MINIMAL RESOLUTIONS OF GORENSTEIN ORBIFOLDS IN DIMENSION THREE

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We give the complete solution of constructing $c_1 = 0$ resolutions of 3-dimensional Gorenstein quotient singularities, and verify the equality between the Euler numbers of resolutions and the orbifold Euler characteristics of quotient singularities.

1. INTRODUCTION

We shall discuss the construction of minimal resolutions of orbifolds with Gorenstein quotient singularities, and the computation of their Euler characteristic. By an orbifold in this paper we shall mean a quasi-projective analytic variety with at most quotient singularities. It is also called a $V$-manifold introduced by Satake in some other literatures [23]. In this article, we shall only concern those singularities which are quotients of $\mathbb{C}^*$ by finite subgroups of $SL_n(\mathbb{C})$. For $n = 2$, these singularities were classified by Klein in 1872 (or so) in his work on the invariant theory of regular solids in $\mathbb{R}^3$ [13]. The Klein’s classification is the well-known A–D–E type singularities. They also arise in many other contexts, to mention only representations of finite groups, topology and complex algebraic geometry. In recent years, the Kleinian singularities have also appeared in the study of conformal field theory in physics. For the higher dimension $n$, these Klein-type singularities are called the Gorenstein quotient singularities. For a complex variety $X$ with Gorenstein quotient singularities, the canonical sheaf $\omega_X$ of $X$ is a locally free $\mathcal{O}_X$-sheaf. By a minimal resolution $\tilde{X}$ of $X$ we shall always mean a desingularization

$$\rho: \tilde{X} \to X$$

such that the canonical bundle of $\tilde{X}$ is equal to $\rho^* \omega_X$. In this paper, we shall complete the ongoing program of constructing minimal resolutions of 3-folds with Gorenstein quotient singularities, and confirming their Euler numbers being given by formula of “orbifold Euler characteristic”. The orbifold Euler number of a manifold $M$ quotiented by a finite group of symmetries $G$ is defined by the expression:

$$\chi(M, G) = \frac{1}{|G|} \sum_{gh=hg} \chi(M^g \cap M^h)$$

where the summation is over the commuting pairs $g, h$ in $G$, and $M^g, M^h$ are the fixed-point sets of $g, h$ respectively. This formula was introduced by Dixon et al. [6] in string theory as the correct Euler characteristic for the quotient of $M$ by the group $G$. By some elementary

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calculation [8], an equivalent definition of the orbifold Euler characteristic is given by

$$\chi(M, G) = \sum_{[g]} \chi(M^g/Z_g)$$  \hspace{1cm} (2)

where $[g]$ is the conjugacy class of $g$, and $Z_g$ its centralizer. The above expression is known to be the same as the Euler characteristic in equivariant K-theory for $K_0(M)$ [1], and in a certain sense $\chi(M/G)$ could be regarded as some kind of "Euler characteristic" for the "loop space" of $M/G$ [14]. However for the physicists' interest, $M/G$ is a n-fold with the trivial canonical sheaf. A minimal resolution $\tilde{M}/\tilde{G}$ with its Euler number given by

$$\chi(\tilde{M}/\tilde{G}) = \chi(M, G)$$

would be expected to exist, say at least for $n = 2$ and 3. In the surface case, this above equality was verified by Hirzebruch and Hörer in [8] with $\tilde{M}/\tilde{G}$ the minimal resolution in surface theory. For the higher dimension $n$, one obvious difficulty for the problem should be the absence of a general theory for the existence of "minimal" one among all the resolutions of a singularity as the surface singularity does. However, for $n = 3$, much progress has been made on the above resolution $\tilde{M}/\tilde{G}$, and together with their important applications to the study of Calabi-Yau spaces [16, 22, 25]. In particular, when $G$ is abelian, the toric geometry [5, 12, 18] has provided an effective method for the construction of minimal resolutions, and the above equality can be easily shown [16, 19, 22]. The result of a general group $G$ is less complete till now. However, substantial progress has recently been obtained on the minimal resolutions $\tilde{M}/\tilde{G}$ for non-abelian groups $G$, among which are the trihedral group [10] and two simple groups [15, 20], (the type (C), (H), (I) respectively in the classification of Section 3, and for a review, see [21] and references therein). Hence a complete solution for this minimal resolution problem for $n = 3$ is expected to be found. In this present paper we show that this is indeed the case. Our results will be stated in the Main Theorem and Theorem 1 of Section 2. Even though the previous unknown cases are all solvable groups, the list is still long. Working on the solution by cases is quite cumbersome and not very illuminating. Our method of solving the problem will go through an induction procedure based on examining whether the group $G$ contains the center of $SL_3(C)$, and studying the relation of orbifold Euler characteristics in this process. Then we can reduce the problem to only two cases, of which the solution can be obtained by methods in toric geometry.

The following is a summary of the contents of this article: In Section 2, we shall review some well-known results on orbifold Euler characteristic, focussing on the general notion of "orbifold Euler number" of a variety with quotient singularities. We shall state the main result of this paper, and explain the procedure of reducing the problem to its local formulation. We shall give the classification of finite groups $G$ of $SL_3(C)$ in Section 3 for easy reference. In Section 4, we shall consider the cases when the finite subgroup $G$ contains the center of $SL_3(C)$. By some inductive processes, one can reduce the problem to the cases for $G$ not containing the center of $SL_3(C)$. Together with the known results for the solution of minimal resolutions, only type (B) and (D) solvable groups in the classification are remained to be solved. The solutions for these two cases will be given in Sections 5 and 6 respectively. The results will depend again on an induction procedure of reducing the
general cases to some special ones to which the method of toric geometry can effectively apply.

Notation. For a topological space \( X \),

\[ \chi(X) : \text{the Euler number of } X \]
\[ X^g : \{ x \in X | g(x) = x \} \text{ for a map } g: X \to X. \]

For \( g \) in a finite group \( G \), and \( S \) a subset of \( G \), we denote

\[ Z_g : \text{the centralizer of } g \]
\[ [g] : \text{the conjugacy class of } g \]
\[ Z(G) : \text{the center of } G \]
\[ Cl(G) : \text{the set of conjugacy classes of } G \]
\[ N_S : \text{the normalizer of } S. \]

2. ORBIFOLD EULER CHARACTERISTIC

In this section, we shall refine the quantities of (1) and (2) to a more general notion of orbifold Euler number for a variety with Gorenstein quotient singularities, and then describe the main result of this paper.

Let \( X \) be a \( n \)-dimensional orbifold with only Gorenstein quotient singularities. Denote \( \text{Sing}(X) \) the singular set of \( X \). For an element of \( x \) of \( X \), there is a finite subgroup \( \pi_x \) of \( SL_n(\mathbb{C}) \) such that

\[ (X,x) \simeq (\mathbb{C}^n/\pi_x, \hat{0}) \]

as germs of analytic spaces. The group \( \pi_x \) is characterized as the fundamental group of \( \mathbb{U}_x - \text{Sing}(X) \) for a sufficiently small neighborhood \( \mathbb{U}_x \) of \( x \) in \( X \). It is known that

\[ \chi(\mathbb{C}^n, \pi_x) = |Cl(\pi_x)|. \]

Define the map

\[ e: X \to \hat{\mathbb{Z}}, \quad e(x) = |Cl(\pi_x)| \]

which is a function with its image consisting of a finite number of positive integers. Then \( e^{-1}(1) = X - \text{Sing}(X) \), and the closure of \( e^{-1}(k) \) is an analytic subspace of \( X \) for \( k \in \mathbb{Z} \). For the purpose of later discussions, we define the following topological invariant of \( X \):

Definition. The orbifold Euler characteristic \( \chi^o(X) \) is the expression:

\[ \chi^o(X) = \sum_{k \geq 1} k \chi(e^{-1}(k)). \]

The right-hand side is indeed a finite sum by setting Euler number of the empty set to be zero. Since the Euler number of \( X \) is given by

\[ \chi(X) = \sum_{k \geq 1} \chi(e^{-1}(k)) \]
$\chi^s(X)$ and $\chi(X)$ are different for a general singular space $X$, and equal when $X$ is non-singular. Note that if $X$ is a finite quotient of a smooth manifold, the above $\chi^s(X)$ is the same as the formula (1) or (2) in Section 1. Actually this relation can be stated in the following lemma, a fact which should be known to specialists (e.g. [8] or [19, Theorem 2]). We repeat it here for the sake of completeness.

**Lemma 1.** Let $M$ a $n$-dimensional smooth variety acted by a finite group $G$ with codim $M^s > 2$ for every non-trivial element $g$ in $G$. If $M/G$ has only Gorenstein quotient singularities, then

$$\chi^s(M/G) = \chi(M, G).$$  \hspace{1cm} (4)

**Proof.** Let $p : M \to X := M/G$ be the natural projection. For $m \in M$ and $x = p(m)$, the canonical sheaf of $M/G$ is trivial near $x$. The group $\pi_x$ in (3) in this situation is the isotropy subgroup $G_m$ at $m$:

$$\pi_x = G_m.$$ \hspace{1cm} (5)

Denote

$$\mathcal{S} = \{G_m \mid m \in M\}$$

$$M(H) = \{m \in M \mid G_m = H\} \quad \text{for} \ H \in \mathcal{S}.$$  

$M$ is the disjoint union of $M(H)$ for $H \in \mathcal{S}$. Since an element $g$ sends $M(H)$ to $M(gHg^{-1})$, $M(H)$ is stable under the action of the normalizer $N_H$ of $H$. The closure $\overline{M(H)}$ of $M(H)$ is equal to the intersection of all $M^s$ with $g$ in $H$. Hence $\overline{M(H)}$ is a closed submanifold of $M$, and $\overline{M(H)} - M(H)$ a finite union of its proper submanifolds. The quotient group $N_H/H$ acts on $\overline{M(H)}$, and freely on $M(H)$. There exists a sequence of $G$-invariant closed subspaces of $M$:

$$M = M_0 \supset \cdots \supset M_k \supset M_{k+1} = \emptyset$$

such that $M_i = M_{i+1} \cup G \cdot M(H_i)$ and $M_i - M_{i+1} = G \cdot M(H_i)$ for a complete set of representatives, $H_i$ ($1 \leq i \leq k$), in $\mathcal{S}$ acted by $G$ through conjugation. Then we have the corresponding sequence of closed subspaces of $X$:

$$X = M_0/G \supset M_1/G \supset \cdots \supset M_k/G \supset M_{k+1}/G = \emptyset$$

and

$$X = \bigsqcup_{i=1}^{k} M(H_i)/N_{H_i}.$$  

By (5),

$$\chi^s(X) = \sum_{i=1}^{k} |Cl(H_i)| \chi(M(H_i)/N_{H_i})$$

$$= \sum_{i=1}^{k} |Cl(H_i)| |H_i|/|N_{H_i}| \chi(M(H_i))$$

$$= \frac{1}{|G|} \sum_{H \in \mathcal{S}} |Cl(H)| |H| \chi(M(H)).$$
On the other hand, for a commuting pair \((g, h) \in G \times G\), \(M^g \cap M^h\) is a disjoint union of \(M(H)\) for those \(H\) containing \(g, h\). Therefore

\[
\chi(M^g \cap M^h) = \sum_{g, h \in H \in \mathcal{F}} \chi(M(H)).
\]

By (1),

\[
\chi(M, G) = \frac{1}{|G|} \sum_{g \in H \in \mathcal{F}} \sum_{h \in H} \chi(M(H)) = \frac{1}{|G|} \sum_{H \in \mathcal{F}} \sum_{g, h \in H, gh = hg} \chi(M(H)) = \frac{1}{|G|} \sum_{H \in \mathcal{F}} |C(H)| |H| \chi(M(H)).
\]

Therefore we have the equality (4). \(\square\)

For the rest of this paper, we shall consider only the case for \(n = 3\). The purpose of this article is to show the following result:

**Main Theorem.** Let \(X\) be a 3-dimensional orbifold with only Gorenstein quotient singularities. Then there exists a minimal resolution \(\tilde{X}\) of \(X\) with

\[
\chi(\tilde{X}) = \chi^a(X).
\]

The above problem is essentially a local one as one has the following lemma.

**Lemma 2.** Let \(X\) be the same as in the Main Theorem. Assume that for every \(x \in X\), there exist a small neighborhood \(U_x\) of \(x\) in \(X\) and its resolution \(\tilde{U}_x\) with

\[
\omega_{\tilde{U}_x} = \tilde{\omega}_{U_x}, \quad \chi(\tilde{U}_x) = e(x).
\]

Then \(X\) has a minimal resolution \(\tilde{X}\) such that the equality (6) holds.

**Proof.** The singular set \(\text{Sing}(X)\) consists of finite isolated points and possibly some curve \(\mathcal{C}\) having a finite number of singularities \(x_i, 1 \leq i \leq l\). For \(x \in \mathcal{C}\) and \(x \neq x_i\), the 3-dimensional representation of \(\pi_x\) in (3) has a fixed line in \(\mathbb{C}^3\), hence there is an unique minimal resolution of \(\mathbb{C}^3/\pi_x\), given by the minimal resolution for surface singularity normal to \(\text{Sing}(\mathbb{C}^3/\pi_x)\). Then one can patch the local data \(\{\tilde{U}_x\}_{x \in X}\) to obtain the resolution \(\tilde{X}\). Since the Euler number of an odd dimensional differential manifold is equal to zero, the Mayer–Vietoris argument will show that the Euler number of \(\tilde{X}\) equals to \(\chi^a(X)\). \(\square\)

Therefore our Main Theorem is reduced to the local situation of a group \(G \subset SL_3(\mathbb{C})\) acting on \(\mathbb{C}^3\):

**Theorem 1.** Let \(G\) be a finite group of \(SL_3(\mathbb{C})\). Then there exists a resolution \(\mathbb{C}^3/G\) of \(\mathbb{C}^3/G\) with the trivial canonical bundle, and

\[
\chi(\mathbb{C}^3/G) = \chi(\mathbb{C}^3, G).
\]
3. CLASSIFICATION OF FINITE GROUP OF $SL_3(C)$

We denote the generator of the center of $SL_3(C)$ by

$$W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \omega = \frac{1}{2}(-1 + \sqrt{-3}).$$

The classification of finite subgroups of $SL_3(C)$ (up to linear equivalence) was done by Blichfelt [3,4] in 1917 and Miller et al. [17] in 1916. Here is the list of the classification obtained by them (see also in [10,26]).

(A) (Abelian) A diagonal subgroup consisting of matrices $[\alpha, \beta, \gamma]$.

(B) A group consisting of matrices

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}.$$

(C) A group generated by an abelian group $H$ in (A) and the transformation

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(D) A group generated by an abelian group $H$, $T$ in (C) and a transformation

$$R = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}, \quad abc = -1.$$

(E) Group of order 108 generated by $T, S, V$:

$$S = [1, \omega, \omega^2], \quad V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

(F) Group of order 216 generated by (E) and the transformation

$$\frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}.$$

(G) Group of order 648 generated by (E) and the transformation

$$[\epsilon, \epsilon, \epsilon \omega], \quad \epsilon^3 = \omega^2.$$

(H) Icosahedral group: simple group of order 60 generated by $T, E_2, E_3$:

$$E_2 = [1, -1, -1]$$

$$E_3 = \frac{1}{2} \begin{pmatrix} -1 & \mu_- & \mu_+ \\ \mu_- & \mu_+ & -1 \\ \mu_+ & -1 & \mu_- \end{pmatrix}, \quad \mu_{\pm} = \frac{1}{2}(-1 \pm \sqrt{5}).$$
(H*) Group of order 180 generated by (H) and W.

(I) Simple group of order 168 generated by $S_7, T, U$:

$$S_7 = [\beta, \beta^2, \beta^3], \quad \beta^7 = 1$$

$$U = \frac{1}{\sqrt{-7}} \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}, \quad a = \beta^2 - \beta^3, \quad b = \beta^2 - \beta^4, \quad c = \beta - \beta^6.$$

(I*) Group of order 504 generated by (I) and W.

(J) Group $G$ of order 1080 generated by (H) and $E_4$:

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & -\omega^2 & 0 \end{pmatrix}$$

with center $Z(G) = \langle W \rangle$ and $G/Z(G) \sim$ the alternating group $A_6$.

4. GROUPS CONTAINING CENTER OF $SL_3(C)$

In this section, we are going to reduce the cases of Theorem 1 to those which do not contain $W$. First we need a simple fact on finite group theory:

**Lemma 3.** Let $C$ be a subgroup of the center of $G$ with $|C| = p$, $p$ a prime. Let

$$\pi : G \rightarrow G/C$$

be the natural projection, and $\pi_*$ the induced map on the conjugacy classes of $G$ and $G/C$,

$$\pi_* : \text{Cl}(G) \rightarrow \text{Cl}(G/C).$$

Then for $g \in G$ and $x = [\pi(g)] \in \text{Cl}(G/C)$, we have

$$|\pi_*^{-1}(x)| |N_{\pi^{-1}(\pi(g))}/Z_{\pi^{-1}(\pi(g))}| = p. \quad (8)$$

As a consequence, $|\pi_*^{-1}(x)| = 1$ or $p$, and $|\pi_*^{-1}(x)| = p$ if and only if $N_{\pi^{-1}(\pi(g))} = Z_{\pi^{-1}(\pi(g))}$.

**Proof.** Let $g$ be an element of $G$, and $x = [\pi(g)]$. Every element in $\pi_*^{-1}(x)$ has a representative in $\pi^{-1}(\pi(g))$. Since the centralizers of elements in $\pi^{-1}(\pi(g))$ are the same, and all equal to $Z_{\pi^{-1}(\pi(g))}$, $N_{\pi^{-1}(\pi(g))}/Z_{\pi^{-1}(\pi(g))}$ acts freely on $\pi^{-1}(\pi(g))$. Then $\pi_*^{-1}(x)$ is in one-one correspondence with the collection of $N_{\pi^{-1}(\pi(g))}/Z_{\pi^{-1}(\pi(g))}$-orbits in $\pi^{-1}(\pi(g))$. Hence

$$|\pi_*^{-1}(x)| |N_{\pi^{-1}(\pi(g))}/Z_{\pi^{-1}(\pi(g))}| = |\pi^{-1}(\pi(g))| = p.$$

The rest follows immediately. \qed

The above lemma can be explicitly realized in the following example.

**Example.** There are 14 conjugacy classes of the group $G$ of type (E), which are represented by the elements

$$\{I, W, W^2, S, ST\} \cup \{V^i W^k \mid 1 \leq i < 3, 0 \leq k < 2\}.$$

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We have
\[ V^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad VTV^{-1} = S^2, \quad VSV^{-1} = T \quad (9) \]

\[ TS^k = W^k S^k T, \quad k \in \mathbb{Z}. \]

The subgroup generated by \( S \) and \( T \) is a normal subgroup of \( G \) of index 4, which consists of all the order 3 elements of \( G \). Hence it contains all the order 3 diagonal matrices with the determinant equal to one. We set \( C = Z(G) = \langle W \rangle \) in Lemma 3. By (9), one obtains the conjugacy classes of \( S \) and \( ST \) in \( G \):

\[ [S] = \{ S^j W^k, T^j W^k \mid 1 \leq j \leq 2, 0 \leq k \leq 2 \} \]
\[ [ST] = \{ S^i T^j W^k \mid 1 \leq i, j \leq 2, 0 \leq k \leq 2 \} \]

and each consists of 12 elements. Hence \( \langle S, T \rangle \) is a disjoint union of \( Z(G) \), \( [S] \), and \( [ST] \); and for any order 3 element \( g \) in \( G - Z(G) \), one has \( |Z_g| = 9 \). Hence for \( 1 \leq i \leq 3 \), the only order 3 elements in \( Z_{Vi} \) are in \( Z(G) \). Therefore

\[ Z_{Vi, W_k} = Z_{Vi} = \langle V, W \rangle, \quad |[V^i W^k]| = 9. \]

Since the orders of elements in \( G \) are 1, 2, 3, 4, 6, 12, Sylow's theorem implies

\[ G = Z(G) \cup [S] \cup [ST] \cup \{ [V^i W^k] \mid 1 \leq i \leq 3, 0 \leq k \leq 3 \}. \]

By \( |G - \langle S, T \rangle| = 81 \), the above relation is indeed the conjugacy decomposition of \( G \). For the quotient group \( G/Z(G) \), there are six conjugacy classes, represented by

\[ I, \pi(S), \pi(ST), \pi(V), \pi(V^2), \pi(V^3) \]

among which the classes with only one element in the \( \pi \)-inverse are \( \pi(S) \) and \( \pi(ST) \).

Let \( G \) be a finite subgroup of \( SL_3(\mathbb{C}) \) containing \( W \), and \( \bar{G} \) the quotient of \( G \) by the subgroup generated by \( W \),

\[ \pi : G \to \bar{G} := G/\langle W \rangle \]
\[ g \mapsto \bar{g} := \pi(g). \]

It is known that \( C^3/\langle W \rangle \) has a unique minimal resolution, denoted by \( \mathcal{W} \). It is equal to the \( -3 \) hyperplane bundle over the projective plane \( \mathbb{P}^2 \):

\[ p : \mathcal{W} = H^{-3} \to \mathbb{P}^2 \]

with the zero section as the exceptional divisor in \( \mathcal{W} \). The linear action of \( G \) on \( C^3 \) induces one for the above line bundle, on which \( W \) acts trivially. Hence we have the bundle action:

\[ \bar{G} \times \mathcal{W} \to \mathcal{W} \]
\[ \downarrow \]
\[ \bar{G} \times \mathbb{P}^2 \to \mathbb{P}^2 \]

which is compatible with the action of \( \bar{G} \) on \( C^3/\langle W \rangle \). It is easy to see that the holomorphic volume form on \( \mathcal{W} \) is invariant under \( \bar{G} \). Hence \( \mathcal{W}/\bar{G} \) has the trivial canonical sheaf.
Lemma 4. The following equality holds:

\[ \chi(C^3, G) = \chi(\mathcal{W}, \tilde{G}). \]  

(10)

Proof. For \( g \in G \), the centralizer of \( \tilde{g} \) in \( \tilde{G} \) is given by

\[ Z_{\tilde{g}} = N_{x^{-1}(g)}Z_g \]

(11)

hence by Lemma 3,

\[ |Z_{\tilde{g}}| = 1 \text{ or } 3. \]

Also we have

\[ \chi(C^3, G) = |Cl(G)| = \sum_{[\tilde{g}] \in Cl(G)} |\pi_{\tilde{g}}^{-1}([\tilde{g}])| = \sum_{[\tilde{g}] \in Cl(G)} \frac{3}{|Z_{\tilde{g}}|} \]

\[ \chi(\mathcal{W}, \tilde{G}) = \sum_{[\tilde{g}] \in Cl(G)} \chi(\mathcal{W}^g/Z_{\tilde{g}}) \]

Therefore the equality (10) will follow from the relations:

\[ \chi(\mathcal{W}^g) = 3 \text{ for } \tilde{g} \in \tilde{G} \]

\[ \chi(\mathcal{W}^g/Z_{\tilde{g}}) = 1 \text{ if } |Z_{\tilde{g}}| = 3. \]

First we describe the fixed point set \( \mathcal{W}^g \). Let \( \mathcal{C}^3 \) be the blow-up of \( C^3 \) at origin, identified with the \(-1\) hyperplane bundle over \( P^2 \). Regard \( \mathcal{W} \) as the quotient \( \mathcal{C}^3 \) by the multiplication of \( \omega \):

\[ \mathcal{C}^3 \rightarrow \mathcal{W} = \mathcal{C}^3/\langle \omega \rangle. \]

(12)

Consider a linear subspace \( L \) of \( C^3 \) which is an eigenspace of \( g \) with the eigenvalue \( \mu \). The proper transform of \( L \) in \( \mathcal{C}^3 \) is mapped onto a submanifold \( B(L) \) of \( \mathcal{W} \) via (12). Then \( B(L) \) intersects the exceptional divisor of \( \mathcal{W} \) at a submanifold \( P(L) \), which is isomorphic to the projective space for the linear space \( L \). Define

\[ F_\mu = \begin{cases} P(L) & \text{if } \mu^3 \neq 1 \\ B(L) & \text{otherwise}. \end{cases} \]

One has

\[ \mathcal{W}^g = \bigsqcup \{ F_\mu | \mu: \text{eigenvalues of } g \}. \]

Note that \( g \) has 3, 2, or 1 distinct eigenvalues according to the maximal dimension of eigenspaces being 1, 2 or 3. Then it is easy to see that \( \chi(\mathcal{W}^g) = 3 \). In the situation for \( |Z_{\tilde{g}}| = 3 \), one has

\[ hgh^{-1} = gW \text{ or } gW^2, \quad \text{for } h \in N_{x^{-1}(g)} - Z_g. \]

For an eigenspace \( L \) of \( g \) with eigenvalue \( \mu \), the above \( h \) sends \( L \) to a different linear subspace \( h(L) \) of \( C^3 \). However, \( h(L) \) is the eigenspace of \( hgh^{-1} \) with the eigenvalue \( \mu \). This implies the eigenspaces of \( g \) are all one-dimensional, which are acted transitively by \( \langle h \rangle \). Therefore \( \chi(\mathcal{W}^g/Z_{\tilde{g}}) = 1. \]

We may regard the minimal resolutions for \( \mathcal{W}/\tilde{G} \) as minimal resolutions of \( C^3/G \). It remains to show the equality (7) holds. By Lemmas 2 and 4, we need only to study the local problem near a point \( x \) of \( \mathcal{W} \). We have the isomorphism:

\[ (\mathcal{W}, x) \simeq (C^3, \tilde{G}_x) \]
for some embedding of the isotropy subgroup \( \tilde{G}_x \) into \( SL_3(\mathbb{C}) \). If \( \tilde{G}_x \) also contains \( W \), we may apply the above procedure again, then eventually arrive at the situation where \( W \) is not in the groups we shall work with. Therefore the conclusion of Theorem 1 follows from the group \( G \) in Section 4 with \( W \notin G \). Since type (E), (F), (G), (H*), (I*), (J) groups all contain \( W \), Theorem 1 is now equivalent to the following one.

**Theorem 2.** Let \( G \) be a group of type \((A), (B), (C), (D), (H)\) or \((I)\), and \( W \notin G \). Then there exists a minimal resolution with

\[
\chi(C^3/G) = \chi(C^3, G).
\]

Among the above groups, the results of Theorem 2 have been known for type (A) \([16,19,22]\), (C) \([10]\), (H) \([20]\) and (I) \([15]\). So only type (B) and (D) are needed to consider. We shall prove these two cases in the next two sections. In this way we show Theorem 2, hence follow Theorem 1 and the Main Theorem. As both of these remaining cases are solvable groups, the following two lemmas will be useful for our later discussion, one is on the conjugacy classes and the other on the minimal resolutions, both related to certain solvable groups we shall deal with.

**Lemma 5.** Let \( G \) be a finite group, \( H \) and \( L \) its subgroups with \( H \) abelian and normal. Assume the following conditions are satisfied:

1. \( G \) is the semidirect product of subgroups \( L \) by \( H \).
2. The order of \( Z_i \cap L \) is a prime for \( y \in L, y \neq 1 \).

Then

\[
|Cl(G)| = |H/L| + \sum |(Z_i \cap H)|
\]

Here the summation \( l \) is over the representatives for non-trivial elements in \( Cl(L) \), and \( H \) acts on \( L \) by conjugation.

**Proof.** For \( y \in L \), it induces an automorphism \( \varphi_y \) of \( H \) by conjugation:

\[
\varphi_y : H \to H, \quad \varphi_y(x) = yxy^{-1}.
\]

Every element of \( G \) can be written uniquely as a product of \( xy \) with \( x \in H \) and \( y \in L \). We have

\[
(x'y)(xy)(x'y)^{-1} = \{\varphi_{y'}(x') \varphi_{y'y}^{-1}(x'y^{-1})\} (y'y^{-1})
\]

for \( x, x' \in H \), \( y, y' \in H \). Let \( \{l\} \) be a collection of elements in \( L \) which represents all the non-trivial classes in \( Cl(L) \). By (13), it follows

\[
|Cl(G)| = |H/L| + \sum |(Hl)/H(Z_i \cap L)|
\]

here \( H(Z_i \cap L) \) acts on \( Hl \) by conjugation, which can also be described in the following form under the identification of \( Hl \) with \( H \):

\[
H(Z_i \cap L) \times H \to H, \quad (x'y', x) \mapsto \varphi_{y'}(x)x'y(x'y)^{-1}.
\]

Introduce the \((Z_i \cap L)\)-equivariant group automorphism of \( H \),

\[
\Phi_l : H \to H, \quad \Phi_l(x') = x'\varphi_l(x'^{-1})
\]
Then \( \text{coker}(\Phi_i) \) is an abelian group with the induced \((Z_i \cap L)\)-action. By (15), we have
\[
|H|/|H(Z_i \cap L)| = |\text{coker}(\Phi_i)/(Z_i \cap L)|. \tag{17}
\]
By the assumption (2), one can easily see that \( Z_i \cap L \) equals to the cyclic group generated by \( l \) with its order being a prime number. Since \( \ker(\Phi_i) \) is \( Z_i \cap H \), the number of \((Z_i \cap L)\)-orbits in \( H \) lying over a \((Z_i \cap L)\)-orbit in \( \text{Im}(\Phi_i) \) via (16) is the same, and it is equal to \( |Z_i \cap H| \).

Hence
\[
|H|/(Z_i \cap L) = |Z_i \cap H| \cdot |\text{Im}(\Phi_i)/(Z_i \cap L)|.
\]
One can also show that the \( Z_i \cap L \) acts trivially on \( \text{coker}(\Phi_i) \), and
\[
|H|/(Z_i \cap L) = |\text{coker}(\Phi_i)/(Z_i \cap L)| \cdot |\text{Im}(\Phi_i)/(Z_i \cap L)|.
\]
Therefore
\[
|\text{coker}(\Phi_i)/(Z_i \cap L)| = |Z_i \cap H|.
\]
By (14) and (17), our result follows. \( \square \)

**Lemma 6.** Let \( G \) be a finite subgroup of \( SL_n(\mathbb{C}) \), and \( K \) be a normal subgroup of \( G \). If \( C^\#/K \) is a minimal resolution of \( C^*/K \),
\[
\rho : C^\#/K \to C^*/K
\]
such that the action of \( G/K \) on \( C^*/K \) can be lifted to \( C^\#/K \):
\[
\begin{array}{c}
G/K \times C^\#/K \\
\downarrow \\
G/K \times C^*/K
\end{array} \hspace{1cm}
\begin{array}{c}
C^\#/K \\
\downarrow \\
C^*/K
\end{array}
\]
then the quotient of \( C^\#/K \) by \( G/K \) has the trivial canonical sheaf.

**Proof:** All we need to show is the existence of a \( G/K \)-invariant holomorphic volume form on \( C^\#/K \). Denote \((z_1, \ldots, z_n)\) the coordinates of \( C^* \). The \( G \)-invariant volume form \( \prod dz_i \) induces one on the non-singular part of \( C^*/K \), denoted by \( \Omega \), which is invariant under \( G/K \).

Through the birational morphism \( \rho \), \( \Omega \) extends to a holomorphic volume form of \( C^\#/K \), hence it is invariant under \( G/K \). \( \square \)

**Definition.** The resolution \( C^\#/K \) in the above lemma will be called a \( G/K \)-invariant minimal resolution of \( C^*/K \).

### 5. MINIMAL RESOLUTION FOR TYPE (B) GROUPS

In this section, we shall verify Theorem 2 for the type (B) group \( G \) with \( W \not\in G \). For
\[
g = \begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \in G
\]
we define
\[ \varphi_1(g) = \varphi_2(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]
Denote
\[ J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

First we shall follow a similar argument as in Section 4 to reduce the general type (B) groups \( G \) to those not containing \( J \).

**Lemma 7.** Let \( G \) be a group of type (B), and \( J \in G \). Denote \( \overline{G} \) the quotient of \( G \) by \( \langle J \rangle \), and \( \mathcal{F} \) the minimal resolution of \( C^3/\langle J \rangle \). Then \( \overline{G} \) acts on \( \mathcal{F} \) with
\[ \omega_{\mathcal{F}/\overline{G}} = \mathcal{O}_{\mathcal{F}/\overline{G}}, \quad \chi(C^3, G) = \chi(\mathcal{F}, \overline{G}). \]
Furthermore, any singularity of \( \mathcal{F}/\overline{G} \) is locally isomorphic to a quotient of \( C^3 \) by some type (A) or (B) group.

**Proof.** Let \( \mathcal{A}_i \) be the minimal resolution of \( C^2/\langle id \rangle \). Then \( \mathcal{A}_1 \) is \(-2\) hyperplane bundle over \( \mathbb{P}^1 \):
\[ \mathcal{A}_1 = H^2 \to \mathbb{P}^1. \]
For a non-trivial linear subspace \( L \) of \( C^2 \), it determines a projective subspace of \( \mathbb{P}^1 \), and the fiber over it, \( B(L) \subset \mathcal{A}_1 \). The intersection of \( B(L) \) with the zero section will be denoted by \( B(L) \). Now the minimal resolution of \( C^3/\langle J \rangle \) is given by
\[ \mathcal{F} \simeq C \times \mathcal{A}_1 \]
which is acted by \( \overline{G} \). Since \( \mathcal{F} \) has the trivial canonical bundle, the canonical sheaf of \( \mathcal{F}/\overline{G} \) is trivial by Lemma 6. By Lemma 3,
\[ |Z_\mu| = 1 \quad \text{or} \quad 2. \]
Then the equality of \( \chi(C^3, G) = \chi(\mathcal{F}, \overline{G}) \) will follow from the relations:
\[ \chi(\mathcal{F}^\mu) = 2 \quad \text{for} \quad g \in \overline{G} \]
\[ \chi(\mathcal{F}^\mu/\mathbb{Z}_\mu) = 1 \quad \text{if} \quad |Z_\mu| = 2. \]
For \( g \) in \( G \), the element in \( \overline{G} \) determined by \( g \) is denoted by \( \overline{g} \). Let \( L \) be an eigenspace of \( \varphi_2(g) \) in \( C^2 \) with eigenvalue \( \mu \). Define the following submanifold of \( \mathcal{F} \):
\[ F_\mu = \begin{cases} C \times B(L) & \text{if} \quad \varphi_1(g) = 1, \mu^2 = 1 \\ C \times \mathbb{P}(L) & \text{if} \quad \varphi_1(g) = 1, \mu^2 \neq 1 \\ 0 \times B(L) & \text{if} \quad \varphi_1(g) \neq 1, \mu^2 = 1 \\ 0 \times \mathbb{P}(L) & \text{if} \quad \varphi_1(g) \neq 1, \mu^2 \neq 1. \end{cases} \]
then
\[ \mathcal{F}^g = \bigsqcup \{ F_\mu | \mu: \text{eigenvalues of} \ \varphi_2(g) \}. \]
Therefore \( \chi(\mathcal{F}^g) = 2. \) Furthermore, if \( x \) is an element of \( \mathcal{F} \) fixed by \( \overline{g} \), the product structure (18) is preserved by the action of \( \overline{g} \). This shows the singularities of \( \mathcal{F}/\overline{G} \) are the quotients of
type (A) or (B). When \(|Z_g| = 2\), there is an element \(h \in G\) such that \(h \in Z_g\) and \(hgh^{-1} = gJ\). For an eigenspace \(L\) for \(\varphi_2(g)\), we have \(L \neq h(L)\), hence \(\varphi_2(g)\) has only one-dimensional eigenspaces, which are permuted by \(h\). Therefore \(\chi(F^d/Z_\delta) = 1\).

As argued in Section 4, the problem of constructing minimal resolutions of \(C^3/G\) is reduced to those for \(F/G\). By Lemmas 2 and 6, one needs only to study the local problem near a point \(x\) of \(F\):

\[
(W', x) \cong (C^3, G_x)
\]

with \(G_x\) a type (A) or (B) group into \(SL_2(C)\). By inductive processes, one will arrive at the situation where both \(W\) and \(J\) are not in the groups by which the quotient singularities are defined. Therefore Theorem 2 for the type (B) group \(G\) will follow from those \(G\) with \(J, W \notin G\).

Now we are going to determine the structure of such groups.

**Lemma 8.** For a type (B) non-abelian group \(G\) with \(W, J \notin G\), \(G\) is generated by an abelian group \(H\) of type (A) and the transformation \(R\)

\[
R = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & b \\
0 & c & 0
\end{pmatrix}
bc = 1.
\]

In this situation, \(G\) is the semidirect product of \(H\) and \(<R>\) with

\[
|\text{Cl}(G)| = \frac{1}{2}(|H| + 3|H \cap Z_R|).
\]

**Proof.** We have the following exact sequence of groups:

\[1 \to G_1 \to G \xrightarrow{\varphi_1} \mu_k \to 1\]

where

\[G_1 = \{g \in G | \varphi_1(g) = 1\}\]
\[\mu_k = \{\alpha \in C^* | \alpha^k = 1\}\].

The group \(G_1\) is now a subgroup of \(SL_2(C)\), hence one of the A–D–E series in Klein's classification. Since \(J\) is not in \(G_1\), the only possibility for \(G_1\) is the cyclic group generated by

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \zeta & 0 \\
0 & 0 & \zeta^{-1}
\end{pmatrix}, \quad \zeta = e^{2\pi i/r}, \ r \geq 3.
\]

Let \(\delta\) be an element in \(G\) such that \(\varphi_1(\delta)\) is the generator of \(\mu_k\). Since \(\delta\) normalizes \(G_1\) and \(G\) is non-abelian, it has the form:

\[
\delta = \begin{pmatrix}
\delta_1 & 0 & 0 \\
0 & 0 & \delta_3 \\
0 & \delta_2 & 0
\end{pmatrix}.
\]

Therefore \(\delta^2\) is in the center \(Z(G)\) of \(G\). Indeed, one can see that \(Z(G)\) is a cyclic group generated by \(\delta^2\), the order of which is an odd integer by the assumption \(J \notin G\). Therefore the
order $k$ of $\delta$ is an even integer with $k/2$ being odd. Let $R$ be the element $\delta^{k/2}$ in $G$. Then $R$ has the expression (19). Since $G_1$, $\delta^2$ and $R$ generate $G$, we obtain the description of $G$ as the semidirect product of its diagonal subgroup $H$ and $\langle R \rangle$, which satisfy the conditions of Lemma 5 with $L = \langle R \rangle$. Since $L$ acts freely on $H - (H \cap Z_R)$, one can easily see that
\[ |H/L| = \frac{1}{2}(|H| + |H \cap Z_R|). \]

By Lemma 5, we obtain the expression of $|Cl(G)|$. □

Now we assume the group $G$ of type (B) is the one in Lemma 8. By Lemma 6, the construction of a minimal resolution of $C^3/G$ is reduced to the existence of a $\langle R \rangle$-invariant minimal toric resolution of $C^3/H$. Now we are going to describe the procedure to obtain such resolution of $C^3/H$. Here we consider $C^3/H$ as a toroidal compactification for the torus $T := C^*^3/H$.

First let us recall the construction of minimal toric resolutions of $C^3/H$ in [19]. Let $\Delta$ be the convex hull in $\mathbb{R}^3$ spanned by the standard basis $e^1, e^2, e^3$. The center of $\Delta$ is
\[ w := \frac{1}{3} \sum_{i=1}^3 e_i. \]

Consider the subset of $\Delta$,
\[ \mathcal{V}_H := \Delta \cap \exp^{-1}(H), \quad (20) \]

where
\[ \exp: \mathbb{R}^3 \to C^*^3, \quad \exp \left( \sum_{i=1}^3 a_i e_i \right) = [e^{2a_1}, e^{2a_2}, e^{2a_3}]. \]

A minimal toric resolution $X_\Sigma$ of $C^3/H$ is corresponding to a triangulation $\Sigma$ of $\Delta$ with $\mathcal{V}_H$ as the set of vertices in $\Sigma$. The toric divisors $D_v$ in $X_\Sigma$ are parametrized by $v$ in $\mathcal{V}_H$, and the exceptional divisors are those $D_v$'s for $v \notin \{e_i\}_{i=1}^3$. The dual relation of intersections among toric divisors of $X_\Sigma$ is indicated by the triangulation $\Sigma$. Now if $L$ is a subgroup of $SL_3(\mathbb{C})$ which normalizes $H$, it induces a $L$-action of $C^3/H$ permuting the toric divisors. Then it corresponds to a $L$-action of $\mathcal{V}_H$ such that the restriction of the exp map defines a $L$-equivariant map
\[ \exp_{\text{rest}}: \mathcal{V}_H \to H \]

injective on $\mathcal{V}_H - \{e_i\}_{i=1}^3$. Since $X_\Sigma$ has the following $T$-orbit decomposition:
\[ X_\Sigma = T \bigcup_{\sigma \in \Sigma} o(\sigma), \quad (21) \]

where $o(\sigma)$ is the $T$-orbit corresponding to $\sigma$ with $\dim o(\sigma) = 2 - \dim \sigma$, one can conclude that the $L$-invariant minimal toric resolutions are those $X_\Sigma$ with $L$-invariant $\Sigma$. For a $L$-invariant $\Sigma$, $X_\Sigma$ is called a $L$-invariant minimal toric resolution of $C^3/H$.

In the case we are now considering, the group $L$ is $\langle R \rangle$. Note that $R$ acts on $\mathcal{V}_H$ by permuting the coordinates. In this case a $\langle R \rangle$-invariant $\Sigma$ can always be found. Actually, one can start with the triangle $\Delta'$ spanned by the vertices $e^1, e^3, \frac{1}{2}e^1 + \frac{1}{2}e^2$. Consider a triangulation for the convex hull generated by $\Delta' \cap \mathcal{V}_H$ using the elements in $\Delta' \cap \mathcal{V}_H$ as 0-simplices, then reflect this triangulation by $R$. Then the complement of their union in $\Delta$ is either the empty set, or a 2-simplex $\{v_1, v_2, v_3\}$ with $v_i \in \mathcal{V}_H$ and $v_1, v_2$ in the segment between $e^1$ and $e^2$. From this description, one can obtain a $\langle R \rangle$-invariant triangulation $\Sigma$ of
Lemma 9. Let $X$ be a $(R)$-invariant minimal toric resolution of $\mathbb{C}^3/H$. Then
\[ \chi(X) = |H \cap Z_R| \]

Proof. By the decomposition (21), the fixed-point set of $R$ is the union of those $o(\sigma)^R$ with $R(\sigma) = \sigma$ for $\sigma \in \Sigma$. Hence
\[ \chi(X) = \sum_{\sigma \in \Sigma, R(\sigma) = \sigma} \chi(o(\sigma)^R). \]

It is easy to see that the subgroup $H \cap Z_R$ of $H$ is a cyclic generated by $[e^{2\pi i/d}, e^{2\pi i/d}, e^{2\pi i(d-2)/d}]$ for some positive integer $d$. Therefore the elements in $V_H$ fixed by $R$ are
\[ w^j = \frac{i}{d} e^1 + \frac{i}{d} e^2 + \frac{d - 2i}{d} e^3, \quad 0 \leq i \leq \left[ \frac{d}{2} \right]. \]

Note that $w^0 = e^3$. For $1 \leq j \leq \left[ \frac{d}{2} \right]$, only one of the following situations occurs for the pair $w^{j-1}, w^j$:

(i) 1-simplex $\{w^{j-1}, w^j\} \in \Sigma$,
(ii) 2-simplices $\{w^{j-1}, p, q\}, \{w^{j-1}, p, q\} \in \Sigma$, for some $p, q$ with $R(p) = q$.

Consider the case (i). There exist elements $v, \tilde{v}$ in $V_H$ such that $\tilde{v} = R(v)$, and $\{\tilde{v}, w^{j-1}, w^j\}$ and $\{v, w^{j-1}, w^j\}$ are 2-simplices in $\Sigma$. We have the relation:
\[ (\tilde{v}, w^{j-1}, w^j) = (v, w^{j-1}, w^j) \begin{pmatrix} -1 & 0 & 0 \\ m & 1 & 0 \\ m' & 0 & 1 \end{pmatrix} \]
for some integers $m, m'$ with $m + m' = 2$. Let $(x_i)_{i=1}^3, (y_i)_{i=1}^3$ be the local coordinates of $X$ corresponding to $\{\tilde{v}, w^{j-1}, w^j\}, \{v, w^{j-1}, w^j\}$ respectively. One has
\[ y_1 = x_1^{j-1}, \quad y_2 = x_1^{j-1} x_2, \quad y_3 = x_1^{j-1} x_3. \]

The local defining equations of $D_{w^{j-1}}, D_{w^j}$ are:
\[ D_{w^{j-1}}: \quad x_2 = 0, \quad y_2 = 0 \]
\[ D_{w^j}: \quad x_3 = 0, \quad y_3 = 0. \]

One can describe the $T$-orbits in the above toric divisors, and their corresponding expressions for $l$:
\[ o(w^{j-1}): \quad x_2 = 0, \quad (x_1, x_3) \in \mathbb{C}^* \quad l(x_1, x_3) = (x_1^{j-1} x_1^{j-1} x_3) \]
\[ o(w^j): \quad x_3 = 0, \quad (x_1, x_2) \in \mathbb{C}^* \quad l(x_1, x_2) = (x_1^{j-1} x_1^{j-1} x_2) \]
\[ o(\{w^{j-1}, w^j\}): \quad x_2 = x_3 = 0, \quad x_1 \in \mathbb{C}^* \quad l(x_1) = x_1^{j-1}. \]

Hence we see that
\[ \chi(o(w^{j-1})') = \chi(o(w^j)') = 0, \quad \chi(o(\{w^{j-1}, w^j\})') = 2. \]

Now we consider the case (ii) in (24). Let $(z_i)_{i=1}^3$ be the local coordinates of $X$ associated to $\{w, p, q\}$. Then the map $l$ is given by
\[ l(z_1, z_2, z_3) = (z_1, z_3, z_2) \]
and the local defining equation for $D_{\omega^j}$ and $D_\rho \cap D_\eta$ are:

$$

D_{\omega^j}: \ z_1 = 0 \\
D_\rho \cap D_\eta: \ z_2 = z_3 = 0.

$$

The $T$-orbits in $D_{\omega^j} \cup D_\rho \cup D_\eta$ invariant under $I$ are:

$$

o(\omega^j): \ z_1 = 0, \ (z_2, z_3) \in \mathbb{C}^2 \\
o(\{p, q\}): \ z_2 = z_3 = 0, \ z_1 \in \mathbb{C} \\
o(\{\omega^j, p, q\}): \ z_1 = z_2 = z_3 = 0.

$$

Hence

$$

\chi(o(\omega^j)^\ell) = \chi(o(\{p, q\})^\ell) = 0, \quad \chi(o(\{\omega^j, p, q\})^\ell) = 1. \tag{26}

$$

Similarly,

$$

\chi(o(\omega^{j-1})^\ell) = 0, \quad \chi(o(\{\omega^{j-1}, p, q\})^\ell) = 1. \tag{27}

$$

From (25)-(27), the contribution on the right-hand side of (23) for the $T$-orbits intersecting the segment between $\omega^{j-1}$ and $\omega^j$ is equal to 2 for $1 \leq j \leq [\frac{d}{2}]$. Therefore we obtain the result (22) for even $d$. When $d$ is odd, the only terms left-out on the right-hand side of (23) are the $T$-orbits associated to the simplices:

$$

\{r_1, r_2, w^{(d-1)/2}, r_1, r_2\}.

$$

The sum of their contribution in $\chi(X_{\mathbb{F}}^\mathcal{X})$ is equal to 1, hence we obtain our result. □

By Lemma 9 and the well-known fixed-point formula [9], one has:

$$

\chi(X_{\mathbb{F}}/\langle R \rangle) = \frac{1}{2} \sum_{I \in C(R)} \chi(X_{\mathbb{F}}^I) = \frac{1}{2} (|H| + |H \cap Z_\eta|).

$$

Therefore

$$

\chi(X_{\mathbb{F}}/\langle R \rangle) = \frac{1}{2} (|H| + 3|H \cap Z_\eta|)

$$

which is equal to $|C(|G|)$ by Lemma 8. Hence we obtain the following result:

**Proposition 1.** For a group $G$ generated by an abelian group $H$ of type (A) and the transformation $R$ of (19), let $X_{\mathbb{F}}$ be a $\langle R \rangle$-invariant minimal toric resolution of $C^3/H$. Then minimal resolutions of $X_{\mathbb{F}}/\langle R \rangle$ are the minimal resolutions $C^3/G$ of $C^3/G$ with the equality (7).

Hence we have shown that Theorem 2 holds for groups of type (B).

**Remark.**

(I) The above argument for Proposition 1 holds also for groups $G$ in the proposition but without the requirement $W \notin \mathcal{X}$.

(II) With same argument as in Proposition 1, one can also show that Theorem 2 holds for the type (C) groups, which has been known by [10].
6. MINIMAL RESOLUTIONS OF TYPE (D) GROUPS

In this section, we shall verify Theorem 2 for a group $G$ of type (D) with $W \notin G$. Let $H$ be the diagonal subgroup in $G$. Replacing the transformation $R$ in $G$ by $T^{-1}R^2TR$ if necessary, we may assume the entry $a$ of $R$ is $-1$, hence $R^2 = I$. Note that $G$ contains the following diagonal elements:

\[
\begin{align*}
[ -b, -b, b^{-2} ] &= RTR^{-1}T^{-2} \\
[1, -b^3, -b^{-3}] &= [ -b, -b, b^{-2} ]^2T^2[ -b, -b, b^{-2} ]T^{-2}.
\end{align*}
\]

(28)

**Lemma 10.** The transformation $R$ in $G$ can be chosen as

\[
R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}
\]

and the subgroup $H$ is given by

\[
H = \{ [a, \beta, \gamma] \mid a^d = \beta^d = \gamma^d = \alpha \beta \gamma = 1 \}
\]

for some $d$ not divisible by 3. In this situation, $T$ and $R$ generate the subgroup $\langle T, R \rangle$ isomorphic to the symmetric group $S_3$ of 3 elements, and $G$ is the semidirect product of $H$ and $S_3$ with

\[
\left| \text{Cl}(G) \right| = \frac{1}{6}(d^2 + 9d + 8). \tag{29}
\]

**Proof.** Let $m$ be the order of the number $b$. If $m$ is a multiple of 3, by (28),

\[
(\mathcal{K}^{-1}T^{-2})^{2m/3} = W \text{ or } W^2 \in G.
\]

Hence we may assume $m$ is not divisible by 3, hence $b^{3k} = b^{-1}$ for some integer $k$. $G$ contains the element $[1, -b^3, -b^{-3}]^kR$, then the transformation $R$ can be chosen as $a = -1, b = \pm 1$. When $b = 1$, by (28), $[1, -1, -1] \in G$. Replacing $R$ by $[1, -1, -1]R$, we obtain the $R$ with $b = -1$, hence the expression of $R$. So $\langle T, R \rangle$ is isomorphic to the symmetric group $S_3$ of 3 elements.

Let $d$ be the maximal order for elements in $H$, and denote $\zeta = e^{2\pi i/d}$. Since $H$ is invariant under the conjugation of $R$ and $T$, $H$ contains some element with the expression $[\zeta, \zeta^a, \zeta^{d-a-1}]$ for $a \in \mathbb{Z}, a \geq 0$. Let $g_1$ be a such element with the minimal $a$. Consider the subgroup of $H$ consisting of all those elements $[a, \beta, \gamma]$ with $a = 1$. It is a cyclic group, and let $h_1$ be its generator, $h_1 = [1, \zeta^k, \zeta^{d-k}]$ for a positive divisor $k$ of $d$. Then $H$ is generated by $g_1$ and $h_1$. We have $0 \leq a < k < d$. Since

\[
g_1Rg_1^{-1}R = [1, \zeta^{2a+1}, \zeta^{d-2a-1}] \in H \tag{30}
\]

there is an integer $m$ with $\zeta^{2a+1} = \zeta^{km}$. Then

\[
RTg_1h_1^{-1}RTg_1 = [1, \zeta^{a+2}, \zeta^{2d-a-2}] = h_1' \in H
\]

for some non-negative integer $l$ less than $d/k$. By $a + 2 \leq k + 1 \leq d$, we have

\[
lk = a + 2 \leq k + 1.
\]

If $l - 1, k - a + 2$, by (30),

\[
[1, \zeta^3, \zeta^{d-3}] \in H
\]
which implies $k = 3$, $a = 1$, hence $d$ is divisible by 3, and $g^{A^3} = W$ which contradicts our assumption of $W \not\in G$. Therefore $l > 1$. Then it is easy to see that $k = 1, a = 0$, hence we obtain the result for $H$. Then $G$ is the semidirect product of $H$ and $\langle T, R \rangle$, which satisfies the conditions of Lemma 5 with $L = \langle T, R \rangle$. Now $L$ acts $H$ by conjugation, freely on those elements of $H$ with three distinct eigenvalues. Hence

$$|H/L| = 1 + (d - 1) + \frac{1}{6}(d^2 - 1 - 3(d - 1)) = \frac{1}{6}(d^2 + 3d + 2)$$

and for $l \in L$,

$$|Z_1 \cap H| = 1, \quad \text{if order}(l) = 3$$
$$|Z_1 \cap H| = d, \quad \text{if order}(l) = 2.$$

By Lemma 5, we obtain the expression for $|CI(G)|$.

We shall identify the subgroup $\langle S, T \rangle$ of $G$ with $S_3$. The action of $S_3$ on $C^3/H$ is given by permutations of coordinates of $C^3$. By Lemma 6 and the solution of Theorem 2 for type (A),(B) groups, the construction of a minimal resolution of $C^3/G$ is reduced to the existence of a $S_3$-invariant minimal toric resolution $X_\Sigma$ of $C^3/H$. Now the subset $V_H$ of $\Sigma$ in (20) is given by

$$V := V_H = \left\{ \sum_{i=1}^{3} \frac{k_i}{d} e^i | k_i > 0, \sum_i k_i = d \right\}$$

Since $d$ is not divisible by 3, $w$ does not belong to $V$. Now a $S_3$-invariant minimal toric resolution $X_\Sigma$ of $C^3/H$ corresponds to a triangulation $S_3$-invariant $\Sigma$ of $\Delta$ with $V$ as the set of vertices in $\Sigma$. There always exists such a $\Sigma$ for each $d$, e.g. the triangulation of $\Delta$ with all the 2-simplices having its edges parallel to those of $\Delta$. For $d = 2$, this is the only $S_3$-invariant one.

For $d > 2$, there are several such $S_3$-invariant $\Sigma$. One can start with the triangle $\Delta_1$ spanned by $e^1, w, \frac{1}{3} e^1 + \frac{1}{3} e^2$. Consider a triangulation of the convex hull generated by $\Delta_1 \cap V$ having $\Delta_1 \cap V$ as the set of 0-simplices. By the $S_3$ action, one can obtain a $S_3$-invariant triangulation $\Sigma$ of $\Delta$. Here are some examples for $d = 4, 5$. 

![Diagram](image-url)
Hence one obtains \( S_3 \)-invariant minimal toric resolutions \( X_\Sigma \) of \( C^3/H \).

**Lemma 11.** Let \( X_\Sigma \) be a \( S_3 \)-invariant minimal toric resolution of \( C^3/H \). For a non-trivial \( l \) of \( S_3 \), the Euler number of the \( l \) fixed-point set \( X_\Sigma^l \) is given by

\[
\chi(X_\Sigma^l) = \begin{cases} 
1 & \text{if } \text{order}(l) = 3 \\
\frac{d}{3} & \text{if } \text{order}(l) = 2.
\end{cases}
\]  

(31)

**Proof.** By the decomposition (21), we have

\[
\chi(X_\Sigma^l) = \sum_{\sigma \in \Sigma, \text{order}(\sigma) = \text{order}(l)} \chi(\sigma^l).
\]

If \( l \) is an order 3 element, \( l = T \) or \( T^2 \), there is only one simplex in \( \Sigma \) stable under \( l \), which is the 2-simplex containing \( w \). Therefore \( X_\Sigma^l \) consists of only one element, hence we obtain the result. Now suppose \( l \) is an order 2 element. By symmetry, we may assume \( l = RT \). Then the conclusion follows from Lemma 9 and the description of \( H \). \( \square \)

By the above lemma and the fixed-point formula, one has:

\[
\chi(X_\Sigma/S_3) = \frac{1}{6} \sum_{l \in \langle S, T \rangle} \chi(X_\Sigma^l) = \frac{1}{6} (d^2 + 3d + 2).
\]

Since the centralizer \( Z_l \) of every non-trivial element \( l \) in \( S_3 \) is equal to the cyclic group generated by \( l \), we have

\[
\chi(X_\Sigma^l/Z_l) = \begin{cases} 
\frac{d}{3} (d^2 + 3d + 2) & \text{if } \text{order}(l) = 1 \\
1 & \text{if } \text{order}(l) = 3 \\
\frac{d}{3} & \text{if } \text{order}(l) = 2.
\end{cases}
\]  

(32)

Therefore

\[
\chi(X_\Sigma/S_3) = \frac{1}{6} (d^2 + 9d + 8)
\]

which is equal to \( |\text{Cl}(G)| \) by (29). Hence we have the following result:

**Proposition 2.** Let \( X_\Sigma \) be a \( S_3 \)-invariant minimal toric resolution of \( C^3/H \). Then minimal resolutions of \( X_\Sigma/S_3 \) are the minimal resolutions \( C^3/G \) of \( C^3/G \) with the equality (7).

Hence we have shown that Theorem 2 holds for type (D) groups.
REFERENCES

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