Boundedness of rough oscillatory singular integral on Triebel–Lizorkin spaces

Chen Jiecheng, Jia Houyu, Jiang Liya

Department of Mathematics, Zhejiang University, 310028, PR China

Received 4 June 2004
Available online 2 February 2005
Submitted by L. Grafakos

Abstract

We give the boundedness on Triebel–Lizorkin spaces for oscillatory singular integral operators with polynomial phases and rough kernels of the form $e^{iP(x)}\Omega(x)|x|^{-n}$, where $\Omega \in L\log^+ L(S^{n-1})$ is homogeneous of degree zero and satisfies certain cancellation condition.

Keywords: Oscillatory singular integral; Rough kernel; Triebel–Lizorkin spaces

1. Introduction

Let $\Omega(x)$ be an $L^1$ function over the unit sphere $S^{n-1}$ of $\mathbb{R}^n$. We assume that $\Omega(x)$ is homogeneous of degree zero, and satisfies

$$\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0. \quad (1.1)$$
Let $P(x)$ be a real polynomial, the oscillatory singular integral operator $T$ is defined on the test function spaces $S(R^n)$ by

$$Tf(x) = \text{p.v.} \int_{R^n} e^{iP(x-y)} \Omega(x-y)|x-y|^{-n} f(y) \, dy.$$  \hspace{1cm} (1.2)

Ricci and Stein in [15] proved that if $\Omega \in C^1(S^{n-1})$ and satisfies the mean value zero condition (1.1), then $T$ is bounded on $L^p(R^n)$, $1 < p < \infty$, and the norm of $L^p(R^n)$ of $T$ depends only on the degree of $P$, not its coefficients. In fact, the operators they considered are more general, in the sense that they are not necessarily of convolution type. Later Chanillo and Christ [3] proved that these operators are also weak-type $(1,1)$. In 1992, Lu and Zhang [13] improved the result in [15] by assuming a weaker condition $\Omega \in L^r(S^{n-1})$, $1 < r \leq \infty$. And in 2000, Ojanen [14] showed that the operator $T$ is bounded on $L^p(w)$ for certain weights $w(x)$ in a further weaker condition $\Omega \in L^1 \log^+ L(S^{n-1})$ (see also [12] for the case of $w(x) = 1$). For the boundedness of $T$ in the Hardy space $H^1$, under a smoothness condition $\Omega \in C^1(S^{n-1})$, Hu [9] obtained the $H^1_w$ boundedness of $T$ in the one dimension case and Hu and Pan [10] obtained the same result in the case of higher dimension, where $w(x)$ is an $A_1$ weight.

On the other hand, the Triebel–Lizorkin space $\dot{F}_{p,q}^{\alpha,q}(R^n)$ is a unified setting of many well-known function spaces including Lebesgue space $L^p(R^n)$, Hardy space $H^p(R^n)$ and Sobolev spaces $\dot{L}^p_{\alpha}(R^n)$. It is of natural interest to extend the above mentioned results on $T$ to the more general Triebel–Lizorkin spaces $\dot{F}_{p,q}^{\alpha,q}(R^n)$. Thus, the main purpose of this paper is to establish the $\dot{F}_{p,q}^{\alpha,q}(R^n)$ boundedness of operator $T$ given in (1.2) with rough kernel $\Omega \in L^1 \log^+ L(S^{n-1})$. When $P(x) = 0$, $T$ is the classical singular integral operator of convolution type and whose boundedness in various function spaces has been well-studied by many authors (see [2,4,7,8,11] and so on). However, if $P(x)$ is a polynomial and $\Omega \in L^1 \log^+ L(S^{n-1})$, the situation is more involved. To obtain our result, we find an interesting result (Proposition 2.1) which says that if a convolution operator $T$ is bounded on all $L^q(w)$ for $w \in A_1$, then $T$ is automatically a bounded operator in the Triebel–Lizorkin space $\dot{F}_{p,q}^{\alpha,q}$.

We recall the definition of the Triebel–Lizorkin spaces. Choose a function $\phi \in C_0^\infty(R^n)$, such that $0 \leq \phi \leq 1$. Let $\phi_j(\xi) = \phi(2^j \xi)$ satisfy supp$(\phi) \subset \{ \xi \in R^n : \frac{3}{2} \leq |\xi| \leq 2 \}$ and $\phi(\xi) > c > 0$ when $\frac{3}{4} \leq |\xi| \leq \frac{5}{2}$. We denote by $S_j$ the convolution operator whose symbol is $\phi_j(\xi)$. For $\alpha \in R$, $1 < p, q < \infty$ and $f \in S'(R^n)/P(R^n)$, where $P(R^n)$ denotes the class of polynomials on $R^n$, the homogeneous Triebel–Lizorkin spaces $\dot{F}_{p,q}^{\alpha,q}(R^n)$ is the set of all $f$ satisfying

$$\| f \|_{\dot{F}_{p,q}^{\alpha,q}(R^n)} = \left\| \sum_{j=-\infty}^{+\infty} 2^{-\alpha j q} |S_j f|^q \right\|_p^{1/q} < \infty.$$  \hspace{1cm} (1.3)

Let $S_j^*$ be the dual operator of $S_j$; it is easy to see

$$\left\| \left( \sum_{j=-\infty}^{+\infty} 2^{-\alpha j q} |S_j^* f|^q \right)^{1/q} \right\|_p \sim \| f \|_{\dot{F}_{p,q}^{\alpha,q}(R^n)}.$$
The inhomogeneous version of Triebel–Lizorkin spaces is obtained by adding the term $\|\Psi \ast f\|_p$ to the right side of (1.3) and replace $\sum_{j=-\infty}^{\infty}$ with $\sum_{j\geq 1}$, where $\Psi \in S(R^n)$, supp $\Psi \subset \{\xi : |\xi| \leq 2\}$ and $|\hat{\Psi}| \geq c > 0$ if $|\xi| \leq \frac{5}{3}$. This space is denoted $F^{\alpha,q}_p (R^n)$ and it is a space of tempered distribution.

The following properties of above spaces are well-known. Let $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$.

1) $L^p = F^{0,2}_p = \dot{F}^{0,2}_p$;
2) $F^{\alpha,q}_p \sim \dot{F}^{\alpha,q}_p \cap L^p$, and $\|f\|_{F^{\alpha,q}_p} \sim \|f\|_{L^p} + \|f\|_{\dot{F}^{\alpha,q}_p}$ for $\alpha > 0$;
3) $(F^{\alpha,q}_p)^* = F^{-\alpha,q'}_p$ and $(\dot{F}^{\alpha,q}_p)^* = \dot{F}^{-\alpha,q'}_p$;
4) $F^{\alpha,q}_p \subset F^{\alpha,q'}_p$ and $F^{\alpha,q}_p \subset F^{\alpha,q'}_p$, if $q_1 \leq q_2$.

Now, let us state our theorems.

**Theorem 1.1.** Let $\alpha \in \mathbb{R}$, $1 < p, q < \infty$. Let $P(x)$ be a polynomial with $\nabla P(0) = 0$, and $T$ be defined as in (1.2). If $\Omega \in L \log L(S^{n-1})$ and satisfies condition (1.1), then $T$ is bounded on $F^{\alpha,q}_p (R^n)$, that is

$$\|Tf\|_{F^{\alpha,q}_p (R^n)} \leq C (1 + \|\Omega\|_{L \log L(S^{n-1})}) \|f\|_{F^{\alpha,q}_p (R^n)},$$

where $C$ is a constant which depends only on degree of $P(x)$ but not its coefficients.

Since the operator $T$ is bounded on $L^p (R^n)$ (see [14]), applying Theorem 1.1 and the properties (2), (3), we have the following corollary.

**Corollary 1.1.** Suppose $\alpha \in \mathbb{R}$, $1 < p, q < \infty$. Let $T$, $\Omega$ and $P(x)$ be as in Theorem 1.1. Then $T$ is bounded on $F^{\alpha,q}_p (R^n)$.

If $\nabla P(0) \neq 0$, we obtain the boundedness on the inhomogeneous space $F^{\alpha,q}_p (R^n)$ as follows.

**Theorem 1.2.** Let $\alpha \in \mathbb{R}$, $1 < p, q < \infty$. Let $\Omega$ and $T$ be as in Theorem 1.1. If $P(x)$ is a polynomial with $\nabla P(0) \neq 0$, then $T$ is bounded on $F^{\alpha,q}_p (R^n)$, that is

$$\|Tf\|_{F^{\alpha,q}_p (R^n)} \leq C (1 + \|\Omega\|_{L \log L(S^{n-1})}) \|f\|_{F^{\alpha,q}_p (R^n)},$$

where $C$ is a constant which depends on $\alpha$, $p$, $q$, $n$, but not on the coefficients of $P(x)$.

2. Preliminaries

First, we state the following useful proposition.

**Proposition 2.1.** Let $T : S \rightarrow S'$ be a convolution operator. If for some $1 < q < \infty$, the inequality $\|Tf\|_{L^q(w)} \leq A \|f\|_{L^q(w)}$ and $\|Tf\|_{L^q'(w)} \leq A \|f\|_{L^q'(w)}$ hold for all $w \in A_1$, then
where $A_1$ is the Muckenhoupt weight class, then for all $1 < p < \infty$, $s \in \mathbb{R}$, $T$ is a bounded operator on $\dot{F}^{s,q}_p(\mathbb{R}^n)$, and

$$
\|Tf\|_{\dot{F}^{s,q}_p(\mathbb{R}^n)} \leq A\|f\|_{\dot{F}^{s,q}_p(\mathbb{R}^n)}.
$$

**Proof.** Let $\phi_j$ be the same as in the introduction, and assume that

$$
\sum_{j=-\infty}^{+\infty} \phi_j^2(\xi) = 1 \quad \text{for all } \xi \neq 0.
$$

Then for all $f \in S(\mathbb{R}^n)$, we have

$$
f(x) = \sum_{j=-\infty}^{+\infty} S_j^2 f(x),
$$

where $S_j^2 f = \phi_j(\xi) \hat{f}(\xi)$.

Decompose the operator $Tf$ by

$$
Tf(x) = \sum_{k \in \mathbb{Z}} S_k T S_k f(x).
$$

For any $g \in \dot{F}^{-s,q'}_p(\mathbb{R}^n)$,

$$
|\langle Tf, g \rangle| = \left| \sum_{k \in \mathbb{Z}} \langle T S_k, S_k^* g \rangle \right| \leq \|g\|_{\dot{F}^{-s,q'}_p} \left( \sum_{k \in \mathbb{Z}} 2^{-skq} |T S_k f|^q \right)^{\frac{1}{q}}.
$$

Hence

$$
\|Tf\|_{\dot{F}^{s,q}_p(\mathbb{R}^n)} \leq \left\| \left( \sum_{k \in \mathbb{Z}} 2^{-skq} |T S_k f|^q \right)^{\frac{1}{q}} \right\|_p.
$$

If $q \leq p$, we choose a function $u(x) \in L_{\mathbb{R}^n}^{\left(\frac{q}{q'}\right)}$ with $\|u\|_{L_{\mathbb{R}^n}^{\frac{q}{q'}}} = 1$, such that

$$
\left\| \left( \sum_{k \in \mathbb{Z}} 2^{-skq} |T S_k f|^q \right)^{\frac{1}{q}} \right\|_p = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} 2^{-skq} |T S_k f|^q(x) u(x) \, dx.
$$

Pick $1 < r < \left(\frac{p}{q}\right)'$, then for a.e. $x \in \mathbb{R}^n$, $M(u')(x) < \infty$. Thus $(M(u'))^\frac{1}{r} \in A_1$. Using the $L^q$ weighted estimate for operator $T$, it follows

$$
\int_{\mathbb{R}^n} |T S_k f|^q(x) u(x) \, dx \leq \int_{\mathbb{R}^n} |T S_k f|^q(x) (M(u'))^\frac{1}{r} \, dx
$$

$$
\leq A \int_{\mathbb{R}^n} |S_k f|^q(x) (M(u'))^\frac{1}{r} \, dx.
$$

Thus,
\[ \sum_{k \in \mathbb{Z}} 2^{-skq} \int_{\mathbb{R}^n} |TS_k f|^q(x) u(x) \, dx \leq A \sum_{k \in \mathbb{Z}} 2^{-skq} \int_{\mathbb{R}^n} |S_k f|^q(x) (M(u'))^{\frac{1}{r}} \, dx \]
\[ \leq A \sum_{k \in \mathbb{Z}} 2^{-skq} |S_k f|^q \left\| M(u') \right\|_{L^\frac{q}{q'}} \]
\[ \leq A \| f \|_{\tilde{F}^q_p(\mathbb{R}^n)}. \]

Therefore \( \| Tf \|_{\tilde{F}^q_p(\mathbb{R}^n)} \leq A \| f \|_{\tilde{F}^q_p(\mathbb{R}^n)} \) holds if \( q \leq p \). By duality, we can obtain the same result if \( q \geq p \). The proof of Proposition 2.1 is completed.

**Remark 2.1.** If we set \( P(x) = 0 \) in (1.2), then for any \( 1 < q < \infty \), \( \| T \|_{q,w} \leq C \| f \|_{q,w} \) holds for all \( w \in A_q \) provided that \( \Omega \in L^\infty(S^{n-1}) \) (see [6]). From Proposition 2.1, we see that \( T \) is bounded on \( \tilde{F}^q_p(R^n) \) for all \( s \in \mathbb{R}, 1 < p, q < \infty \).

To prove our results, we also need the following lemmas.

We denote by \( C^\rho(R^n) \) the Zygmund spaces (see [17]).

**Lemma 2.1** [17, p. 141]. Let \( s \in \mathbb{R}, 0 < p < \infty, 0 < q \leq \infty \) and \( \rho > \max(s, \frac{n}{\min(p, q)} - s) \).

Then \( g \in C^\rho(R^n) \) is a multiplier for \( F^q_p(R^n) \). In other words, \( f \rightarrow gf \) yields a bounded linear mapping from \( F^q_p(R^n) \) into itself and there exists a positive constant \( c \) such that

\[ \| gf \|_{F^q_p(R^n)} \leq c \| g \|_{C^\rho(R^n)} \| f \|_{F^q_p(R^n)} \]

holds for all \( g \in C^\rho(R^n) \) and all \( f \in F^q_p(R^n) \).

**Lemma 2.2** (Van der Corput [16]). Suppose \( \phi(t) \) is real-valued and smooth in \( (a,b) \), and that \( |\phi(t)| \geq 1 \) for all \( t \in (a, b) \). Then

\[ \left| \int_a^b e^{i\lambda \phi(t)} \, dt \right| \leq C_k \lambda^{-\frac{1}{4}} \]

holds when

(i) \( k \geq 2 \), or
(ii) \( k = 1 \) and \( \phi'(t) \) is monotonic.

The bound \( C_k \) is independent of \( \phi \) and \( \lambda \).

**Lemma 2.3** [15]. Let \( P(x) = \sum_{|\beta| \leq d} a_\beta x^\beta \) be a polynomial of degree \( d \) in \( \mathbb{R}^n \), and \( \varepsilon < 1/d \). Then

\[ \int_{|x| \leq 1} |P(x)|^{-\varepsilon} \, dx \leq A_\varepsilon \left( \sum_{|\beta| = d} |a_\beta| \right)^{-\varepsilon}. \]

The bound \( A_\varepsilon \) depends on \( n \) and \( \varepsilon \), but not on the coefficients \( \{a_\beta\} \).
Lemma 2.4 [15]. Let $P(x) = \sum_{|\beta| = d} a_\beta x^\beta$ be a polynomial of degree $d$ in $\mathbb{R}^n$, and $\varepsilon < 1/d$. Then

$$\int_{|x|=1} |P(x)|^{-\varepsilon} \, dx \leqslant A_\varepsilon \left( \sum_{|\beta| = d} |a_\beta| \right)^{-\varepsilon}.$$ 

The bound $A_\varepsilon$ depends on $n$ and $\varepsilon$, but not on the coefficients $\{a_\beta\}$.

Now according to [5], we decompose $\Omega(x)$ as follows. Let

$$\theta_0 = \{ x' \in S^{n-1} : |\Omega(x')| \leqslant 1 \},$$
$$\theta_d = \{ x' \in S^{n-1} : 2^{d-1} \leqslant |\Omega(x')| \leqslant 2^d \} \quad (d \geqslant 1),$$
$$\tilde{\Omega}_d(x) = \Omega(x) \chi_{\theta_d}(x),$$
$$\Omega_d(x) = \tilde{\Omega}_d(x) - \frac{\int_{S^{n-1}} \tilde{\Omega}_d(x) \, dx}{\omega_n}.$$ 

Then we have

$$\sum_{d \geqslant 0} \Omega_d(x) = \Omega(x), \quad \int \Omega_d(x) \, dx = 0,$$
$$\| \Omega_d \|_\infty \leqslant C 2^d, \quad \| \Omega_d \|_{L^1} \leqslant C 2^d |\theta_d|, \quad \sum_{d \geqslant 0} d 2^d |\theta_d| \leqslant C \| \Omega \|_{L^1} \log^+ L.$$ 

For $f \in S(\mathbb{R}^n)$, write

$$T_f(x) = \sum_{d \geqslant 0} \sum_{k \in \mathbb{Z}} e^{iP(x)} \frac{\Omega_d(x)}{|x|^n} X_{2^k-1 < |x| \leqslant 2^k} \ast f(x) = \sum_{d \geqslant 0} \sum_{k \in \mathbb{Z}} T_d^k f(x).$$

Lemma 2.5. Let $\sigma_d^k(x) = e^{iP(x)} \frac{\Omega_d^k(x)}{|x|^n} X_{2^k-1 < |x| \leqslant 2^k}(x), \quad k \in \mathbb{Z}$. Then for all $1 < p < \infty$,

there holds

$$\| | \sigma_d^k | \ast |f| \|_p \leqslant \| \Omega_d \|_{L^1} \| f \|_p.$$ 

Proof. We note that

$$| \sigma_d^k | \ast |f| (x) \leqslant \int_{S^{n-1}} |\Omega_d(y')| \int_{2^k-1}^{2^k} \left| f(x - ry') \right| \frac{1}{r} \, dr \, d\sigma(y').$$ 

Hence

$$\| | \sigma_d^k | \ast |f| \|_p \leqslant \int_{\mathbb{R}^n} \int_{S^{n-1}} |\Omega_d(y')| \int_{2^k-1}^{2^k} \left| f(x - ry') \right| \frac{1}{r} \, dr \, d\sigma(y') \, dx \leqslant \| \Omega_d \|_{L^1} \| f \|_p. \quad \square$$
Lemma 2.6. For $\alpha \in \mathbb{R}$, $1 < p, q < \infty$, there holds
\[
\| T_k^d f(x) \|_{\dot{F}_p^\alpha,\dot{Q}^q \left( \mathbb{R}^n \right)} \leq \| \Omega_d \|_{L^1} \| f \|_{\dot{F}_p^\alpha,\dot{Q}^q \left( \mathbb{R}^n \right)}.
\] (2.1)

Proof. By the same proof in Proposition 2.1, for any $g \in \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^n)$, we have
\[
\langle T_k^d f, g \rangle = \left\| \left( \sum_j 2^{-aq(j+k)} |T_k^d S_{j+k} f|^q \right)^{\frac{1}{q'}} \left( \sum_j 2^{aq(j+k)} |S_{j+k}^* g|^q \right)^{\frac{1}{q'}} \right\|_{p'}.
\]
It follows
\[
\| T_k^d f \|_{\dot{F}_p^\alpha,\dot{Q}^q \left( \mathbb{R}^n \right)} \leq \left\| \left( \sum_j 2^{-aq(j+k)} |T_k^d S_{j+k} f|^q \right)^{\frac{1}{q'}} \right\|_{p'}.
\]
By Lemma 2.5,
\[
\sup_j 2^{-a(j+k)} |T_k^d S_{j+k} f| \leq \| \Omega_d \|_{L^1} \sup_j 2^{-a(j+k)} |S_{j+k} f| \leq \left| \tilde{\sigma}_d \right| \left( x \right) = |\sigma_d \left( -x \right)|.\] (2.2)
Since $p > 1$, there exists a function $g \in L^{p'}$ with $\| g \|_{p'} = 1$ such that
\[
\left\| \sum_j 2^{-a(j+k)} |T_k^d S_{j+k} f| \right\|_{p'} \leq \sum_j \left\| 2^{-a(j+k)} |T_k^d S_{j+k} f| \right\|_{p'} \leq \left\| \sum_j 2^{-a(j+k)} |S_{j+k} f| \right\|_{p'} \left\| g \right\|_{p'},
\]
where $|\tilde{\sigma}_d \left( x \right)| = |\sigma_d \left( -x \right)|$. Using Lemma 2.5, we obtain
\[
\left\| \sum_j 2^{-a(j+k)} |T_k^d S_{j+k} f| \right\|_{p'} \leq \| \Omega_d \|_{L^1} \left\| \sum_j 2^{-a(j+k)} |S_{j+k} f| \right\|_{p'}.\] (2.3)
Thus (2.1) follows immediately by using an interpolation between (2.2) and (2.3). \(\square\)

3. Proofs of the theorems

In this section, we will prove Theorems 1.1 and 1.2.

Let $P(x) = \sum_{|\alpha| \leq m} a_\alpha x^\alpha$. Without loss of generality we may assume that
\[
\sum_{|\alpha| = m} |a_\alpha| = 1.
\]
If this is not the case, let $A = \left(\sum_{|\alpha|=m} |a_\alpha|\right)^{1/2}$; we can write $P(x)$ as follows:

$$P(x) = \sum_{|\alpha| \leq m} \frac{a_\alpha}{A^m} (Ax)^\alpha := Q(Ax).$$

Then a change of variable gives

$$Tf\left(\frac{x}{A}\right) = \text{p.v.} \int_{\mathbb{R}^n} e^{Q(x-y)} \frac{\Omega(x-y)}{|x-y|^n} f\left(\frac{y}{A}\right) dy.$$ 

Since $\|f\|_{L^p(\mathbb{R}^n)} \sim A^{-\alpha + \frac{n}{p}}$, it is enough to consider the case $A = 1$.

When $m = 0$, the phase function in $T$ is identically zero and $T$ is the usual convolution type singular integral operator with rough kernel. Thus (1.4) holds (see [11]). In fact, using the same proof in [11], we can see, for any $\varepsilon > 0$, the truncated operator $T_\varepsilon$ is also bounded on $\dot{F}^{\alpha,q}_p (\mathbb{R}^n)$.

**Proof of Theorem 1.1.** As usual, write

$$Tf(x) = e^{iP(x)} \frac{\Omega(x)}{|x|^n} \chi_{|x| \leq 1} * f(x) + e^{iP(x)} \frac{\Omega(x)}{|x|^n} \chi_{|x| > 1} * f(x)$$

$$:= T_0 f(x) + T_\infty f(x).$$

First, let us treat $T_0 f$.

Since $\nabla P(0) = 0$, we write $P(x) = \sum_{|\alpha| = m} a_\alpha x^\alpha + P_{m-1}(x)$ for $m \geq 2$, where $\text{deg}(P_{m-1}) \leq m - 1$ with $\nabla P_{m-1}(0) = 0$. We shall proceed by induction on $m$. When $m = 0$, (1.4) holds for $T_0$. Suppose that (1.4) holds for $T_0$ with the phase function $P_{m-1}(x)$ when $m \geq 2$. To prove $T_0$ satisfies (1.4) when $\text{deg}(P) = m$, rewrite

$$T_0 f(x) = \left[ e^{iP(x)} - e^{iP_{m-1}(x)} \right] \frac{\Omega(x)}{|x|^n} \chi_{|x| \leq 1} * f(x) + e^{iP_{m-1}(x)} \frac{\Omega(x)}{|x|^n} \chi_{|x| \leq 1} * f(x)$$

$$:= I + II.$$

The estimate for $II$ follows from the induction hypothesis. To treat the first term, we write it as

$$I = \sum_{d \geq 0} \sum_{k \leq 0} \int_{2^{k-1} < |x-y| \leq 2^k} \left| e^{iP(x-y)} - e^{iP_{m-1}(x-y)} \right| \frac{\Omega_d(x-y)}{|x-y|^n} f(y) dy$$

$$:= \sum_{d \geq 0} \sum_{k \leq 0} T_{0,k}^d f(x).$$

Note that $|P(x-y) - P_{m-1}(x-y)| \leq \sum_{|\alpha| = m} |a_\alpha| |x-y|^m \leq |x-y|$ when $|x-y| \leq 2^k \leq 1$, it follows, for $k \leq 0$,

$$\left| T_{0,k}^d f(x) \right| \leq \int_{2^{k-1} < |x-y| \leq 2^k} \frac{\Omega_d(x-y)}{|x-y|^n} |f(y)| dy \leq 2^k \|\Omega_d\|_\infty M(f)(x).$$
where $M$ is the Hardy–Littlewood maximal function. For $1 < q < \infty$ and $w \in A_q$, $M$ is bounded on $L^q(w)$. Therefore

$$
\|T_{0,k}f\|_{q,w} \leq 2^k \|\Omega_d\|_\infty \|f\|_{q,w}.
$$

(3.1)

Using Proposition 2.1, we have

$$
\|T_{0,k}f\|_{\dot{F}^{p,q}_\alpha(R^n)} \leq 2^k \|\Omega_d\|_\infty \|f\|_{\dot{F}^{p,q}_\alpha(R^n)}.
$$

(3.2)

By Lemma 2.6, we can get

$$
\|T_{0,k}f\|_{\dot{F}^{p,q}_\alpha(R^n)} \leq \|\Omega_d\|_1 \|f\|_{\dot{F}^{p,q}_\alpha(R^n)}.
$$

(3.3)

So

$$
\|I\|_{\dot{F}^{p,q}_\alpha(R^n)} \leq \sum_{d \geq 0} \sum_{k \leq 0} \|T_{0,k}f\|_{\dot{F}^{p,q}_\alpha(R^n)}
\leq \sum_{d \geq 0} \sum_{k \leq 0} \|T_{0,k}f\|_{\dot{F}^{p,q}_\alpha(R^n)} + \sum_{d \geq 0} \sum_{k \leq 0} \|T_{0,k}f\|_{\dot{F}^{p,q}_\alpha(R^n)}
:= I_1 + I_2.
$$

By (3.3),

$$
I_1 \leq \sum_{d \geq 0} \sum_{k \leq 0} \|\Omega_d\|_1 \|f\|_{\dot{F}^{p,q}_\alpha(R^n)} \leq \sum_{d \geq 0} dN 2^d |\theta_d| \|f\|_{\dot{F}^{p,q}_\alpha(R^n)}
\leq C \|\Omega\|_{L^{\log^+ L}} \|f\|_{\dot{F}^{p,q}_\alpha(R^n)}.
$$

And the estimate for $I_2$ follows from (3.2) if we choose $N$ sufficiently large.

Next, we shall prove that $T_\infty$ satisfies (1.4). Write

$$
T_\infty f(x) = \sum_{d \geq 0} \sum_{k \geq 1} e^{i P(x)} \Omega_d(x) \frac{\chi_{2^k-1 < |x| \leq 2^k}}{|x|^n} f(x) = \sum_{d \geq 0} \sum_{k \geq 1} T_{0,k}^d f(x).
$$

We will use the method in [13] to establish the $L^2$ norm of $T_{0,k}^d$:

$$
T_{0,k}^d f(x) = \int_{2^k-1 < |x-y| \leq 2^k} e^{i P(x-y)} \frac{\Omega_d(x-y)}{|x-y|^n} f(y) \, dy
\leq \int_{S^{n-1}} \Omega_d(\theta) \int_{2^{k-1}}^{2^k} e^{i P(r\theta)} f(x-r\theta) \frac{drd\theta}{r}.
$$

For a fixed $\theta \in S^{n-1}$, let $Y$ be the hyperplane through the origin orthogonal to $\theta$, we have, for $x \in R^n$, $x = z + s\theta$, with $s \in R$, $z \in Y$, and so

$$
\int_{2^{k-1}}^{2^k} e^{i P(r\theta)} f(x-r\theta) \frac{dr}{r} = \int_{2^{k-1}}^{2^k} e^{i P(r\theta)} f(z + (s-r)\theta) \frac{dr}{r}.
$$
It is easy to see
\[ |M_k(u)| \leq C 2^{-k} \chi_{[0,2^{k+1}]}(|u|). \] (3.4)

Now we can write
\[ P(2^k r \theta) - P(2^k r \theta - u \theta) = \sum_{|\alpha|=m} \sum_{\beta: \beta+\gamma=\alpha} 2^{k(m-1)} \mu_{\alpha} u a_{\beta+\gamma} C_{\beta \gamma} \partial_\beta \partial_\gamma + R(r,u) \]
\[ = (2^k r)^{m-1} u \sum_{|\beta|=m-1} \sum_{|\gamma|=1} \theta_\beta a_{\beta+\gamma} C_{\beta \gamma} \partial_\beta \partial_\gamma + R(r,u), \]
where \( C_{\beta \gamma} \) are nonzero constants depending only on \( m \), and \( R(r,u) \) is a polynomial with degree in \( r \) strictly less than \( m-1 \). We have
\[ \left( \frac{\partial}{\partial r} \right)^{m-1} \left( P(2^k r \theta) - P(2^k r \theta - u \theta) \right) = 2^{k(m-1)} (m-1)! u \sum_{|\beta|=m-1} \sum_{|\gamma|=1} \theta_\beta a_{\beta+\gamma} C_{\beta \gamma} \partial_\beta \partial_\gamma. \]

By Lemma 2.2 and using the integration by parts,
\[ |M_k(u)| \leq C 2^{-k} \left( u \sum_{|\beta|=m-1} \sum_{|\gamma|=1} \theta_\beta a_{\beta+\gamma} C_{\beta \gamma} \partial_\beta \partial_\gamma \right)^{-\frac{1}{m-1}}. \] (3.5)

Combining (3.4) and (3.5), we obtain
\[ |M_k(u)| \leq C 2^{-k(1+\delta)} \left( u \sum_{|\beta|=m-1} \sum_{|\gamma|=1} \theta_\beta a_{\beta+\gamma} C_{\beta \gamma} \partial_\beta \partial_\gamma \right)^{-\frac{\delta}{m-1}} \chi_{[0,2^{k+1}]}(|u|), \]
where \( \delta \in (0,1) \). Thus,
\[ \int_R |M_k(u)| \, du \leq C 2^{-k \delta} \int_{|u| \leq 1} \left| u \sum_{|\beta|=m-1} \sum_{|\gamma|=1} \theta_\beta a_{\beta+\gamma} C_{\beta \gamma} \partial_\beta \partial_\gamma \right|^{-\frac{\delta}{m-1}} \, du. \]
Since \( m \geq 2 \) and \( \delta \in (0, 1) \), \( \frac{\delta}{m - 1} < 1 \). Due to Lemma 2.3, it follows
\[
\int_{R} \left| M_k(u) \right| \, du \leq C 2^{-k\delta} \left| u \sum_{|\beta|=m-1} \theta^\beta \sum_{|\gamma|=1} a_{\beta + \gamma} C_{\beta \gamma} \theta^\gamma \right|^{-\frac{2}{m - 1}}.
\]
Thus,
\[
\left\| N_k N_k \right\|_{L^2 \to L^2} \leq C 2^{-k\delta} \left| u \sum_{|\beta|=m-1} \theta^\beta \sum_{|\gamma|=1} a_{\beta + \gamma} C_{\beta \gamma} \theta^\gamma \right|^{-\frac{2}{m - 1}}.
\]
So
\[
\left\| N_k \right\|_{L^2 \to L^2} \leq C 2^{-\frac{k\delta}{2}} \left| u \sum_{|\beta|=m-1} \theta^\beta \sum_{|\gamma|=1} a_{\beta + \gamma} C_{\beta \gamma} \theta^\gamma \right|^{-\frac{2}{m - 1}}.
\]

The Minkowski’s inequality shows that
\[
\left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{L^2} \leq \left( \int_{R} \left\| N_k \right| \, du \right)^{\frac{1}{2}} \, \left\| f \right\|_{L^2}
\]
\[
\leq C 2^{-\frac{\delta}{2}} \left\| f \right\|_{L^2} \left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{L^2}
\]
\[
\leq C 2^{-\frac{\delta}{2}} \left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{L^2} \left\| f \right\|_{L^2} \leq C 2^{-\frac{\delta}{2}} \left\| f \right\|_{L^2}.
\]

Since \( \delta \in (0, 1) \) and \( m \geq 2 \), we can see \( \frac{\delta}{2(m - 1)} < \frac{1}{m} \). Then by Lemma 2.4,
\[
\left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{L^2} \leq C 2^{-\frac{\delta}{2}} \left\| f \right\|_{L^2} \left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{L^2}
\]
\[
\leq C 2^{-\frac{\delta}{2}} \left\| f \right\|_{L^2} \left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{L^2} \leq C 2^{-\frac{\delta}{2}} \left\| f \right\|_{L^2}.
\]

where \( C \) is a constant depending on \( m \).

On the other hand,
\[
\left| T \left( \int_{R} \left| N_k \right| \, du \right) \right| \leq \int_{R} \left| T \left( \int_{R} \left| N_k \right| \, du \right) \right| \, dy \leq C \left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{L^2} \left\| f \right\|_{L^2}.
\]

For \( 1 < q < \infty \) and \( w \in A_q \), we have
\[
\left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{q, w} \leq \left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{q, w} \left\| f \right\|_{q, w}.
\]

Since \( \left\| f \right\|_{q, w} \leq \left\| f \right\|_{L^2} \), we obtain
\[
\left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{q, w} \leq \left\| T \left( \int_{R} \left| N_k \right| \, du \right) \right\|_{L^2} \left\| f \right\|_{L^2} = C \left( \int_{R} \left| N_k \right| \, du \right)^{\eta} \left\| f \right\|_{L^2}.
\]

(3.6)
For any $w \in A_q(R^n)$, there exists an $\varepsilon > 0$ such that $w^{1+\varepsilon} \in A_q(R^n)$. By the virtue of (3.7), we get, for $w \in A_q(R^n)$,

$$\|T_{\infty, k}^d f\|_{q, w^{1+\varepsilon}} \leq \|\Omega_d\|_\infty \|f\|_{q, w^{1+\varepsilon}}.$$  \hfill (3.9)

Therefore, using the interpolation theorem with change of measure (see [1, p. 115]), we interpolate between (3.8) and (3.9), and then there exists a positive constant $\mu$ such that

$$\|T_{\infty, k}^d f\|_{q, w} \leq C \|\Omega_d\|_\infty \|f\|_{q, w}.$$  \hfill (3.10)

Using (3.12) and (3.11), we get

$$I \leq C \|\Omega\|_{L \log L(S_{n})} \|f\|_{p^\mu q(R^n)} \quad \text{and} \quad II \leq C \|f\|_{p^\mu q(R^n)},$$

if we choose $N$ sufficiently large. Therefore Theorem 1.1 is completely proved.

**Proof of Theorem 1.2.** It is enough to consider the case for $P(x) = x$. Write

$$S(f)(x) = p.v. \int_\mathbb{R}^n \frac{\Omega(x - y)}{|x - y|^n} f(y) dy.$$  

Then $T f(x) = e^{ix} S(e^{ix} f(x))$. Noting that $e^{ix}$ is a $C^\infty(R^n)$ function with all its derivatives in $L^\infty(R^n)$, from Lemma 2.1, we have

$$\|e^{ix} f(x)\|_{p^\mu q(R^n)} \leq C \|f(x)\|_{p^\mu q(R^n)}.$$  \hfill (3.13)

Then by Corollary 1.1, the inequality (1.5) is obtained. So the proof of Theorem 1.2 is finished.

**References**