$ satisfies (27) with $(u(t), v(t), p(t), q(t)) = (g(t), \lambda\,h(t), \mu\,h(t), \nu\,h(t))$ for any $(\lambda, \mu, \nu) \in \mathbb{R}^3$ when $\lambda^2 + \mu^2 + \nu^2 = 1$.

Motivated by the above observation, we say that $f(t)$ *generates* an RRMF curve $\mathbf{r}(t)$ if a primitive quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ satisfying (10) exists, such that $f(t) = |\mathcal{A}(t)|^2$ and $\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$. It seems natural, then, to pose the following question.

Question 1. Does a given positive real polynomial $f(t)$ of degree $2m$ generate any RRMF curves? If so, are they *proper* or *degenerate* (see Section 2)? Moreover, what are the necessary conditions (if any) on $f(t)$, and its root structure?

We now address this question in the particular case of *Class I* RRMF curves, when $f(t)$ is square-free and $m = 2$ (the complete analysis of this problem is deferred to a future study). Let $f(t) = t^4 + f_3t^3 + f_2t^2 + f_1t + f_0$. By the change of variables $t \to t - \frac{1}{4}f_3$ we may assume, without loss of generality, that $f_3 = 0$. Then we must have $f(t) = [(t-r)^2 + k^2]\,[(t+r)^2 + l^2]$ for $r, k, l \in \mathbb{R}$ with $k\,l \neq 0$ and $r^2 + (k+l)^2 > 0$.

Proposition 3. *For $f(t)$ as above, the Class I RRMF curves it generates may be categorized as follows*

(a) *$f(t)$ always generates straight lines;*
(b) *$f(t)$ does not generate any planar curves;*
(c) *$f(t)$ generates true space curves if and only if $4r^2 + k^2 + l^2 \pm 6kl < 0$.*

Proof. Let $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ be a *primitive* quaternion polynomial (in normal form) that defines the hodograph of an RRMF curve $\mathbf{r}(t)$ generated by $f(t)$. Since $f_3 = 0$, we must have $u_1 = 0$ in (18) and thus $u(t) = t^2 + u_0$, $v(t) = v_1 t + v_0$, $p(t) = p_1 t + p_0$, $q(t) = q_1 t + q_0$. Note also that the polynomials on the right side of (10) must be of the form $a(t) = t^2 + a_0$, $b(t) = b_1 t + b_0$. Equating coefficients of like powers of t for the numerators and denominators in (10) then gives $(b_1, b_0) = (v_1, v_0)$ and

$$u_0 v_1 - p_0 q_1 + p_1 q_0 = a_0 v_1, \quad 2u_0 + p_1^2 + q_1^2 = 2a_0, \quad p_0 p_1 + q_0 q_1 = 0, \quad u_0^2 + p_0^2 + q_0^2 = a_0^2. \tag{28}$$

The second and fourth of these equations imply that, if $p_1 = q_1 = 0$, then $p_0 = q_0 = 0$ as well. Furthermore, eliminating a_0 between the first and second equations yields

$$p_1 q_0 - p_0 q_1 = \tfrac{1}{2}(p_1^2 + q_1^2) v_1, \tag{29}$$

and solving this together with the third equation – under the assumption that $p_1^2 + q_1^2 \neq 0$ – gives

$$p_0 = -\tfrac{1}{2} v_1 q_1 \quad \text{and} \quad q_0 = \tfrac{1}{2} v_1 p_1. \tag{30}$$

Consider now cases (a)–(c) of the Proposition.

(a) Define $u(t) + \mathrm{i}\, v(t) = [(t-r) - \mathrm{i}\, k]\,[(t+r) + \mathrm{i}\, l]$. Then one can easily verify that $\gcd(u(t), v(t)) = 1$ and $u^2(t) + v^2(t) = f(t)$, and (10) is satisfied with $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i}$ when $a(t) = u(t)$, $b(t) = v(t)$. Thus, Proposition 1 shows that $f(t)$ generates straight lines.

(b) Suppose now that $f(t)$ generates a planar PH curve, satisfying (13) with $(p(t), q(t)) \neq (0, 0)$. From the above arguments we must have $p_1^2 + q_1^2 > 0$, and hence relations (30) hold. Equating coefficients of t^4 on the left and right sides of (13) and using (29) then gives $-(p_1^2 + q_1^2)v_1 = \frac{1}{2}(p_1^2 + q_1^2)v_1$, and hence $v_1 = 0$. Therefore, $p_0 = q_0 = 0$ from (30), so $qp' - q'p = 0$ and (13) implies that $uv' - u'v - pq' + p'q = 0$. In that case, $\mathbf{r}(t)$ must be a (planar) Class III curve.

(c) Finally, suppose that $\mathcal{A}(t)$ generates a proper RRMF space curve. Then, from expression (12) in Farouki and Sakkalis (2010), condition (10) takes the form

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \pm \frac{k}{(t-r)^2 + k^2} \pm \frac{l}{(t+r)^2 + l^2}. \tag{31}$$

Now substituting $a_0 = u_0 + \frac{1}{2}(p_1^2 + q_1^2)$ from the second equation in (28) into the fourth equation, we obtain $4u_0 = v_1^2 - (p_1^2 + q_1^2)$, and comparing coefficients of the t^2 term in the numerators in (31) gives $v_1^2 = (k \pm l)^2$. Substituting these results into the equation

$$2u_0 + v_1^2 + p_1^2 + q_1^2 = k^2 + l^2 - r^2$$

obtained by equating coefficients of t^2 in the denominators in (31), we have $p_1^2 + q_1^2 = -(4r^2 + k^2 + l^2 \pm 6kl)$. Therefore, $f(t)$ generates a proper RRMF space curve if and only if $4r^2 + k^2 + l^2 \pm 6kl < 0$. □

Example 3. The polynomial $f(t) = t^4 + 3t^2 - 6t + 10$, with roots $1 \pm \mathrm{i}$ and $-1 \pm 2\mathrm{i}$, generates both straight lines and true space curves as Class I RRMF quintics. For $\mathcal{A}(t) = t^2 + 1 + (t-3)\,\mathbf{i}$ we have $f(t) = |\mathcal{A}(t)|^2$, and from (3) the components of $\mathbf{r}'(t)$ are $x'(t) = t^4 + 3t^2 - 6t + 10$, $y'(t) = 0$, $z'(t) = 0$. On the other hand, for $\mathcal{A}(t) = t^2 - \frac{1}{2} + (t-3)\,\mathbf{i} + \sqrt{3}t\,\mathbf{j} + \frac{1}{2}\sqrt{3}\,\mathbf{k}$ we again have $f(t) = |\mathcal{A}(t)|^2$ and (3) gives

$$x'(t) = t^4 - 3t^2 - 6t + 8\tfrac{1}{2}, \qquad y'(t) = \sqrt{3}\,(3t^2 - 6t - \tfrac{1}{2}), \qquad z'(t) = \sqrt{3}\,(-2t^3 + 2t - 3).$$

Since $[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t) \neq 0$, this is a true space curve. Both these solutions satisfy (10) with $a(t) = t^2 + 1$, $b(t) = t - 3$ and are thus Class I RRMF quintics.

5. Closure

A complete characterization of all quintic curves with rational rotation-minimizing frames (RRMF quintics) has been developed, through their reduction to normal form by a spatial scaling/rotation transformation. The characterization incorporates a succinct identification of *degenerate* solutions (straight lines and planar PH curves) through condition (13). For *proper* RRMF quintics (true space curves) a new set of solutions, the Class II RRMF quintics, has been identified satisfying the RRMF condition (10) with $\deg(a(t), b(t)) = 1$, as distinct from the previously-known Class I RRMF quintics that satisfy (10) with $\deg(a(t), b(t)) = 2$. As with the Class I RRMF quintics, the Class II RRMF quintics depend upon five free parameters, and their rational RMFs are of somewhat lower degree (six rather than eight).

However, the parameterization (23) of the set of Class II curves is more complicated than the corresponding representation for Class I curves. Concerted efforts to derive simpler generating formulas for Class II curves, or sufficient-and-necessary constraints on the quaternion or Hopf map coefficients of spatial PH curves for a Class II curve (such as are available for Class I curves) have thus far been unsuccessful. This topic deserves further investigation, due to its importance in making these new RRMF curves amenable to the development of algorithms for practical use in animation, spatial path planning, and geometric design.

Finally, a new approach to the RRMF curves has been proposed, based on considering the four polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$ in (2) or (4) to be generated by the decomposition of a positive polynomial $f(t)$ as a sum of four squares. Some preliminary results concerning the construction of Class I RRMF quintics through this approach were presented.

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