ELSEVIER

Topology and its Applications 123 (2002) 533–535

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

A note on actions of the cylinder $S^1 \times R$

José Luis Arraut [a], Paul A. Schweitzer, S.J. [b,*]

[a] *Instituto de Ciências Matemáticas de São Carlos, Universidade de São Paulo, Cx. Postal 668, São Carlos, SP 13560-970, Brazil*

[b] *Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro (PUC-Rio), Rio de Janeiro, RJ 22453-900, Brazil*

Received 30 October 2000; received in revised form 6 May 2001

Dedicated to the memory of Gilles Chatelet (†1999)

Abstract

Spheres and certain other manifolds do not admit actions of $S^1 \times R$ with all orbits 2-dimensional. This is related to the problem of determining the rank of a manifold.
© 2001 Elsevier Science B.V. All rights reserved.

Elon Lima [5] proved in 1963 that any C^2 action of R^2 on the 3-sphere S^3 has a singular orbit, i.e., an orbit of dimension less than two. The question whether the same statement holds for other spheres S^{2n+1} with $\mathrm{Span}(S^{2n+1}) \geqslant 2$ is still an open and apparently very difficult problem.

If we replace R^2 by $S^1 \times R$ the problem becomes much simpler. In fact, the presence of the compact factor S^1 permits the use of standard tools from algebraic topology. In this note we give a proof of the following theorem and a small extension (Theorem 4).

Theorem 1. *Any topological action of $S^1 \times R$ on a rational homology sphere of arbitrary dimension has a singular orbit.*

By rational homology sphere we mean a compact manifold M whose homology with rational coefficients is that of a sphere (of lower dimension in case M has non-empty boundary). Recall that a Lie group action $G \times M \to M$ on a manifold is *locally free* if all orbits have the same dimension as G. Any orbit of lower dimension is called *singular*. The *rank* of a smooth closed manifold M is the largest integer k such that R^k acts locally

* Corresponding author. Supported by the CNPq and the Ministry of Science and Technology of Brazil.
E-mail addresses: arraut@icmsc.sc.usp.br (J.L. Arraut), paul@mat.puc-rio.br (P.A. Schweitzer, S.J.).

0166-8641/01/$ – see front matter © 2001 Elsevier Science B.V. All rights reserved.
PII: S0166-8641(01)00219-X

freely and smoothly on M, while the *span* of M is the largest number of vector fields on M that are linearly independent at each point. The 3-manifolds of rank greater than one are precisely the torus bundles over the circle [7]. In higher dimensions very little is known [1].

The proof of the Theorem is an easy application of the fact that the existence of a flow with no stationary points on a compact finite dimensional ANR (absolute neighborhood retract) implies that its Euler characteristic must vanish together with the two following results of Conner and Floyd on circle actions, which we cite without proof.[1] All cohomology will have coefficients in the rational field Q.

Theorem 2 [3, Theorem 2.4]. *The orbit space of a circle action on a manifold is an ANR.*

Furthermore the orbit space is finite dimensional, as shown by [6].

Theorem 3 [2, Lemma 2.2]. *Let $p : M \to B = M/S^1$ be the projection onto the orbit space of an S^1-action on a compact manifold M with all orbits one-dimensional. Then there is a Gysin exact sequence* (*with rational coefficients*)

$$\cdots \to H^{i-1}B \to H^{i+1}B \to H^{i+1}M \to H^iB \to H^{i+2}B \to H^{i+2}M \to \cdots .$$

Theorem 4. *If a closed connected orientable m-manifold with vanishing rational Betti numbers b_i for $0 < i < k$, where $k = [m/2]$, admits a locally free topological action of $S^1 \times R$, then either $m \equiv 2 \bmod 4$ and $b_k = 2$, or $m \equiv 3 \bmod 4$ and $b_k = k + 1$ or $k + 2$. In particular, k must be odd.*

As in Theorem 1, the same result holds if the manifold is replaced by a rational Poincaré space of rational homological dimension m with the same vanishing of the Betti numbers as before, provided the underlying space is a compact topological manifold with boundary (of higher dimension if the boundary is non-empty, of course).

Proof of Theorem 1. Consider a locally free action of $S^1 \times R$ on a manifold M. The restriction to the first factor gives a locally free S^1-action whose orbit space $B = M/S^1$ is a finite dimensional ANR. Since $S^1 \times R$ is abelian, the action of the second factor descends to a non-singular flow $\phi_t : B \to B$, $t \in R$. Since B is compact, ϕ_t has no fixed points for sufficiently small $t > 0$. Hence its Lefschetz fixed point index, which is well-defined since B is a finite dimensional ANR and thus a Euclidean neighborhood retract, must vanish [4, Proposition 6.6, p. 209]. Since ϕ_t is homotopic to the identity of B, ϕ_0, this index coincides with the Euler characteristic $\chi(B)$ of B [4, Proposition 6.22, p. 212], which must therefore be zero.

Now suppose that M has the rational homology of a sphere of dimension m, so that $H^iM = 0$ except when i is 0 or m, and $H^0M \cong H^mM \cong Q$. We claim that B has the cohomology of a complex projective space, just as if the circle action were free. In fact, $H^0B \cong Q$ and $H^iB = 0$ for sufficiently large i since B is a neighborhood retract

[1] We are grateful to Frank Raymond for these references.

in some R^k. Then the Gysin exact sequence above shows that $H^iB \cong H^{i+2}B$ for every $i \geqslant m$, so that $H^iB = 0$ for $i \geqslant m$. The same exact sequence with $i = m - 1$ then shows that $H^{m-1}B \cong H^mM \cong Q$. Next, for $i = 0$, the homomorphism $H^0B \to H^0M$, induced by the projection $M \to B$, is an isomorphism and $H^{-1}B$ (which does make sense in the Gysin sequence) vanishes, so $H^1B = 0$ as well. Working upwards in dimension by the Gysin sequence with $0 \leqslant i < m - 2$, we find that $H^kB \cong Q$ for k even with $0 \leqslant k < m$ and vanishes for k odd in the same range. Consequently $m - 1$ must be even, and the Euler characteristic of B is just $\chi(B) = (m + 1)/2$, which is greater than zero, yielding a contradiction. □

Proof of Theorem 4. Consider a locally free action of $S^1 \times R$ on a closed connected oriented m-manifold M, and suppose that the rational Betti numbers b_i of M vanish for $0 < i < k$, where $k = [m/2]$. As in the proof of Theorem 1, the quotient of M by the induced action of S^1 must have $\chi(B) = 0$.

If m is even, so that $m = 2k$, then only the Betti numbers $b_0 = 1, b_k$ and $b_{2k} = 1$ of M are non-vanishing. Since M admits a locally free action of R, its Euler characteristic vanishes, so k is odd and $b_k = 2$, as claimed. (This case does in fact occur, since $S^k \times S^k$ admits a locally free action of $S^1 \times R$, with S^1 acting on one factor and R acting on the other.)

Now let m be odd, so that $m = 2k + 1$. As before, the Gysin sequence shows that for $0 \leqslant i < k$, $H^iB \cong Q$ when i is even and vanishes when i is odd. By Poincaré duality, H^iM vanishes for $k + 1 < i < m$, so working downwards from high dimensions in the Gysin sequence shows that for $k < i$, $H^iB \cong Q$ when i is even and less than m, and it vanishes when i is odd or greater than $m - 1$.

Since $\chi(B) = 0$, we must have $H^iB \neq 0$ for some odd value of i; the only possibility is $i = k$, so k must be odd and $H^kB \cong Q^{k+1}$. Setting $i = k - 1$ in the Gysin sequence above gives

$$0 \to Q^{k+1} \to Q^{b_k} \to Q \to \cdots,$$

which shows that b_k must be either $k + 1$ or $k + 2$. □

References

[1] G. Chatelet, H. Rosenberg, Manifolds which admit R^n actions, Publ. Math. IHES 43 (1974) 245–260.
[2] P. Conner, On the action of the circle group, Michigan Math. J. 4 (1957) 241–247.
[3] P. Conner, E.E. Floyd, On orbit spaces of circle actions, Ann. of Math. 67 (1958) 90–98.
[4] A. Dold, Lectures on Algebraic Topology, Springer, Berlin, 1972.
[5] E. Lima, Commuting vector fields on S^3, Ann. of Math. 81 (1965) 70–81.
[6] D. Montgomery, C.T. Yang, The existence of a slice, Ann. of Math. 65 (1957) 108–116.
[7] H. Rosenberg, R. Roussarie, D. Weil, A classification of closed orientable 3-manifolds of rank two, Ann. of Math. 91 (1970) 449–464.