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Optimal route decision with a geometric ground-airborne hybrid model under weather uncertainty

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Abstract

Adverse weather is the dominant cause of delays in the National Airspace System (NAS). Since the future weather condition is only predictable with a certain degree of accuracy, managing traffic in the weather-affected airspace is a challenging task. In this paper, we propose a geometric model to generate an optimal combination of ground delay and route choice to hedge against weather risk. The geometric recourse model (GRM) is a strategic Probabilistic Air Traffic Management (PATM) model that generates optimal route choice, incorporating route hedging and en-route recourse to respond to weather change: hedged routes are routes other than the nominal or the detour one, and recourse occurs when the weather restricted airspace becomes flyable and aircraft are re-routed to fly direct to the destination. Among several variations of the GRM, we focus on the hybrid Dual Recourse Model (DRM), which allows ground delay as well as route hedging and recourses, when the weather clearance time follows a uniform distribution. The formulation of the hybrid DRM involves two decision variables -- ground delay and route choice -- and four parameters: storm location, storm size, maximum storm duration time, and ground-airborne cost ratio. The objective function has two components: expected total ground delay cost and expected total airborne cost. We propose a solution algorithm that guarantees to find the global optimum of the hybrid-DRM. Based on the numerical analysis, we find that ground-holding is effective only when combined with the nominal route. Otherwise, it is optimal to fly on the route determined by the DRM without ground delay. We also find the formula of the threshold ground-airborne cost ratio, which we call the Critical Cost Ratio (CCR), that determines the efficacy of ground delay: the higher the CCR, the more effective the strategies involving ground delay. We conclude that both ground delay and route hedging should be considered together to produce the best ATM decisions.

Keywords: probabilistic air traffic management; ATFM; weather risk hedging; geometric model; minimum cost route decision; GDP; AFP

1. Introduction

There is a growing interest in air traffic management (ATM) strategies that incorporate uncertainty in the national airspace system (NAS). Research in “probabilistic air traffic management” (PATM) seeks to guide decisions on ground-holding or otherwise modifying aircraft four-dimensional trajectories (4DTs) in order to
minimize the expected cost, or to hedge against “worst case” scenarios in the Next Generation Air Transportation System (NextGen).

In this paper, we study the problem of developing a minimum-cost aircraft routing strategy when some weather condition inhibits the use of the nominal route for an indefinite period. In conventional Air Traffic Management (ATM), two options are commonly considered in such a situation; the flight is either held at the origin airports until the nominal route becomes flyable or rerouted to avoid the weather region entirely. The choice between these options is based upon a deterministic and conservative characterization of future weather, often resulting in underutilized airspace and unnecessary delay if the weather clears early.

We propose a geometric model to find an optimal combination of ground holding and route decision when the weather clearance time is stochastic. The route decision takes into account the probability distribution of storm clearance time, the possibility of route hedging, the cost difference between ground delay and airborne, and the recourse opportunities. When facing uncertain weather, there are two potential risks to hedge against: persistence risk and clearance risk. Persistence risk is the risk when we take an “optimistic” route and weather persists, resulting in unplanned re-routing and delay. Clearance risk is the risk when we take a “pessimistic” route and weather clears sooner, resulting in unnecessary extra flight time. To mitigate these risks, we consider the intermediate routing options that may not be optimal under either persistence or clearance, but hedge against either possibility. In doing so, we consider how the route might be adjusted if the weather clears during the course of the flight. We assume that the flight plan can be amended in such an event so that the plane can fly direct to the destination.

We first discuss optimal routing decision based on the geometric optimization model without considering ground holding. In this model, the routing decision is made based on four parameters; nominal route between origin and destination airport, storm location, storm size, and maximum storm duration time. The optimistic route is the nominal one while the pessimistic route goes around the storm. A hedged route is one that is between the optimistic and the pessimistic one. We use the term recourse for a change in a routing that results from the storm clearing. We consider two recourse possibilities. First, the storm may clear before the aircraft reaches it, so that it can be rerouted directly to its destination. This is called first recourse. The storm may instead persist beyond the time when the aircraft reaches it—so that the plane must turn and begin to fly around it-- but clear before the tip of the storm is reached. The aircraft may then be rerouted direct to the destination; we refer to this as second recourse.

In our model, which we term the Geometric Recourse Model (GRM), a triangle is drawn in which the base is the nominal route between the origin and destination, and the vertex is the tip of the storm, which we assume to be a straight line perpendicular to the nominal route. We seek routes that minimize expected total flight cost, which in some cases are hedged routes. Out of several variations of the geometric recourse model, we consider the dual recourse model (DRM). The DRM allows both the first and second recourses to assume greater responsiveness to changing conditions and consequently results in reduced cost. We discuss a set of conditions that guarantees the nominal route to be optimal in the DRM, regardless of the probabilistic nature of the weather clearance time.

Following discussions on the formulation and properties of the DRM, we extend the model with the ground delay option at the origin. The ground-airborne hybrid model, or the hybrid-DRM, finds the optimal combination of ground holding and airborne routing based on the weather characteristics as well as the cost difference between ground holding and additional airborne time. We propose a solution algorithm to find the optimal solution of the hybrid-DRM. In searching for a solution, the algorithm finds the range of the ground-airborne cost ratio for each optimal ground holding and route combination. In model analysis, we find that it is optimal to take positive ground delay only when the ground-airborne cost ratio is below a certain threshold value, which we call the Critical Cost Ratio (CCR), and no ground holding is necessary otherwise. The Critical Cost Ratio (CCR) is found as a formula in weather parameters. We also find that the positive ground holding is optimal only when combined with the nominal route. When ground holding is inappropriate, the hybrid-DRM reduces to the DRM, which determines the optimal route choice.

In our models, the probabilistic nature of the weather is represented with its clearance time. Although it is more realistic to assume that the weather condition either improves or deteriorates gradually rather than changes
instantaneously, the weather clearance time is an effective and practical representation of the time when the airspace becomes flyable. A similar idea applies to the storm represented as a perpendicular line to the nominal course. The line should be understood as a control point to let the traffic go through or not, behind which convective weather exists.

2. Literature review

There have been numerous efforts to address weather-related disruptions in air traffic management. Earlier traffic flow management models such as Bertsimas (1998) and Goodhart (1999), often have a deterministic setting. More recently, Nilim et al. (2001, 2002, 2004) proposed a dynamic aircraft routing model with robust control. Their research adopted shortest-path algorithms in a grid structure, by discretizing time into stages when the routing decisions are made, and airspace as a two-dimensional grid. The weather condition in each potential storm region is assumed and modeled as a Markovian process with two states: 0 (No storm) and 1 (Storm). The transition matrix is estimated based on the historical weather forecasts. Optimization results show a promising improvement compared to simply flying around the storm.

In the air transportation system, however, the frequent routing adjustments entailed by such an approach may place undue workload on controllers and pilots. Moreover, the Markovian assumption is of doubtful validity in the context of convective weather. Two of the goals in our study are to set up a model that has the flexibility to adopt a variety of probability distributions of storm clearance times, and to limit re-routing decisions to a reasonable number.

Bertsimas et al. (2000) proposed a two-stage optimization model based on a dynamic network flow approach. The authors set up a multi-aircraft optimization model minimizing the weather delay cost, based on a deterministic weather scenario. One important aspect of their study is that the cost function covers all components of aircraft operation costs, such as fixed cost, ground holding cost, and airborne cost.

From the air traffic management perspective, it is ideal to utilize both the Ground Delay Program (GDP) and airborne rerouting to mitigate weather related disruptions, especially since ground delay is less costly than extra flight time. In this paper, we first consider a routing decision model without ground delay when the weather is stochastic, and extend the model with the ground delay option in the later part.

3. Geometric Recourse Model (GRM)

3.1. Geometric recourse model concept

Consider the problem of routing a single flight in the presence of a single storm. Given an origin and destination pair, assume there is a linear storm of known size blocking the direct route at a certain location. Based on those five parameters- origin (O), destination (D), storm-route intersection ($S_L$), and storm tip ($S_T$), construct a triangle $ODS_T$, where the nominal route is the base $OD$ and storm size is the altitude $S_L S_T$, as illustrated in Figure 1.

Note that while the storm has two tips, we choose the one nearer to $S_L$, since this is the one that the aircraft would be routed around. We refer to the base $OD$ as the nominal route, the altitude $S_L S_T$ as the front of the storm, and the vertex $S_T$ as the tip of the storm. The route $OS_T D$, which goes around the storm, is called the detour route. Upon departure, the aircraft may set a course along the nominal route, the detour route, or one in between.
During the course of the flight, aircraft may be re-routed to fly direct to the destination when the storm clears; we refer to such route changes as *recourse*. Depending on the timing of storm clearance, there are three recourse scenarios as illustrated in Figure 2: (a) recourse if the storm clears before the aircraft reaches it; (b) recourse at the storm front if the storm persists until after the aircraft reaches it, but clears as the aircraft flies along the storm front toward the tip; or (c) no recourse because the storm persists until after the aircraft reaches the tip of the storm.

We define the case (a) as the *first recourse*, the case (b) as the *second recourse*, and the case (c) as *no recourse*. Given the geometric setup, the objective is to find the route that minimizes the expected total flight cost, where choosing a route is equivalent to choosing an angle between zero and the base angle $\beta$. Although such a decision variable is intuitive, the resulting objective function involves complex trigonometric terms that make it difficult to analyze. Instead, we propose a ratio-based model in which the complexity is reduced without loss of generality.

In the ratio-based model, the nominal route and weather parameters are expressed as ratios to the nominal route as illustrated in Figure 3. In other words, we define the unit of distance as the distance of the nominal route, and the unit of time as the time required to fly that route. Now we introduce a new decision variable $x$, which is the distance from the origin to the storm front along the course set from the origin. Defining the unit of distance such that the aircraft cruises at a constant speed of 1, the ratio-based model is formulated as follows.

1: flight time (equivalent to distance) of nominal route between origin and destination

$\alpha$: storm distance from origin in units of nominal route flight time: $0 < \alpha < 1$

$\beta$: storm size in units of nominal route flight time: $\beta > 0$

$\mu$: random variable representing the storm clearance time with probability density function $p(\mu)$
$x$: distance to the storm along the course set from origin in units of nominal route flight time: $\alpha \leq x \leq \sqrt{\alpha^2 + \beta^2}$

Figure 3. Ratio-based Geometric Model

There are several variations of geometric recourse model. The Single Recourse Model (SRM) considers the GRM with the first recourse only. The Dual Recourse Model (DRM) considers both the first and second recourse in the GRM. The DRM is more responsive to changing weather conditions than the SRM, since it allows for immediate rerouting of a flight moving along the storm region when the storm burns off. In this paper, we first introduce the DRM, and extend our discussion to the ground-airborne hybrid DRM, in which ground delay is considered as well as the first and second recourses. Combined with ground delay, the hybrid DRM provides sophisticated and flexible environment to reduce weather related risks.

4. Dual Recourse Model (DRM)

4.1. Formulation of the Dual Recourse Model (DRM)

We first discuss the Dual Recourse Model (DRM), which allows recourses both before and at the storm region. The optimization model is formulated as follows.

$$
\min_x \int_0^x \left( \mu + \sqrt{1 + \mu^2 - 2\mu \frac{\alpha}{x}} \right) p(\mu) d\mu + \int_x^{x+\beta - \sqrt{x^2 - \alpha^2}} \left( \mu + \sqrt{\left(\mu - x + \sqrt{x^2 - \alpha^2}\right)^2 + (1 - \alpha)^2} \right) p(\mu) d\mu \\
+ \int_{x+\beta - \sqrt{x^2 - \alpha^2}}^{\infty} \left( x + \beta - \sqrt{x^2 - \alpha^2} + \sqrt{(1 - \alpha)^2 + \beta^2} \right) p(\mu) d\mu
$$

s.t. \hspace{1mm} $\alpha \leq x \leq \sqrt{\alpha^2 + \beta^2}$, where $0 < \alpha < 1$, $\beta > 0$

(1)

In the objective function, the first integral is the expected total flight cost when the first recourse is taken, the second integral is the cost when the second recourse is taken, and the third integral is the cost when no recourse is possible.

Although the weather clearance time can stretch to infinity in theory, it is reasonable to assume that the weather forecast contains the latest possible time the weather clears, upon which the restricted airspace recovers its full capacity. We define a weather parameter for such a time limit, which we call the maximum storm duration time $T$. $T$ is the latest possible time that the storm will remain. With the introduction of $T$, it is clear that when $T \leq \alpha$, the GRM in general has the optimal solution $x^* = \alpha$ with expected total cost of 1: If the storm is guaranteed to clear before the aircraft reaches it by flying on the nominal route, then flying on the nominal route results in the shortest travel time to the destination with no weather interruption.
4.2. Nominal route theorem

The Dual Recourse Model (DRM) has a unique property that guarantees the nominal route to be optimal when the maximum storm duration time is below a specific threshold value.

**Nominal Route Theorem.**

Given the second recourse option, the geometric recourse model yields the optimal solution \( x^* = \alpha \), if \( \alpha \leq T \leq \sqrt{\alpha^2 + \beta^2} \).

**Proof.**

It is trivial that \( x^* = \alpha \), if \( T \leq \alpha \). \( (2) \)

If \( \alpha < T \leq \sqrt{\alpha^2 + \beta^2} \), the objective function \( f_d(x) \) becomes the following.

\[
 f_d(x) = \int_{0}^{x} \left( \mu + \sqrt{\mu^2 + 1 - 2\mu x} \right) p(\mu) \, d\mu + \int_{x}^{T} \left( \mu + \sqrt{(1-\alpha)^2 + (\mu - x + \sqrt{x^2 - \alpha^2})^2} \right) p(\mu) \, d\mu \quad (3)
\]

Then, we have

\[
 f_d(x) - f_d(\alpha) = \int_{0}^{x} \left( \sqrt{\mu^2 + 1 - 2\mu x} - \sqrt{\mu^2 + 1 - 2\mu} \right) p(\mu) \, d\mu + \int_{x}^{T} \left( \sqrt{(1-\alpha)^2 + (\mu - x + \sqrt{x^2 - \alpha^2})^2} - \sqrt{(1-\alpha)^2 + (\mu - \alpha)^2} \right) p(\mu) \, d\mu \quad (4)
\]

To show \( (3) \) is positive for \( \forall x \in (\alpha, T] \), we show that each integrand in \( (4) \) is positive. It is clear that

\[
 \sqrt{\mu^2 + 1 - 2\mu x} - \sqrt{\mu^2 + 1 - 2\mu} > 0. \quad (5)
\]

Since \( \left( \mu^2 + 1 - 2\mu x \right) - \left( (1-\alpha)^2 + (\mu - \alpha)^2 \right) > 0 \) when \( \alpha < \mu \leq x \), we have

\[
 \sqrt{\mu^2 + 1 - 2\mu x} - \sqrt{(1-\alpha)^2 + (\mu - \alpha)^2} > 0. \quad (6)
\]

Similarly, \( \sqrt{(1-\alpha)^2 + (\mu - x + \sqrt{x^2 - \alpha^2})^2} - \sqrt{(1-\alpha)^2 + (\mu - \alpha)^2} > 0 \), where \( x < \mu \leq T \). \( (7) \)

From \( (5) \), \( (6) \) and \( (7) \), we have \( f_d(x) - f_d(\alpha) > 0, \forall x \in (\alpha, T] \). Therefore,

\[ x^* = \alpha \text{ if } 0 < T \leq \sqrt{\alpha^2 + \beta^2}. \quad (Q.E.D) \]

\(^2\) The nominal route theorem can be proved based on the geometry of the model as well.
The nominal route theorem states that it is always optimal to fly on the nominal route when the maximum storm duration time doesn’t exceed the flight time to the tip of the storm on the detour route. However, flying on the nominal route doesn’t necessarily mean that the flight cost is 1. In fact, we expect flights on the nominal route are likely to utilize the second recourse option, resulting in the flight cost larger than 1. As shown in the proof, the nominal route theorem holds regardless of the probabilistic nature of the weather clearance time.

4.3. Uniform weather distribution

In the following analyses, we assume that the weather (storm) clearance time follows a uniform distribution ranging between 0 and $T$. The uniform distribution assumption not only makes the model analytically tractable, but it is a reasonable choice from the practical perspectives as well. A forecast of convective weather is included in several weather forecast products published by the National Oceanic and Atmospheric Administration (NOAA)’s Storm Prediction Center (SPC). One of the forecasts that is widely used both in practice and research is the convective outlook watch. According to the SPC, roughly 1,000 watches are published each year to address possible severe weather conditions in the subsequent few hours, and each convective activity is associated with a probability. The probability value provided in the forecast may be translated to the weather clearance probability we assume in the model.

It should also be noted that while we maintain the assumption of a uniform distribution for clearance time, our results apply equally to the case where the clearance time distribution is uniform only up to the latest time when a flight can reach the tip of the storm. If the storm persists beyond that time, the details of its clearance time distribution are no longer important, since the flight cost is independent from the timing of the weather clearance.

From the modeling perspectives, the expected total cost of the DRM is in general difficult to analyze since it is not integrable analytically for most of standard probability distributions. With the uniform distribution assumption, the objective function is integrable, resulting in a closed form formula. Instead of relying on simulations, we now can obtain solutions of the DRM for a large set of input parameters to extract critical knowledge of proposed PATM strategy based on analytic approach.

5. Ground-Airborne Hybrid DRM (hybrid-DRM)

The ground-airborne hybrid DRM concerns hedging against weather risk through ground delay in addition to route hedging. In this model, aircraft have the option to wait on the ground at the origin airport before taking off on the route with minimum expected flight cost. Optimal ground delay decisions are affected not only by the weather parameters included in the previous models but also by the ground-airborne cost ratio. It is conventional wisdom that the unit cost of ground delay is less than that of extra airborne time, but their ratio varies from flight to flight. The effect of the cost ratio is critical in determining the optimal ground delay: if the ground delay costs significantly less than airborne, then it will be best to wait on the ground for en route weather to clear. On the other hand, if the ratio is close to 1, an early take-off will be preferred.

The formulation of the hybrid geometric recourse model involves two decision variables -- the ground delay amount and the route choice -- and four parameters: storm location, storm size, maximum storm duration time, and ground-airborne cost ratio. The objective is to find the combination of ground delay and route choice that minimizes the expected total cost. In the following sections, we first discuss the formulation and properties of the hybrid-DRM. We then investigate how the optimal solutions and associated expected costs vary with model parameters. As mentioned earlier, the weather clearance time is assumed to follow a uniform distribution between 0 and $T$.

5.1. Hybrid Dual Recourse Model (hybrid-DRM) formulation

Adopting the ratio-based framework where variables and parameters are defined in units of nominal route flight distance, the hybrid DRM is formulated as follows.
\(\alpha\): storm location from origin in units of nominal route: \(0 < \alpha < 1\)
\(\beta\): storm size in units of nominal route: \(\beta > 0\)
\(c\): ground-airborne cost ratio: \(0 < c < 1\)
\(x\): distance to the storm front along the course set from the origin in units of nominal route: \(\alpha \leq x \leq \sqrt{\alpha^2 + \beta^2}\)
\(g\): ground delay amount: \(0 \leq g \leq T - \alpha\)
\(\mu\): random variable representing the storm clearance time with probability density function \(p(\mu)\).

\[
\begin{align*}
\min_x & \quad \int_0^g (1 + c\mu) p(\mu) \, d\mu + \int_g^{g+x} \left( cg + \mu + \sqrt{1 + \mu^2 - 2\mu \frac{\alpha}{x}} \right) p(\mu) \, d\mu + \\
& \quad \int_g^{g+x+\sqrt{x^2-\alpha^2}} \left( cg + x + \sqrt{(x - \sqrt{x^2 - \alpha^2})^2 + (1 - \alpha)^2} \right) p(\mu) \, d\mu + \\
& \quad \int_g^{\infty} \left( cg + x + \sqrt{(x + \beta - \sqrt{x^2 - \alpha^2})^2 + (1 - \alpha)^2 + \beta^2} \right) p(\mu) \, d\mu
\end{align*}
\]

s.t. \(\alpha \leq x \leq \sqrt{\alpha^2 + \beta^2}, \ g \geq 0\) \hspace{1cm} (8)

In the formulation, we introduce an additional parameter \(c\), which is the ground-airborne cost ratio ranging between 0 and 1. We also introduce an additional decision variable \(g\), which is the ground delay amount. In the objective function, the first integral represents the case when the storm clears while the aircraft waits on the ground, after which the aircraft departs and flies along the nominal route with a total flight cost of 1. The second integral is the case when the aircraft is able to take the first recourse after spending \(g\) time units on the ground and departing on a course specified by \(x\). Likewise, the third and the fourth integral are the cases when the aircraft takes the second recourse or no recourse after spending \(g\) time units on the ground, respectively.

With the uniform distribution assumption on the weather clearance time, the objective function in (8) becomes (9), where \(f_d(x|\alpha, \beta, T)\) and \(f_h(x, g|c, \alpha, \beta, T)\) is the expected total cost of the DRM and the hybrid-DRM respectively.

\[
f_h(x, g|c, \alpha, \beta, T) = \int_{0}^{T-g} \left(1 + c\mu\right) \, d\mu + \frac{T-g}{T} f_d(x|\alpha, \beta, T - g)
\]

(9)

The first term in \(f_h\) represents the case when the weather clears before the end of the assigned ground delay \(g\), and aircraft departs on the nominal route to continue to the destination without interruption. The second term is the case when the aircraft waits on the ground for \(g\) time units, and then flies on the route determined by the DRM. Note that the effect of the ground delay is equivalent to reduction in the maximum storm duration time, and the optimal route choice is determined based on \(\alpha, \beta\) and \(T - g\). Since the weather clearance time follows a uniform distribution between zero and the maximum duration time \(T\), \(f_d(x|\alpha, \beta, T - g)\) is the expected total cost of the DRM when the storm clearance probability is \(\frac{1}{T-g}\). In other words, \(f_d(x|\alpha, \beta, T - g)\) will yield the optimal route choice based on the conditional probability that the storm didn’t clear in the first \(g\) time units. Since we seek an optimal route choice based on the unconditional probability, the correction term \(\frac{T-g}{T}\) is required.

Based on (9), we now propose a systematic way to determine the route choice portion of the hybrid DRM from the solutions of the DRM. Finding the solutions of the hybrid DRM based on the solutions of the DRM is particularly useful, since \(f_h\) is neither convex nor concave with multiple local minima in some cases, and the DRM solution is obtainable for a wide range of weather parameter values. Before embarking on the discussion of the solution algorithms, we first assert the relationship between the DRM and the hybrid DRM in the following theorem.
**Hybrid DRM Theorem.**

If $x^*$ and $g^*$ are the optimal solutions of the hybrid-DRM with cost function $f_h(x, g|c, \alpha, \beta, T)$, then $x^*$ is the optimal solution of the DRM with cost function $f_d(x|\alpha, \beta, T - g^*)$.

**Proof.**

Let $f_d(x|\alpha, \beta, T - g^*)$ and $f_h(x, g|c, \alpha, \beta, T)$ be the cost function of the DRM and the hybrid DRM, respectively.

Suppose that there exists $x_0$ and $g^*$ such that they are the optimal solutions of the hybrid-DRM, while $x_0$ is not an optimal solution of the DRM given the same weather parameter values. Then, there exists $x$ satisfying the following two conditions.

\[
\begin{align*}
  f_d(x|\alpha, \beta, T - g^*) &< f_d(x_0|\alpha, \beta, T - g^*) \\
  f_h(x, g^*|c, \alpha, \beta, T) &> f_h(x_0, g^*|c, \alpha, \beta, T)
\end{align*}
\]  

(10) From (9), we have

\[
  f_h(x, g^*|c, \alpha, \beta, T) - f_h(x_0, g^*|c, \alpha, \beta, T) = \frac{T - g^*}{T} f_d(x|\alpha, \beta, T - g^*) - \frac{T - g^*}{T} f_d(x_0|\alpha, \beta, T - g^*) < 0
\]

(11)

which contradicts (11).

Therefore, $x_0$ is an optimal solution of $f_d(x|\alpha, \beta, T - g^*)$ and $f_h(x, g^*|c, \alpha, \beta, T)$. (Q.E.D.)

The theorem states that the optimal route choice of the hybrid-DRM is determined by the DRM with maximum storm duration time $T$ reduced by the ground delay time $g$. In the following section, we discuss the solution algorithm for the hybrid DRM based on the hybrid DRM theorem.

**5.2. Solving the hybrid Dual Recourse Model (hybrid-DRM)**

To solve the hybrid model, we propose an algorithm to find the range of cost ratio $c$ that makes a specific combination of $x$ and $g$ optimal. The idea is as follows: given the weather parameters, find the DRM solution. Continue solving the DRM for the same storm location and size, but with maximum storm duration time varying from the current value to $\alpha$ in decrement of $\delta$. The difference between the initial and reduced value of the maximum storm duration time is thus the ground delay $g$. This process generates a set of expected total cost values of the hybrid DRM as functions of $c$ for given $\alpha, \beta$, and $T$. For each member in the set, find the range of $c$ that makes that member the minimum among all members in the entire set. If such a range exists, then the $x$ and $g$ associated with the current cost are the optimal solution of the hybrid DRM when the cost ratio is in that range. If no such range exists, then the selected $x$ and $g$ are never optimal. The pseudo code is presented below.

Given $(\alpha_0, \beta_0, T_0)$

Select $\delta$ and generate the index set $I$

$I = \{0, 1, \ldots, \frac{T_0 - \alpha_0}{\delta}\}$

Create the parameter set $P$ and the ground delay set $G$

$P = \{p_i|p_i = (\alpha_0, \beta_0, T_0 - i\delta), i \in I\}, G = \{g_i|g_i = i\delta, i \in I\}$

Find the DRM solution set $S_D$

$S_D = \left\{x_i|x_i = \text{argmin}_x f_d(x|p_i), p_i \in P, i \in I\right\}$

Generate the set of expected total costs of the hybrid DRM $F_H$, where

$F_H = \{f_h(x_i, g_i|c, p_i)|x_i \in S_D, p_i \in P, g_i \in G, i \in I\}$

For each $i \in \{0, 1, \ldots, \frac{T_0 - \alpha}{\delta}\}$
Find the range of $c$ satisfying $f_h(x_i, g_j | c, p_i) < f_h(x_j, g_j | c, p_j)$ for $j \neq i$ where $i, j \in I$, where $0 < c < 1$.

If such range of $c$ exists, which we call $C_i$, then store $(\alpha_0, \beta_0, T_0, x_i, g_i, C_i)$ in the hybrid DRM solution set. Else, repeat with the next $i$.

End

The pseudo code finds the solution set for one combination of weather parameters, and we repeat the code to find the solution set for a dense grid of parameter combinations. In our case, we created the parameter grid spaced by 0.05 in the $\alpha - \beta - T$ space, where $0 < \alpha < 1, 0 < \beta \leq \beta_{\text{max}}, 0 < T \leq T_{\text{max}}$. By setting $\delta = 0.05$, define the cost matrix $H$ as follows.

$$
H = \begin{pmatrix}
0 & \text{arg min}_x f_a(x | \alpha_0, \beta_0, T_0) & \text{arg min}_x f_d(x | \alpha_0, \beta_0, T_0 - \delta) & \cdots & \text{arg min}_x f_d(x | \alpha_0, \beta_0, \alpha_0) \\
\text{arg min}_x f_a(x | \alpha_0, \beta_0, T_0) & f_d(x | \alpha_0, \beta_0, T_0 - \delta) & \cdots & f_d(x | \alpha_0, \beta_0, \alpha_0) \\
f_h(x^*, 0 | c, \alpha_0, \beta_0, T_0) & f_h(x^*, \delta | c, \alpha_0, \beta_0, T_0 - \delta) & \cdots & f_h(x^*, T_0 - \alpha_0 | c, \alpha_0, \beta_0, \alpha_0)
\end{pmatrix}
$$

The cost matrix $H$ is a 4 by $\frac{T_0 - \alpha_0}{\delta}$ matrix, where each column contains the cost information when the ground delay specified in the first row is taken. In the matrix, the first row is the ground delay, the second row is the optimal route choice $x^*$ based on the reduced maximum storm duration time. The third row is the expected total cost associated with the route using the DRM. The fourth row is the expected total cost of the hybrid DRM $f_h^n$, when the ground delay in the first row and the route in the second are chosen.

In essence, the matrix $H$ is the search space to find the optimal solution when $\alpha = \alpha_0, \beta = \beta_0$ and $T = T_0$. For each element in the fourth row, we find the range of the cost ratio $c$ that makes the element minimum among all elements in the fourth row. If such a range exists, the ground delay and the route choice in the same column of the selected element is the optimal solution when $c$ is in that range. By repeating this process for all elements in the fourth row, we obtain the solution set of the hybrid DRM.

In Figure 4, three examples of the hybrid DRM solution are presented. In the plot, dotted lines show the upper and the lower bound of the decision variable, and solid lines show the optimal solutions. In the first case when $\alpha = 0.8, \beta = 1.65, T = 1.8$, the optimal route is always the nominal one, while the optimal ground delay decreases as the cost ratio increases. In the second case when $\alpha = 0.05, \beta = 0.5, T = 2.55$, the optimal choice is to take some positive ground delay to fly the nominal route until $c = 0.4$. Once the cost ratio exceeds 0.4, the ground delay option is discarded, and it is optimal to take off on an intermediate route immediately. The third case is similar to the second one, with the detour route being optimal instead of an intermediate one, once the ground delay cost is over a threshold value.

Case I) $\alpha = 0.8, \beta = 1.65, T = 1.8$
Case II) $\alpha=0.05$, $\beta=0.5$, $T=2.55$

Case III) $\alpha=0.95$, $\beta=1$, $T=3$

Figure 4. Hybrid DRM Solution Examples

In Table 1, the optimal route of the hybrid-DRM is categorized into three groups: the nominal, intermediate, and detour route. Likewise, the optimal ground delay is categorized into three groups: the minimum (zero), intermediate, and maximum delay. The optimal solution of the hybrid model therefore is grouped into nine categories. Out of nine categories, there are four categories that are never optimal: those which involve taking positive ground delay and then flying on an intermediate or the detour route.

Table 1. Existence of Optimal Solutions of the Hybrid DRM

| $g_{\text{min}} = 0$ | $Y$ | $Y$ | $Y$ |
| $g \in (0, T - \alpha)$ | $Y$ | $N$ | $N$ |
| $g_{\text{max}} = T - \alpha$ | $Y$ | $N$ | $N$ |

Based on our observations from Figure 4 and Table 1, we conclude that a solution that involves positive ground delay is optimal only when combined with the nominal route. Otherwise, the optimal choice is to depart...
without delay to fly the DRM optimal route—which may be nominal, intermediate or detour. In addition, we find that there exists a unique threshold value of the ground-airborne cost ratio that triggers elimination of the ground delay option. In the following sections, we discuss these findings in more detail, and further characterize the solution of the hybrid DRM.

5.3. Optimal solution of the hybrid DRM (hybrid-DRM)

We seek to further characterize the relationship between the solutions of the hybrid-DRM and its input parameters. As discussed in the previous section, if the cost ratio is below a certain threshold value, the optimal solution is the combination of positive ground delay and the nominal route. On the other hand, if the cost ratio is over the threshold value, an immediate take-off without ground delay is optimal and the optimal route is determined by the DRM.

We call the unique cost ratio which, when exceeded, yields an optimal solution with no ground delay the Critical Cost Ratio (CCR). The CCR defines the set of cases in which ground delay is too expensive to be part of the optimal strategy for avoiding en route weather. The higher the CCR is, the more promising the weather avoidance strategy that involves ground delay becomes.

Now, the solution of the hybrid-DRM solution is summarized in (13).

\[
(g^*, x^*) = \begin{cases} 
(> 0, \alpha) & \text{if } 0 < c \leq \text{CCR}(\alpha, \beta, T) \\
(0, \text{argmin}_x f_d(x|\alpha, \beta, T)) & \text{if } \text{CCR}(\alpha, \beta, T) < c < 1 
\end{cases}
\]

(13)

To find the optimal ground delay \( g^* \), we utilize the fact that positive ground delay is always associated with the nominal route. In other words, the optimal ground delay \( g^* \) is the one that minimizes the expected total cost of the hybrid-DRM when \( x = \alpha \). Therefore, we can find \( g^* \) by solving the following.

\[
\min f_h(\alpha, g|c, \alpha, \beta, T) \quad \text{s.t. } 0 < g \leq T - \alpha, \text{ where } 0 < c < 1, 0 < \alpha < 1, \beta > 0, T > \alpha.
\]

(14)

We obtain \( f_h(\alpha, g|c, \alpha, \beta, T) \) from (9), which results in a piecewise function as shown below.

\[
f_h(\alpha, g|c, \alpha, \beta, T) = \begin{cases} 
 h_1(g, c, \alpha, \beta, T) & g = T - \alpha \\
 h_2(g, c, \alpha, \beta, T) & T - \alpha < g < T - \alpha \\
 h_3(g, c, \alpha, \beta, T) & 0 < g < T - \alpha - \beta 
\end{cases}
\]

(15)

, where

\[
h_1(g, c, \alpha, \beta, T) = 1 + c \left( g - \frac{g^2}{2T} \right)
\]

(16)

\[
h_2(g, c, \alpha, \beta, T) = \frac{g(1-c)(g-2T)+2g+(2-\alpha)g-(g-T+c)\sqrt{(g-T+\alpha)^2+(1-\alpha)^2} \ln \frac{1-\alpha}{T-g-a+\sqrt{(1+\alpha)^2+(g-T+\alpha)^2}}}{2T}
\]

(17)

\[
h_3(g, c, \alpha, \beta, T) = \frac{(\alpha+\beta+\sqrt{(1-\alpha)^2+\beta^2})(2\alpha-2T+2g+\beta)+c(g-2T)-\alpha\beta-2T+\beta+(1-\alpha)^2 \ln \frac{1-\alpha}{\beta+\sqrt{(1-\alpha)^2+\beta^2}}}{2T}
\]

(18)

\( f_h(\alpha, g|c, \alpha, \beta, T) \) is a continuous piecewise function with respect to \( g \), and not always convex nor concave. To further characterize the solution of the optimization problem in (14), we first study characteristics of each piece.

\footnote{When \( T \leq \alpha \), the combination of no ground delay and the nominal route is always optimal regardless of the ground-airborne cost ratio. In fact, \( T \leq \alpha \) is the necessary and satisfying condition for the combinations of zero ground delay and nominal route to be optimal.}
in (15). The first piece \( h_1 \) is the expected total cost when the maximum ground delay \( T - \alpha \) is taken, which is equivalent to \( 1 + \frac{c(T-\alpha)}{2} \left( 1 + \frac{\alpha}{T} \right) \). \( h_2 \) is the expected total cost when \( T - \alpha - \beta \leq g < T - \alpha \) and is a convex function. In fact, \( h_2 \) is not only convex but non-monotonic\(^4\) with a unique interior minimum, which we can obtain from the first order condition as shown in (19).

\[
\frac{\partial h_2}{\partial g} = -\frac{1 + (1 + c)g + T - cT + \sqrt{1 + g^2 - 2gT + T^2 - 2\alpha + 2ga - 2Ta + 2a^2}}{T} = 0, \text{where } T - \alpha - \beta \leq g^* < T - \alpha
\]  

(20)

Therefore, \( h_2 \) has the unique interior solution as follows.

\[
g_2^* = \frac{-1 + c - 2cT + c^2T + \alpha + \sqrt{1 - 2c + 4c^2 - 2\alpha + 2c^2 + 2c^2a^2}}{-2c + c^2} \]  

(21)

\( g_2^* \) is the optimal solution of \( f_h(\alpha, g|c, \alpha, \beta, T) \) when \( \alpha < T \leq \alpha + \beta \), since the cost function consists of two pieces \( h_1 \) and \( h_2 \).

\( h_3 \) is the expected total cost when \( 0 \leq g < T - \alpha - \beta \) and is a non-monotonic concave function.\(^5\) The optimal solution occurs either at the lower or the upper bound in this case when \( T > \alpha + \beta \).

\[
g_3^* = \begin{cases} 
0 & \text{if } f_h(\alpha, 0|c, \alpha, \beta, T) \leq f_h(\alpha, T - \alpha - \beta|c, \alpha, \beta, T) \\
\left( T - \alpha - \beta \right) & \text{if } f_h(\alpha, 0|c, \alpha, \beta, T) > f_h(\alpha, T - \alpha - \beta|c, \alpha, \beta, T)
\end{cases}
\]  

(22)

The zero ground delay condition \( f_h(\alpha, 0|c, \alpha, \beta, T) \leq f_h(\alpha, T - \alpha - \beta|c, \alpha, \beta, T) \) in (21) is reduced to (23).

\[
0 < \alpha < 1, \ \beta > 0, \ T > \alpha + \beta, \ c \geq \frac{2(-1 + \alpha + \beta)}{T + \alpha + \beta} + 2 \sqrt{1 - 2a + 2a^2 + \beta^2} \]  

(24)

Likewise, the non-zero ground delay condition \( f_h(\alpha, 0|c, \alpha, \beta, T) > f_h(\alpha, T - \alpha - \beta|c, \alpha, \beta, T) \) in (21) is reduced to (25).

\[
0 < \alpha < 1, \ \beta > 0, \ T > \alpha + \beta, \ 0 < c < \frac{2(-1 + \alpha + \beta)}{T + \alpha + \beta} + 2 \sqrt{1 - 2a + 2a^2 + \beta^2} \]  

(26)

From (22) and (23), we can rewrite (21) as follows.

\[
g_3^* = \begin{cases} 
0 & \text{if } c \geq \frac{2(-1 + \alpha + \beta)}{T + \alpha + \beta} + 2 \sqrt{1 - 2a + 2a^2 + \beta^2} \]  

\( T - \alpha - \beta \) & \text{if } 0 < c < \frac{2(-1 + \alpha + \beta)}{T + \alpha + \beta} + 2 \sqrt{1 - 2a + 2a^2 + \beta^2}
\]

where \( 0 < \alpha < 1, \ \beta > 0, \ T > \alpha + \beta \)

(27)

When \( T \geq \alpha + \beta \), the cost function \( f_h(\alpha, g|c, \alpha, \beta, T) \) is concave when \( 0 \leq g < T - \alpha - \beta \) and convex when \( T - \alpha - \beta \leq g \leq T - \alpha \). The fact that the cost function is a continuous function yields the optimal solution either at the lower bound of the concave piece \( h_3 \) or in the interior of the convex piece \( h_2 \). In other words, the optimal solution is the following when \( T \geq \alpha + \beta \).

\(^4\) \( h_2(g, c, \alpha, \beta, T) \) is a non-monotonic since \( \frac{\partial h_2}{\partial g} < 0 \) when \( \alpha = 0.35, \beta = 1, T = 0.8, c = 0.23, g = 0.33 \), while \( \frac{\partial h_2}{\partial g} > 0 \) when \( \alpha = 0.31, \beta = 1, T = 0.63, c = 0.62, g = 0.32 \). We also have \( \frac{\partial^2 h_2}{\partial g^2} = -\frac{1 + c - 2cT + c^2T + (1 + g^2 - 2gT + T^2 - 2\alpha + 2ga - 2Ta + 2a^2)\beta}{T} > 0 \) and \( h_2 \) is a convex function with respect to \( g \).

\(^5\) \( h_3(g, c, \alpha, \beta, T) \) is not monotonic since \( \frac{\partial h_3}{\partial g} = -\frac{1 + c - 2cT + c^2T + (1 + g^2 - 2gT + T^2 - 2\alpha + 2ga - 2Ta + 2a^2)\beta}{T} < 0 \) when \( \alpha = 0.625, \beta = 2, T = 3.125, c = 0.5, g = 0.25 \), while \( \frac{\partial h_3}{\partial g} > 0 \) when \( \alpha = 0.56, \beta = 0.625, T = 3.1, c = 0.5, g = 0.375 \). We have \( \frac{\partial^2 h_3}{\partial g^2} = -\frac{\zeta}{T} < 0 \), and \( h_3 \) is concave.
Neither of the conditions in (25) can be reduced analytically. However, we already found those conditions in (23): when $h_3$ has the minimum at the upper bound, $f_h(\alpha, g_2^*|c, \alpha, \beta, T)$ has the optimal solution in $h_2$. In other words, the condition (23) is the condition for $g_2^*$ to be optimal.

In summary, the optimal ground delay is shown below where $g_{\text{int}}^* = g_2^*$.

$$g^* = \begin{cases} 
0, & f_h(\alpha, 0|c, \alpha, \beta, T) \leq f_h(\alpha, g_2^*|c, \alpha, \beta, T) \\
g_2^*, & f_h(\alpha, 0|c, \alpha, \beta, T) > f_h(\alpha, g_2^*|c, \alpha, \beta, T)
\end{cases}$$  \hspace{1cm} (28)

When $\alpha < T \leq \alpha + \beta$, it is always optimal to take ground delay at all cost levels. Recall that the DRM in general has the optimal solution of $\alpha$ when $0 < T \leq \sqrt{\alpha^2 + \beta^2}$ according to the nominal route theorem. With the ground delay option added, we have a larger set of weather parameters that guarantees the nominal route to be optimal when weather clearance time follows a uniform distribution. In other words, the ground delay option always reduces expected flight cost when $\alpha < T \leq \alpha + \beta$.

5.4. The Critical Cost Ratio (CCR)

Now, we introduce the critical cost ratio theorem.

**Critical Cost Ratio Theorem.**

The critical cost ratio $c_0$ is the following.

$$c_0 = \begin{cases} 
\frac{2(\alpha + \beta - 1 + \sqrt{(1-\alpha)^2 + \beta^2})}{T + \alpha + \beta}, & T > \alpha + \beta \\
1, & \alpha < T \leq \alpha + \beta
\end{cases}$$

**Proof.**

When $\alpha < T \leq \alpha + \beta$, it is always optimal to take positive ground delay according to (26), which means that $c_0 = 1$.

Given $T > \alpha + \beta$, let $c_1 = \frac{2(\alpha + \beta - 1 + \sqrt{(1-\alpha)^2 + \beta^2})}{T + \alpha + \beta}$. Suppose that $0 < c_0 < c_1 < 1$. Then, there must exist $c_0 < c < c_1$ satisfying $g^* = 0$ from (13) as well as $g^* > 0$ from (26), which is a contradiction. Therefore,

$$c_0 \geq c_1.$$  \hspace{1cm} (30)

Now, suppose that $0 < c_1 < c_0 < 1$. Then, there must exist $c_1 < c < c_0$ satisfying $g^* > 0$ from (13). From (26) we also have that $g^* = 0$, which establishes a contradiction. Therefore,

$$c_0 \leq c_1.$$  \hspace{1cm} (31)

From (27) and (28), we have $c_0 = c_1 = \frac{2(\alpha + \beta - 1 + \sqrt{(1-\alpha)^2 + \beta^2})}{T + \alpha + \beta}$. (Q.E.D)
From the critical ratio theorem, the solution to the hybrid DRM is further characterized as follows.

\[
(g^*, x^*) = \begin{cases} 
(0, \arg\min_x f_d(x|\alpha, \beta, T)) & c > \frac{2(-1+\alpha+\beta)}{T+\alpha+\beta} + 2 \sqrt{\frac{1-2\alpha+\alpha^2+\beta^2}{(T+\alpha+\beta)^2}}, T > \alpha + \beta \\
(g_{int}^*, \alpha) & 0 < c \leq \frac{2(-1+\alpha+\beta)}{T+\alpha+\beta} + 2 \sqrt{\frac{1-2\alpha+\alpha^2+\beta^2}{(T+\alpha+\beta)^2}}, T > \alpha + \beta \\
(g_{int}^*, \alpha) & 0 < c < 1, \ \alpha < T \leq \alpha + \beta 
\end{cases}
\]  

(32)

where \(g_{int}^* = -\frac{1+\alpha+\beta+2\sqrt{1-2\alpha+\alpha^2+\beta^2}}{2c+\alpha^2+c^2} \). Once the weather parameters and the cost ratio are known, one can determine if the optimal solution requires ground hold or not from the CCR formula. If positive ground delay is required, \(g_{int}^*\) provides the required ground delay amount while the optimal route is the nominal one. If no ground delay is required, one can determine the optimal route based on the DRM.

The CCR formula always yields a positive value, and can reach over 1. Since the ground-airborne cost ratio is never larger than 1, we treat the weather parameters satisfying \(c_0 > 1\) to have the CCR of 1, in which case, it is always optimal to take ground holding to fly on the nominal route. Since the CCR is always 1 when \(T \leq \alpha + \beta\), consider the case when \(T > \alpha + \beta\). By rearranging terms in (33), we obtain the corresponding set of weather parameters as shown below.

\[0 < \alpha < 1, \ \beta > \sqrt{2\alpha - \alpha^2}, \ \alpha + \beta < T < -2 + \alpha + \beta + 2\sqrt{1-2\alpha + \alpha^2 + \beta^2} \quad (34)\]

Before discussing the sensitivity of the CCR, let us consider limiting cases with respect to each weather parameter. When \(\alpha \to 0\), or when the storm is very near the origin, we have \(c_0 = \frac{2(-1+\beta+\sqrt{1+\beta^2})}{T+\beta}\). In Figure 5, we show the contour plot of those CCR’s in the \(\beta - T\) plane. We observe that the benefit of ground delay is limited when the storms are small, unless the cost ratio is small. The large region with CCR over 1 indicates that there are many combinations of the storm size and maximum duration time that will benefit from ground delay even when the ground delay option is fairly costly. \(^6\) When \(\alpha \to 1\), or when the storm is very near the destination, we have \(c_0 = \frac{4\beta}{1+T+\beta}\). In Figure 6, we show the contour plot of those CCR’s in the \(\beta - T\) plane. We observe trends similar to those in the previous plot. \(^7\)

\[\text{Figure 5. Contour Plot of CCR when } \alpha \to 0\]

\(^6\) The contour line associated with CCR value 1 is \(T = -2 + \beta + 2\sqrt{1+\beta^2}\).

\(^7\) The contour line associated with CCR value 1 is \(T \to -1 + 3\beta\).
From the limiting cases of storm location, we conclude that the storm located near the origin or the destination is more likely to take ground delay even when the cost ratio is high. There exist, however, weather conditions when the immediate take-off is optimal at a fairly low cost ratio, especially when the storm is small.

When $\beta \to 0$, the CCR is zero, while it is larger than 1 when $\beta \to \infty$. Such results follow our intuition that ground delay is not appropriate for very small storms, while it is always justifiable for very large ones.

When $T = \alpha + \beta$, we have $c_0 = \frac{-1+\alpha+\beta+\sqrt{(1-\alpha)^2+\beta^2}}{\alpha+\beta}$. Since $\frac{-1+\alpha+\beta+\sqrt{(1-\alpha)^2+\beta^2}}{\alpha+\beta} > \frac{2(\alpha+\beta-1+\sqrt{(1-\alpha)^2+\beta^2})}{T+\alpha+\beta}$, the CCR when $T = \alpha + \beta$ is an upper bound to the CCR in general. In other words, the cost ratio to eliminate the ground delay option is the highest among all CCR’s associated with a specific combination of $\alpha$ and $\beta$, when $T$ is $\alpha + \beta$. In Figure 7, we show the contour plot of such CCR’s in the $\alpha - \beta$ plane. In the plot, we observe that CCR is always larger than 1 when $\beta$ is larger than 1, which indicates that ground delay is always appropriate for storms larger than 1, if the maximum storm duration time is close to the flight time to the tip of the storm on the nominal route without recourse. We also observe that the CCR is more sensitive to the storm size than to storm location, except when the location is very near the origin airport. When $T \to \infty$, we have $c_0 \to 0$, which follows our intuition; for storms expected to last for a longer period time, it is optimal not to take any ground delay to fly on the route determined by the DRM.
5.5. Sensitivity analysis

We first discuss the sensitivity of the critical cost ratio \( c_0 = \frac{2(\alpha+\beta-1+\sqrt{(1-\alpha)^2+\beta^2})}{T+\alpha+\beta} \). Since \( \frac{\partial c_0}{\partial T} < 0 \), ground delay becomes less attractive for storms expected to last longer. Likewise, \( \frac{\partial c_0}{\partial \beta} > 0 \) suggests that ground delay becomes more valuable with a larger storm even at a relatively higher cost. The sensitivity of the storm location, however, needs a closer look. \( c_0 \) is not monotonic with respect to \( \alpha \), yet the partial second derivative is positive since \( \frac{\partial^2 c_0}{\partial \alpha^2} > 0 \). Therefore, \( c_0 \) is convex with respect to \( \alpha \), which means for storms located near the origin or the destination, the optimal solution is more likely to include ground delay. It also means that an intermediate and the detour route are more likely to be optimal for storms in the vicinity of the nominal route midpoint, although the nominal may still be preferable if ground delay cost is low.

As discussed in section 5.4, If the optimal solution includes ground delay, optimal amount of such delay is

\[
g^*_{\text{int}} = \frac{-1+c-2cT+c^2T+\alpha+\sqrt{-2c+c^2-2a+6c\alpha-2c^2\alpha+4c^2\alpha^2+2c^2\alpha^2}}{-2c+c^2}.
\]

Note that \( g^*_{\text{int}} \) does not depend on the storm size \( \beta \), which seems counter-intuitive. In fact, the effect of \( \beta \) is captured in the critical cost ratio \( c_0 \).

\( g^*_{\text{int}} \) is a concave monotonic decreasing function with respect to \( c \) since \( \frac{\partial g^*_{\text{int}}}{\partial c} > 0 \) and \( \frac{\partial^2 g^*_{\text{int}}}{\partial c^2} < 0 \). The decreasing trend is consistent with the observations in the numerical analysis. The concavity suggests that the amount of ground delay decreases at an increasing rate as the CCR increases.

As for the storm location \( \alpha \), we have \( \frac{\partial g^*_{\text{int}}}{\partial \alpha} < 0 \) and \( \frac{\partial^2 g^*_{\text{int}}}{\partial \alpha^2} > 0 \). Therefore, \( g^*_{\text{int}} \) is a convex monotonic decreasing function with respect to \( \alpha \). Therefore, the optimal ground delay decreases with distance of the storm from the origin, and the rate of decrease is more rapid for storms located near the origin. It also suggests that for storms located near the destination, ground delay has limited value.

As for the maximum storm duration time, we have \( \frac{\partial g^*_{\text{int}}}{\partial T} = 1 \), and \( g^*_{\text{int}} \) is linearly increasing function with respect to \( T \). In other words, the amount of ground delay increases at exactly the same rate as \( T \). Such result follows out intuition that the ground delay amount essentially reduces the maximum storm duration time.

5.6. Performance analysis of DRM and hybrid-DRM

The performance analysis of hybrid DRM is based on the cost reduction from using the model compared to the DRM. Since the DRM is an upper bound of the hybrid-DRM, the solution of the hybrid-DRM is never worse than that of the DRM.\(^8\) The performance metric is defined as the percentage cost saving as shown below.

\[
S(hDRM) = 1 - \frac{\text{Optimal Expected Total Cost of hybrid-DRM}}{\text{Optimal Expected Total Cost of DRM}}
\]

In Figure 8, the cumulative distribution function plot of \( S(hDRM) \) is shown for \( c = 0.1, 0.2, \ldots, 0.9 \). When \( c = 0.1 \), almost 80% of all cases we tested have cost savings greater than 5% when ground delay is allowed. This percentage decreases as the cost ratio increases, reaching about 22% when \( c = 0.9 \). Likewise, we observe that nearly 50% of tested cases realize more than 25% saving when \( c = 0.1 \), while none of tested cases show this level of saving when \( c = 0.9 \). The CDFs are also seen to be quite linear and nearly parallel when they approach their limiting value of 1.

The average and maximum cost savings are plotted against the storm parameters in Figure 9. The average and maximum saving is obtained for each parameter by aggregating the other parameters.

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\(^8\) In theory, the optimal expected total cost of DRM can be obtained from hDRM by assuming \( c \to \infty \).
Figure 8. Cumulative Distribution Function of $S(hDRM)$ With Various Cost Ratio $c$

Figure 9. Average and Maximum $S(hDRM)$ With Respect To Weather Parameter
In the $\alpha - S(h\text{DRM})$ plot, we observe that both the average and the maximum cost savings reduce as $\alpha$ increases. Such trend suggests that the addition of ground delay is beneficial for storms located closer to the origin. The overall shape however, is nearly flat, which indicates that the effect of ground delay is better understood in combinations of weather parameters than solely the location of the storm. The $\beta - S(h\text{DRM})$ plot shows that both the average and the maximum cost saving increases as $\beta$ increases, which suggests that larger storms may realize greater benefit from the ground delay option. Note that when the storm is smaller than 0.4, the average saving is below 3% even though the maximum saving can be high. Such a difference indicates that there are few cases involving small storms in which taking ground delay can significantly reduce costs. The $T - S(h\text{DRM})$ plot shows similar trends to those observed in the $\beta - S(h\text{DRM})$ plot, suggesting that the ground delay option is more useful for storms with longer duration time. The insensitivity of savings in the region where $T$ is larger than 2 indicates that a longer duration time doesn’t always result in greater savings from employing ground delay, but rather also depends on other factors such as the cost ratio.

6. Conclusions and future research

In this paper, a probabilistic air traffic management strategy is proposed and studied, which aims to reduce the risks associated with weather uncertainty in the airspace. The proposed strategy seeks to achieve a minimum expected total cost, when a set of operational and control capabilities such as hedged routes and recourses are available. A geometric model is adopted to incorporate operational flexibilities including the first and the second recourses as well as route hedging. The simple geometric setup enables us to conduct complex analyses and gain critical knowledge both in the quantitative and qualitative aspects.

Among numerous variations of the geometric recourse model, we first study the Dual Recourse Model (DRM), which allows both the first and the second recourses. According to the nominal route theorem, regardless of the weather clearance time distribution, the nominal route is always optimal if the flight time to the tip of the storm on the detour route doesn’t exceed the maximum storm duration time. It is the second recourse option that makes the nominal route optimal in many cases, which points to the value of having a responsive air traffic management system that can reroute planes quickly to respond to changing weather condition.

The ground-airborne hybrid model allows weather avoidance through ground holding in addition to the recourses and route hedging. Since the cost ratio between the ground and airborne delay is not fixed, we treat the cost ratio as an additional model parameter and find the optimal combination of the ground delay and the route choice to minimize the total expected cost. Assuming a uniformly distributed weather clearance time, the optimal route of the hybrid model is determined by the geometric recourse model with the maximum storm duration time reduced by the ground delay amount.

We identify the ground-airborne cost ratio threshold, which we call the Critical Cost Ratio (CCR), that triggers elimination of the ground delay option. If the current cost ratio is below the CCR, it is optimal to take positive ground delay and fly on the nominal route. On the other hand, if it is above the CCR, immediate take-off without ground delay is optimal, and the optimal route is determined by the DRM. The formula of the CCR is obtained in closed form in terms of the storm location, size, and the maximum duration time.

In the sensitivity analysis, we find that storms located near the origin or the destination is more likely to benefit from ground delay than those in the midway. For storms about halfway between the origin and destination, non-nominal routes are especially useful unless the cost ratio is fairly low. It is also found that the optimal ground delay is larger when the storm is either located near the origin or associated with larger maximum duration time. The optimal ground delay is invariant to storm size, however, except insofar as the size affects the CCR.

Cost savings of the hybrid DRM over the DRM depends on the cost ratio. At 30% cost ratio, over 60% of all cases we tested show savings of more than 10%, with the maximum saving of nearly 45%. At a 50% cost ratio, nearly 45% of the cases show at least a 10% savings from taking ground delay, with the maximum savings of 35%.
The ground delay is thus expected to continue to be a valuable strategy for avoiding en route weather even when flight routing becomes more flexible and responsive to changing conditions.

We conclude that the capability to respond to weather change during the course of the flight such as the recourse option is most effective in reducing cost associated with uncertain weather. The first recourse enables an efficient use of the detour route, while the second recourse combined with the nominal route is one of the most effective ways to save cost when optimal. The ground delay option adds another cost saving opportunity from expanded use of the nominal route, although its benefit depends on the ground-airborne cost ratio. It is not rare for immediate take-off on the hedged routes to be optimal, and we have identified the weather and the cost ratio combinations for such cases. The hedged routes are especially useful in cases when the storm is of size smaller than the nominal route length and the CCR is low.

There are several variations of the geometric recourse model that we consider in future research. One can change the assumptions on the weather, for example, by introducing a non-uniform weather probability distribution. Other storm geometries, including storms that are not orthogonal to the nominal route, and moving storms, may also be considered. The choice of routes might be extended as well to consider curvilinear ones that begin at a narrower angle relative to the nominal route but splay outward as the aircraft approaches an active storm. We can also introduce a variable cost structure for operational capabilities. For example, the extensive use of the nominal route in the DRM and the hybrid-DRM is due to the availability of the second recourse at no additional cost. However, it is reasonable to suspect that the last minute change of the flight plan very near the restricted airspace might involve a higher cost than the first recourse or ground-holding.

The development of future air traffic management strategies not only involves finding solution algorithms for complex optimization models but also requires extensive analyses to extract critical insights and policy implications for the air transportation system users and the air navigation service provider. Although the expected total cost functions of the geometric models we discussed are obtainable in analytic forms, this may not be the case for other variations, and different methods such as numeric simulation may be needed to find solutions. We expect many of the qualitative insights gained from our research to also hold in more general and complex settings in which the analytical methods are no longer tractable.

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