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On operations of soft sets

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ABSTRACT

Soft set theory, proposed by Molodtsov, has been regarded as an effective mathematical tool to deal with uncertainties. In this paper, first we prove that certain De Morgan's law hold in soft set theory with respect to different operations on soft sets. Then, we discuss the basic properties of operations on soft sets such as intersection, extended intersection, restricted union and restricted difference. Moreover, we illustrate their interconnections between each other. Also we define the notion of restricted symmetric difference of soft sets and investigate its properties. The main purpose of this paper is to extend the theoretical aspect of operations on soft sets.

1. Introduction

Researchers studying to solve complicated problems in economics, engineering, environmental science, sociology, medical science and many other fields deal with the complex problems of modeling uncertain data. While some mathematical theories such as probability theory, fuzzy set theory [1,2], rough set theory [3,4], vague set theory [5] and the interval mathematics [6] are useful approaches to describing uncertainty, each of these theories has its inherent difficulties as mentioned by Molodtsov [7]. Consequently, Molodtsov [7] proposed a completely new approach for modeling vagueness and uncertainty in 1999. This approach called soft set theory is free from the difficulties affecting existing methods. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields.

Works on soft set theory has been progressing rapidly since Maji et al. [8] introduced several operations of soft sets. Since then, Pei and Miao [9] and Ali et al. [10] introduced and studied several soft set operations as well. Soft set theory has also potential applications in many fields including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory. Especially it has been successfully applied to soft decision making in [11–15]. Aktaş and Çağman [16] studied the basic concepts of soft set theory and compared soft sets to fuzzy and rough sets. They also defined and studied soft groups, soft subgroups, normal soft subgroups and soft homomorphisms. Since then, many authors [17–25] have studied the soft algebraic structures and soft operations as well.

In this paper, we try to find an answer to the question how the classical set operations and their interrelations between each other correspond to soft set operations. While studying with this aim, we have seen that although there are some similarities, there are some apparent dissimilarities, too. The paper is organized as follows: First we prove that a certain De Morgan's law hold in soft set theory with respect to different operations on soft sets defined by Maji et al. [8], Pei and Miao [9] and Ali et al. [10]. Then, we discuss the basic properties of operations on soft sets such as intersection, extended

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intersection, restricted union, restricted difference and we illustrate the interconnections between each other. Finally, we define the notion of restricted symmetric difference of soft sets and investigate its properties with a corresponding example. This paper can be classified as a theoretical study of soft sets.

2. Preliminaries

In this section, we recall some basic notions in soft set theory. Let U be an initial universe set and E_U be the set of all possible parameters under consideration with respect to U. The power set of U (i.e., the set of all subsets of U) is denoted by $P(U)$ and A is a subset of E. Usually parameters are attributes, characteristics or properties of objects in U. In what follows, E_U (simply denoted by E) always stands for the universe set of parameters with respect to U, unless otherwise specified.

Molodtsov [7] defined the soft set in the following manner:

Definition 1 ([7]). Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U. A pair (F, E) is called a soft set over U, where F is a mapping of E into the set of all subsets of the set U.

In other words, a soft set over U is a parameterized family of subsets of U. For $\varepsilon \in E$, $F(\varepsilon)$ may be considered as the set of ε-elements of the soft set (F, E) or as the set of ε-approximate elements of the soft set.

Definition 2 ([8]). For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a *soft subset* of (G, B), denoted by $(F, A)\widetilde{\subset}(G, B)$, if it satisfies:

(i) $A \subset B$;
(ii) $\forall \varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations.

Similarly (F, A) is called a *superset* of (G, B) if (G, B) is a soft subset of (F, A). This relation is denoted by $(F, A)\widetilde{\supset}(G, B)$.

Definition 3 ([8]). Two soft sets (F, A) and (G, B) over a common universe U are called *soft equal* if $(F, A)\widetilde{\subset}(G, B)$ and $(G, B)\widetilde{\subset}(F, A)$.

Definition 4 ([10]). The *relative complement of a soft set* (F, A) is denoted by $(F, A)^r$ and is defined by $(F, A)^r = (F^r, A)$, where $F^r : A \to P(U)$ is a mapping given by $F^r(\alpha) = U \setminus F(\alpha)$, for all $\alpha \in A$.

Definition 5 ([8]). A soft set (F, A) over U is said to be a *null soft set* denoted by Φ, if $\forall e \in A$, $F(e) = \emptyset$ (null set).

Since some researchers are in some conflict about a null soft set due to its notation, we prefer to use Φ_A instead of Φ for the null soft set of (F, A) as Ali et al. [10] used.

Definition 6 ([8]). A soft set (F, A) over U is said to be an *absolute soft set* denoted by $\widetilde{A}$, if $\forall e \in A$, $F(e) = U$.

Note that we use the notation $\mathcal{U}_A$ instead of $\widetilde{A}$ as in [10] throughout this paper.

Definition 7 ([8]). If (F, A) and (G, B) are two soft sets over a common universe U, then "(F, A) *AND* (G, B)" denoted by $(F, A) \wedge (G, B)$ is defined by $(F, A) \wedge (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cap G(y)$ for all $(x, y) \in A \times B$.

Definition 8 ([8]). If (F, A) and (G, B) are two soft sets over a common universe U, then "(F, A) *OR* (G, B)" denoted by $(F, A) \vee (G, B)$ is defined by $(F, A) \vee (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cup G(y)$ for all $(x, y) \in A \times B$.

Definition 9 ([8]). Let (F, A) and (G, B) be two soft sets over a common universe U. The *union* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions: (i) $C = A \cup B$; (ii) for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

This relation is denoted by $(F, A)\widetilde{\cup}(G, B) = (H, C)$.

Definition 10 ([8]). The *intersection* of two soft sets (F, A), (G, B) over a common universe set U is the soft set (H, C), where $C = A \cap B$, and $\forall e \in C$, $H(e) = F(e)$ or $G(e)$, (as both are the same set). We write $(F, A)\widetilde{\cap}(G, B) = (H, C)$.

Pei and Miao [9] defined an alternative definition for intersection of soft sets as following:

Definition 11 ([9]). Let (F, A) and (G, B) be two soft sets over a common universe U. The *intersection* of (F, A) and (G, B) is denoted by $(F, A) \cap (G, B)$, and is defined as $(F, A) \cap (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.

Since the notation of soft set intersection of Pei and Miao [9] is similar to the intersection of sets in classical set theory, thus may mislead the readers, we denote "(F, A) intersection (G, B)" by "$(F, A) \Cap (G, B)$" as Ali et al. used in [10]. In addition to the above definition, Ali et al. [10] introduced a new definition for intersection, called *extended intersection* as following:

Definition 12 (*[10]*)**.** Let (F, A) and (G, B) be two soft sets over a common universe U. The *extended intersection* of (F, A) and (G, B) is defined to be the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cap G(e) & \text{if } e \in A \cap B. \end{cases}$$

This relation is denoted by $(F, A) \sqcap_{\varepsilon} (G, B) = (H, C)$.

Definition 13 (*[10]*)**.** Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The *restricted difference* of (F, A) and (G, B) is denoted by $(F, A) \smallsmile_{\mathcal{R}} (G, B)$, and is defined as $(F, A) \smallsmile_{\mathcal{R}} (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \setminus G(c)$.

Definition 14 (*[10]*)**.** Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The *restricted union* of (F, A) and (G, B) is denoted by $(F, A) \cup_{\mathcal{R}} (G, B)$, and is defined as $(F, A) \cup_{\mathcal{R}} (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cup G(c)$.

3. De Morgan's laws in soft set theory

Proposition 1 (*[10], Theorem 4.1*)**.**

(1) $((F, A) \cup_{\mathcal{R}} (G, B))^r = (F, A)^r \Cap (G, B)^r$.
(2) $((F, A) \Cap (G, B))^r = (F, A)^r \cup_{\mathcal{R}} (G, B)^r$.

In Proposition 1, Ali et al. [10] showed the De Morgan's law for restricted union, intersection and relative complement. We illustrate in Theorem 1 how De Morgan's type of results hold in soft set theory for AND-operation, OR-operation and relative complement.

Theorem 1. *Let (F, A) and (G, B) be two soft sets over the same universe U. Then we have the following;*

(i) $((F, A) \vee (G, B))^r = (F, A)^r \wedge (G, B)^r$.
(ii) $((F, A) \wedge (G, B))^r = (F, A)^r \vee (G, B)^r$.

Proof. (i) We prove part (i) of the Theorem 1. By using a similar technique, part (ii) can be proved, too. Suppose that $(F, A) \vee (G, B) = (O, A \times B)$. Therefore, $((F, A) \vee (G, B))^r = (O, A \times B)^r = (O^r, A \times B)$. Now,

$$\begin{aligned} (F, A)^r \wedge (G, B)^r &= (F^r, A) \wedge (G^r, B), \\ &= (J, A \times B), \quad \text{where } J(x, y) = F^r(x) \cap G^r(y). \end{aligned}$$

Let $(\alpha, \beta) \in A \times B$. Then, by Definition 4,

$$\begin{aligned} O^r(\alpha, \beta) &= U \setminus O(\alpha, \beta) \\ &= U \setminus [F(\alpha) \cup G(\beta)] \\ &= [U \setminus F(\alpha)] \cap [U \setminus G(\beta)] \\ &= F^r(\alpha) \cap G^r(\beta) \\ &= J(\alpha, \beta). \end{aligned}$$

Since O^r and J are indeed the same set-valued mapping, $((F, A) \vee (G, B))^r = (F, A)^r \wedge (G, B)^r$. □

Example 1. Suppose that U is the set of houses under consideration, A and B are both parameter sets. Let there be four houses in the universe U given by $U = \{h_1, h_2, h_3, h_4\}$. And $A = \{expensive, modern\}$, $B = \{modern\}$. The soft sets (F, A) and (G, B) describe the "attractiveness of the houses". For the sake of ease of designation, we use e, instead of *expensive* and m instead of *modern*. The soft set (F, A) is defined as following: $F(e)$ means *expensive* houses, $F(m)$ means *modern* houses. The soft set (F, A) is the collection of approximations as below:

$$(F, A) = \{(e, \{h_1, h_2\}), (m, \{h_4\})\}.$$

The soft set (G, B) is defined as $G(m)$, which means the *modern* houses. The soft set (G, B) is the collection of approximations as below:

$$(G, B) = \{(m, \{h_1, h_4\})\}.$$

First we handle the left-hand side of Theorem 1(i). Let $(F, A) \vee (G, B) = (H, C)$, where $H(x, y) = F(x) \cup G(y)$ for all (x, y) in $A \times B$. Then,

$$(H, C) = \{((e, m), \{h_1, h_2, h_4\}), ((m, m), \{h_1, h_4\})\} \quad \text{and}$$
$$((F, A) \vee (G, B))^r = (H, C)^r = \{((e, m), \{h_3\}), ((m, m), \{h_2, h_3\})\}.$$

Now we handle the right-hand side of the equality. By Definition 4,

$$(F,A)^r = \{(e,\{h_3,h_4\}),(m,\{h_1,h_2,h_3\})\}, \qquad (G,B)^r = \{(m,\{h_2,h_3\})\}.$$

Then, by Definition 7,

$$(F,A)^r \wedge (G,B)^r = \{((e,m),\{h_3\}),((m,m),\{h_2,h_3\})\}.$$

This shows that $((F,A)\vee(G,B))^r = (F,A)^r \wedge (G,B)^r$.

4. Properties of operations on soft sets and their interrelations between each others

In this section, we illustrate the basic properties of operations on soft sets, such as union operation proposed by Maji et al. [8], restricted union, restricted intersection, restricted difference and extended intersection proposed by Ali et al. [10] and intersection by Pei and Miao [9].

Theorem 2. *Properties of the union* ($\widetilde{\cup}$) *operation*

(a) $(F,A)\widetilde{\cup}((G,B)\widetilde{\cup}(H,C)) = ((F,A)\widetilde{\cup}(G,B))\widetilde{\cup}(H,C)$ [8].
(b) $(F,A)\widetilde{\cup}\mathcal{U}_A = \mathcal{U}_A$ [9], $(F,A)\widetilde{\cup}\mathcal{U}_E = \mathcal{U}_E$ [24], $(F,A)\widetilde{\cup}\Phi_A = (F,A)$ [9].
(c) (F,A) *needs not be a soft subset of* $(F,A)\widetilde{\cup}(G,B)$. *But if* $(F,A)\widetilde{\subset}(G,B)$, *then* $(F,A)\widetilde{\subset}(F,A)\widetilde{\cup}(G,B)$, *moreover* $(F,A) = (F,A)\widetilde{\cup}(G,B)$.
(d) $(F,A)\widetilde{\cup}(G,A) = \Phi_A \Leftrightarrow (F,A) = \Phi_A$ *and* $(G,A) = \Phi_A$.
(e) $(F,A)\widetilde{\cup}((G,B)\Cap(H,C)) = ((F,A)\widetilde{\cup}(G,B))\Cap((F,A)\widetilde{\cup}(H,C))$ [9].
(f) $((F,A)\Cap(G,B))\widetilde{\cup}(H,C) = ((F,A)\widetilde{\cup}(H,C))\Cap((G,B)\widetilde{\cup}(H,C))$ [9].

Proof. (c) Let $(F,A)\widetilde{\cup}(G,B) = (H,C)$, where $C = A\cup B$ and

$$H(e) = \begin{cases} F(e) & \text{if } e\in A\setminus B,\\ G(e) & \text{if } e\in B\setminus A,\\ F(e)\cup G(e) & \text{if } e\in A\cap B\end{cases}$$

for all $e\in C$. It is obvious that if $e\in A\cap B$, then $H(e) = F(e)\cup G(e)$, thus $F(e)$ and $H(e)$ need not be the same approximations. Thus, (F,A) needs not be a soft subset of $(F,A)\widetilde{\cup}(G,B)$.

Now let $(F,A)\widetilde{\subset}(G,B)$. Then, it is clear that $A \subset A\cup B = A$. We need to show that $F(e)$ and $H(e)$ are the same approximations for all $e\in A$. Let $e\in A$, then $e\in A\cap B = A$, since $A\subset B$ implies $A\setminus B = \emptyset$. Thus, $H(e) = F(e)\cup G(e) = F(e)\cup F(e) = F(e)$, as $G(e)$ and $F(e)$ are the same approximations for all $e\in A$. This follows that H and F are the same set-valued mapping for all $e\in A$, as required.

(d) Suppose that $(F,A)\widetilde{\cup}(G,A) = (H,A)$, where $H(x) = F(x)\cup G(x)$ for all $x\in A$. Since $(H,A) = \Phi_A$ from the assumption, $H(x) = F(x)\cup G(x) = \emptyset \Leftrightarrow F(x) = \emptyset$ and $G(x) = \emptyset \Leftrightarrow (F,A) = \Phi_A$ and $(G,A) = \Phi_A$ for all $x\in A$.

Now assume that $(F,A) = \Phi_A$ and $(G,A) = \Phi_A$ and $(F,A)\widetilde{\cup}(G,A) = (H,A)$. Since $F(x) = \emptyset$ and $G(x) = \emptyset$ for all $x\in A$, $H(x) = F(x)\cup G(x) = \emptyset$ for all $x\in A$. Therefore, $(F,A)\widetilde{\cup}(G,A) = \Phi_A$ by Definition 5. □

Proposition 2. $(F,A)\widetilde{\cup}(G,A) = (F,A)\cup_{\mathcal{R}}(G,A)$.

Proof. It is obvious when considering the parameter sets of the soft sets together with Definitions 9 and 14. □

Theorem 3. *Properties of the restricted union* ($\cup_{\mathcal{R}}$) *operation*

(a) $(F,A)\cup_{\mathcal{R}}((G,B)\cup_{\mathcal{R}}(H,C)) = ((F,A)\cup_{\mathcal{R}}(G,B))\cup_{\mathcal{R}}(H,C)$.
(b) $(F,A)\cup_{\mathcal{R}}\mathcal{U}_A = \mathcal{U}_A$ [24], $(F,A)\cup_{\mathcal{R}}\mathcal{U}_E = \mathcal{U}_A$ [24], $(F,A)\cup_{\mathcal{R}}\Phi_A = (F,A)$ [24], $(F,A)\cup_{\mathcal{R}}\Phi_E = (F,A)$ [24].
(c) $(F,A)\widetilde{\not\subseteq}(F,A)\cup_{\mathcal{R}}(G,B)$, *in general. But if* $(F,A)\widetilde{\subset}(G,B)$, *then* $(F,A)\widetilde{\subset}(F,A)\cup_{\mathcal{R}}(G,B)$, *moreover* $(F,A) = (F,A)\cup_{\mathcal{R}}(G,B)$.
(d) $(F,A)\cup_{\mathcal{R}}(G,A) = \Phi_A \Leftrightarrow (F,A) = \Phi_A$ *and* $(G,A) = \Phi_A$.
(e) $(F,A)\cup_{\mathcal{R}}((G,B)\Cap(H,C)) = ((F,A)\cup_{\mathcal{R}}(G,B))\Cap((F,A)\cup_{\mathcal{R}}(H,C))$.
(f) $((F,A)\Cap(G,B))\cup_{\mathcal{R}}(H,C) = ((F,A)\cup_{\mathcal{R}}(H,C))\Cap((G,B)\cup_{\mathcal{R}}(H,C))$.
(g) $(F,A)\cup_{\mathcal{R}}((G,B)\sqcap_{\varepsilon}(H,C)) = ((F,A)\cup_{\mathcal{R}}(G,B))\sqcap_{\varepsilon}((F,A)\cup_{\mathcal{R}}(H,C))$.
(h) $((F,A)\sqcap_{\varepsilon}(G,B))\cup_{\mathcal{R}}(H,C) = ((F,A)\cup_{\mathcal{R}}(H,C))\sqcap_{\varepsilon}((G,B)\cup_{\mathcal{R}}(H,C))$.

Proof. (a) First, we investigate the left-hand side of the equality. Suppose that $(G,B)\cup_{\mathcal{R}}(H,C) = (T,B\cap C)$, where $T(x) = G(x)\cup H(x)$ for all $x\in B\cap C\neq\emptyset$. And assume $(F,A)\cup_{\mathcal{R}}(T,B\cap C) = (W,A\cap(B\cap C))$, where $W(x) = F(x)\cup T(x) = F(x)\cup(G(x)\cup H(x))$ for all $x\in A\cap(B\cap C)\neq\emptyset$.

Now consider the right-hand side of the equality. Suppose that $(F,A)\cup_{\mathcal{R}}(G,B) = (M,A\cap B)$, where $M(x) = F(x)\cup G(x)$ for all $x\in A\cap B\neq\emptyset$. And let $(M,A\cap B)\cup_{\mathcal{R}}(H,C) = (N,(A\cap B)\cap C)$, where $N(x) = M(x)\cup H(x) = (F(x)\cup G(x))\cup H(x)$ for all $x\in(A\cap B)\cap C\neq\emptyset$. Since W and N are the same mapping for all $x\in A\cap(B\cap C) = (A\cap B)\cap C$, the proof is completed.

(c) Since $A\not\subseteq A\cap B$ without any extra condition being given, $(F,A)\widetilde{\not\subseteq}(F,A)\cup_{\mathcal{R}}(G,B)$ in general. Now assume that (F,A) is a soft subset of (G,B) and $(F,A)\cup_{\mathcal{R}}(G,B) = (H,A\cap B = C)$, where $H(x) = F(x)\cup G(x)$ for all $x\in C$. Then, $(F,A)\widetilde{\subset}(G,B) \Leftrightarrow$

$A \subset A \cap B = A$ and $F(e)$ and $G(e)$ are the same approximations for all $e \in A \Leftrightarrow H(e) = F(e) \cup G(e) = F(e) \cup F(e) = F(e)$ for all $e \in A$. Thus, F and H are the same set-valued mapping for all $e \in A$, so the proof is completed.

(d) It follows from Proposition 2 and Theorem 2(d).

(e) First, we handle the left-hand side of the equality. Suppose that $(G, B) \Cap (H, C) = (T, B \cap C)$, where $T(x) = G(x) \cap H(x)$ for all $x \in B \cap C$. Let $(F, A) \cup_{\mathcal{R}} (T, B \cap C) = (W, A \cap (B \cap C))$, where $W(x) = F(x) \cup T(x) = F(x) \cup (G(x) \cap H(x))$ for all $x \in (A \cap B) \cap C$.

Now consider the right-hand side of the equality. Assume that $(F, A) \cup_{\mathcal{R}} (G, B) = (M, A \cap B)$, where $M(x) = F(x) \cup G(x)$ for all $x \in A \cap B \neq \emptyset$. And let $(F, A) \cup_{\mathcal{R}} (H, C) = (N, A \cap C)$, where $N(x) = F(x) \cup H(x)$ for all $x \in A \cap C \neq \emptyset$. Suppose that $(M, A \cap B) \Cap (N, B \cap C) = (K, (A \cap B) \cap (A \cap C)) = (K, (A \cap B) \cap C)$, where $K(x) = M(x) \cap N(x) = (F(x) \cup G(x)) \cap (F(x) \cup H(x)) = F(x) \cup (G(x) \cap H(x))$ for all $x \in (A \cap B) \cap C$. Since W and K are the same set-valued mapping, the proof is completed.

(f) By similar techniques used to prove (e), (f) can be illustrated, and is therefore omitted.

(g) Suppose that $(G, B) \sqcap_{\varepsilon} (H, C) = (T, B \cup C)$, where

$$T(e) = \begin{cases} G(e) & \text{if } e \in B \setminus C, \\ H(e) & \text{if } e \in C \setminus B, \\ G(e) \cap H(e) & \text{if } e \in B \cap C. \end{cases}$$

Assume that $(F, A) \cup_{\mathcal{R}} (T, B \cup C) = (M, A \cap (B \cup C))$, where $M(x) = F(x) \cup T(x)$ for all $x \in A \cap (B \cup C)$. By taking into account the properties of operations in set theory and the definitions of M along with T and considering that T is a piecewise function, we can write the below equalities for M:

$$M(e) = \begin{cases} F(e) \cup G(e) & \text{if } e \in A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C), \\ F(e) \cup H(e) & \text{if } e \in A \cap (C \setminus B) = (A \cap C) \setminus (A \cap B), \\ F(e) \cup (G(e) \cap H(e)) & \text{if } e \in A \cap (B \cap C) \end{cases}$$

for all $e \in A \cap (B \cup C)$.

Now consider the right-hand side of the equality. Suppose that $(F, A) \cup_{\mathcal{R}} (G, B) = (Q, A \cap B)$, where $Q(x) = F(x) \cup G(x)$ for all $x \in A \cap B \neq \emptyset$. Assume $(F, A) \cup_{\mathcal{R}} (H, C) = (W, A \cap C)$, where $W(x) = F(x) \cup H(x)$ for all $x \in A \cap C \neq \emptyset$. Let $(Q, A \cap B) \sqcap_{\varepsilon} (W, A \cap C) = (N, (A \cap B) \cup (A \cap C))$, where

$$N(e) = \begin{cases} Q(e) & \text{if } e \in (A \cap B) \setminus (A \cap C), \\ W(e) & \text{if } e \in (A \cap C) \setminus (A \cap B), \\ Q(e) \cap W(e) & \text{if } e \in (A \cap B) \cap (A \cap C) = A \cap (B \cap C) \end{cases}$$

for all $x \in (A \cap B) \cup (A \cap C)$. By taking into account the definitions of Q and W, we can rewrite N as below:

$$N(e) = \begin{cases} F(e) \cup G(e) & \text{if } e \in (A \cap B) \setminus (A \cap C), \\ F(e) \cup H(e) & \text{if } e \in (A \cap C) \setminus (A \cap B), \\ (F(e) \cup G(e)) \cap (F(e) \cup H(e)) & \text{if } e \in A \cap (B \cap C). \end{cases}$$

This follows that N and M are the same set-valued mapping when considering the properties of operations on set theory, which completes the proof.

(h) By similar techniques used to prove (g), (h) can be illustrated, and is therefore omitted. □

Now we will illustrate Theorem 3(e) with a corresponding example.

Example 2. Let E be the universe set of parameters, A, B, C be the subsets of E such that

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}, \qquad A = \{e_1, e_2, e_5\}, \qquad B = \{e_1, e_4, e_5\} \quad \text{and} \quad C = \{e_1, e_4, e_6\}.$$

Let (F, A), (G, B) and (H, C) be three soft sets over the same universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8\}$ such that

$$(F, A) = \{(e_1, \{h_1, h_3, h_8\}), (e_2, \emptyset), (e_5, \{h_2, h_4, h_7, h_8\})\}.$$
$$(G, B) = \{(e_1, \{h_1, h_4, h_7, h_8\}), (e_4, U), (e_5, \{h_1, h_3\})\}.$$
$$(H, C) = \{(e_1, \{h_2, h_3, h_4, h_8\}), (e_4, \{h_4, h_7\}), (e_6, \{h_1, h_6\})\}.$$

First we handle the left-hand side of the equality. Let $(G, B) \Cap (H, C) = (T, B \cap C)$, where $T(x) = G(x) \cap H(x)$ for all $x \in B \cap C = \{e_1, e_4\}$. Then,

$$(T, B \cap C) = \{(e_1, \{h_4, h_8\}), (e_4, \{h_4, h_7\})\}.$$

And let $(F, A) \cup_{\mathcal{R}} (T, B \cap C) = (W, A \cap (B \cap C))$, where $W(x) = F(x) \cup T(x)$ for all $x \in A \cap (B \cap C) = \{e_1\}$. Then,

$$(W, A \cap (B \cap C)) = (F, A) \cup_{\mathcal{R}} ((G, B) \Cap (H, C)) = \{(e_1, \{h_1, h_3, h_4, h_8\})\}.$$

Now we investigate the right-hand side of the equality. Let $(F, A) \cup_{\mathcal{R}} (G, B) = (M, A \cap B)$, where $M(x) = F(x) \cup G(x)$ for all $x \in A \cap B = \{e_1, e_5\}$. Then,

$$(M, A \cap B) = \{(e_1, \{h_1, h_3, h_4, h_7, h_8\}), (e_5, \{h_1, h_2, h_3, h_4, h_7, h_8\})\}.$$

And let $(F, A) \cup_{\mathcal{R}} (H, C) = (N, A \cap C)$, where $N(x) = F(x) \cup H(x)$ for all $x \in A \cap C = \{e_1\}$. Then,

$$(N, A \cap C) = \{(e_1, \{h_1, h_2, h_3, h_4, h_8\})\}.$$

Let $(M, A \cap B) \Cap (N, A \cap C) = (K, A \cap (B \cap C))$, where $K(x) = M(x) \cap N(x)$ for all $x \in A \cap (B \cap C) = \{e_1\}$. Then,

$$(K, A \cap (B \cap C)) = ((F, A) \cup_{\mathcal{R}} (G, B)) \Cap ((F, A) \cup_{\mathcal{R}} (H, C)) = \{(e_1, \{h_1, h_3, h_4, h_8\})\}.$$

Since W and K are the same set-valued mapping, $(F, A) \cup_{\mathcal{R}} ((G, B) \Cap (H, C)) = ((F, A) \cup_{\mathcal{R}} (G, B)) \Cap ((F, A) \cup_{\mathcal{R}} (H, C))$ is satisfied.

Theorem 4. *Properties of the extended intersection* $(\sqcap_{\varepsilon})$ *operation*

(a) $(F, A) \sqcap_{\varepsilon} ((G, B) \sqcap_{\varepsilon} (H, C)) = ((F, A) \sqcap_{\varepsilon} (G, B)) \sqcap_{\varepsilon} (H, C)$ [24].
(b) $(F, A) \sqcap_{\varepsilon} \mathcal{U}_A = (F, A)$ [24], $(F, A) \sqcap_{\varepsilon} \Phi_A = \Phi_A$ [24].
(c) $(F, A) \sqcap_{\varepsilon} (G, B) \widetilde{\nsubseteq} (G, B)$, *in general. But if* $(F, A) \widetilde{\subset} (G, B)$, *then* $(F, A) \sqcap_{\varepsilon} (G, B) \widetilde{\subset} (G, B)$, *moreover* $(F, A) \sqcap_{\varepsilon} (G, B) = (G, B)$.
(d) $(F, A) \sqcap_{\varepsilon} ((G, B) \cup_{\mathcal{R}} (H, C)) = ((F, A) \sqcap_{\varepsilon} (G, B)) \cup_{\mathcal{R}} ((F, A) \sqcap_{\varepsilon} (H, C))$.
(e) $((F, A) \cup_{\mathcal{R}} (G, B)) \sqcap_{\varepsilon} (H, C) = ((F, A) \sqcap_{\varepsilon} (H, C)) \cup_{\mathcal{R}} ((G, B) \sqcap_{\varepsilon} (H, C))$.

Proof. (c) Since $A \cup B \nsubseteq A$ without any extra condition is given, $(F, A) \sqcap_{\varepsilon} (G, B) \widetilde{\nsubseteq} (F, A)$, in general. Now assume that $(F, A) \widetilde{\subset} (G, B)$ and $(F, A) \sqcap_{\varepsilon} (G, B) = (H, C)$, where

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C = A \cup B$. Since $A \subset B$, then it is obvious that $A \cup B = B \subset B$. Now we need to show that $H(x)$ and $G(x)$ are the same approximations for all $x \in B$. Let $x \in B$, then either $x \in B \setminus A$ or $x \in A \cap B = A$. If $x \in B \setminus A$, then $H(x) = G(x)$, and if $x \in A \cap B = A$, then $H(x) = F(x) \cap G(x) = G(x) \cap G(x) = G(x)$ for all $x \in A$, since $F(x)$ and $G(x)$ are the same approximations for all $x \in A$. Thus $G(x)$ and $H(x)$ are the identical approximations for all $x \in B$, which completes the proof.

(d) Suppose that $(G, B) \cup_{\mathcal{R}} (H, C) = (M, B \cap C)$, where $M(x) = G(x) \cup H(x)$ for all $x \in B \cap C \neq \emptyset$. Assume that $(F, A) \sqcap_{\varepsilon} (M, (B \cap C)) = (N, A \cup (B \cap C))$, where

$$N(e) = \begin{cases} F(e) & \text{if } e \in A \setminus (B \cap C), \\ M(e) & \text{if } e \in (B \cap C) \setminus A, \\ F(e) \cap M(e) & \text{if } e \in A \cap (B \cap C) \end{cases}$$

for all $e \in A \cup (B \cap C)$.

By taking into account the properties of operations in set theory, it follows that,

$$N(e) = \begin{cases} F(e) & \text{if } e \in (A \setminus B) \cup (A \setminus C), \\ G(e) \cup H(e) & \text{if } e \in (B \setminus A) \cap (C \setminus A), \\ F(e) \cap (G(e) \cup H(e)) & \text{if } e \in A \cap (B \cap C). \end{cases}$$

Now consider the right-hand side of the equality. Suppose that $(F, A) \sqcap_{\varepsilon} (G, B) = (T, A \cup B)$, where

$$T(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in A \cup B$. And suppose $(F, A) \sqcap_{\varepsilon} (H, C) = (W, A \cup C)$, where

$$W(e) = \begin{cases} F(e) & \text{if } e \in A \setminus C, \\ H(e) & \text{if } e \in C \setminus A, \\ F(e) \cap H(e) & \text{if } e \in A \cap C \end{cases}$$

for all $e \in A \cup C$.

Let $(T, A \cup B) \cup_{\mathcal{R}} (W, A \cup C) = (P, (A \cup B) \cap (A \cup C))$, where $P(x) = T(x) \cup W(x)$ for all $x \in (A \cup B) \cap (A \cup C)$. By considering the definitions of T and W along with P, we can write below the equalities:

$$P(e) = \begin{cases} F(e) & \text{if } e \in (A \setminus B) \cup (A \setminus C), \\ G(e) \cup H(e) & \text{if } e \in (B \setminus A) \cap (C \setminus A), \\ (F(e) \cap G(e)) \cup (F(e) \cap H(e)) & \text{if } e \in (A \cap B) \cap (A \cap C) \end{cases}$$

for all $e \in (A \cup B) \cap (A \cup C)$. This follows that N and P are the same set-valued mapping. Therefore, the proof is completed.

(e) By using similar techniques which we have used to prove (d), (e) can be illustrated, too, therefore we skip the proof. □

Proposition 3. $(F, A) \sqcap_{\varepsilon} (G, A) = (F, A) \Cap (G, A)$.

Proof. It is obvious when considering the parameter sets of the soft sets together with Definitions 11 and 12. □

Now, we give a corresponding example of part (d) of Theorem 4.

Example 3. Consider the soft sets (F, A), (G, B) and (H, C) in Example 2.

First, we handle the left-hand side of the equality. Let $(F, A) \cup_{\mathcal{R}} (G, B) = (M, A \cap B)$, where $M(x) = F(x) \cap G(x)$ for all $x \in A \cap B = \{e_1, e_5\}$. Then,

$$(M, A \cap B) = \{(e_1, \{h_1, h_3, h_4, h_7, h_8\}), (e_5, \{h_1, h_2, h_3, h_4, h_7, h_8\})\}.$$

And let $(M, A \cap B) \sqcap_{\varepsilon} (H, C) = (N, (A \cap B) \cup C)$. Then,

$$\begin{aligned}(N, (A \cap B) \cup C) &= ((F, A) \cup_{\mathcal{R}} (G, B)) \sqcap_{\varepsilon} (H, C) \\ &= \{(e_1, \{h_3, h_4, h_8\}), (e_4, \{h_4, h_7\}), (e_5, \{h_1, h_2, h_3, h_4, h_7, h_8\}), (e_6, \{h_1, h_6\})\}.\end{aligned}$$

Now, we investigate the right-hand side of the equality. Let $(F, A) \sqcap_{\varepsilon} (H, C) = (T, A \cup C)$, then

$$(T, A \cup C) = \{(e_1, \{h_3, h_8\}), (e_2, \emptyset), (e_4, \{h_4, h_7\}), (e_5, \{h_2, h_4, h_7, h_8\}), (e_6, \{h_1, h_6\})\}.$$

And let $(G, B) \sqcap_{\varepsilon} (H, C) = (W, B \cup C)$, then

$$(W, B \cup C) = \{(e_1, \{h_4, h_8\}), (e_4, \{h_4, h_7\}), (e_5, \{h_1, h_3\}), (e_6, \{h_1, h_6\})\}.$$

Let $(T, A \cup C) \cup_{\mathcal{R}} (W, B \cup C) = (P, (A \cap B) \cup C)$, then

$$\begin{aligned}(P, (A \cap B) \cup C) &= ((F, A) \sqcap_{\varepsilon} (H, C)) \cup_{\mathcal{R}} ((G, B) \sqcap_{\varepsilon} (H, C)) \\ &= \{(e_1, \{h_3, h_4, h_8\}), (e_4, \{h_4, h_7\}), (e_5, \{h_1, h_2, h_3, h_4, h_7, h_8\}), (e_6, \{h_1, h_6\})\}.\end{aligned}$$

Since N and P are the same set-valued mapping, $((F, A) \cup_{\mathcal{R}} (G, B)) \sqcap_{\varepsilon} (H, C) = ((F, A) \sqcap_{\varepsilon} (H, C)) \cup_{\mathcal{R}} ((G, B) \sqcap_{\varepsilon} (H, C))$ is satisfied.

Theorem 5. *Properties of the intersection* ($\Cap$) *operation*

(a) $(F, A) \Cap ((G, B) \Cap (H, C)) = ((F, A) \Cap (G, B)) \Cap (H, C)$ [9].
(b) $(F, A) \Cap \mathcal{U}_A = (F, A)$ [9], $(F, A) \Cap \mathcal{U}_E = (F, A)$ [24], $(F, A) \Cap \Phi_A = \Phi_A$ [9], $(F, A) \Cap \Phi_E = \Phi_A$ [24].
(c) $(F, A) \Cap (G, B) \tilde{\nsubseteq} (F, A)$*, in general. But if* $(F, A) \tilde{\subseteq} (G, B)$*, then* $(F, A) \Cap (G, B) \tilde{\subseteq} (F, A)$*, moreover* $(F, A) \Cap (G, B) = (F, A)$.
(d) $(F, A) \Cap ((G, B) \cup_{\mathcal{R}} (H, C)) = ((F, A) \Cap (G, B)) \cup_{\mathcal{R}} ((F, A) \Cap (H, C))$.
(e) $((F, A) \cup_{\mathcal{R}} (G, B)) \Cap (H, C) = ((F, A) \Cap (H, C)) \cup_{\mathcal{R}} ((G, B) \Cap (H, C))$.
(f) $(F, A) \Cap ((G, B) \tilde{\cup} (H, C)) = ((F, A) \Cap (G, B)) \tilde{\cup} ((F, A) \Cap (H, C))$ [9].
(g) $((F, A) \tilde{\cup} (G, B)) \Cap (H, C) = ((F, A) \Cap (H, C)) \tilde{\cup} ((G, B) \Cap (H, C))$ [9].
(h) $(F, A) \Cap ((G, B) \smile_{\mathcal{R}} (H, C)) = ((F, A) \Cap (G, B)) \smile_{\mathcal{R}} ((F, A) \Cap (H, C))$.
(i) $((F, A) \smile_{\mathcal{R}} (G, B)) \Cap (H, C) = ((F, A) \Cap (H, C)) \smile_{\mathcal{R}} ((G, B) \Cap (H, C))$.

Proof. (c) Let $(F, A) \Cap (G, B) = (H, C)$, where $C = A \cap B$ and $H(x) = F(x) \cap G(x)$ for all $x \in C$. Since H and F do not need to be the same set-valued mapping for all $x \in A \cap B$, $(F, A) \Cap (G, B) \tilde{\nsubseteq} (F, A)$, in general.

Now assume that $(F, A) \tilde{\subseteq} (G, B)$, then it is obvious that $A \cap B = A \subset A$. Now we need to show that $H(e)$ and $F(e)$ are the same approximations for all $e \in A \cap B = A$. Since $(F, A) \tilde{\subseteq} (G, B)$ and $F(e)$ and $G(e)$ are the same approximation for all $e \in A$, it follows that $H(e) = F(e) \cap G(e) = F(e) \cap F(e) = F(e)$ for all $e \in A$, which completes the proof.

(d) First, we investigate the left-hand side of the equality. Suppose that $(G, B) \cup_{\mathcal{R}} (H, C) = (T, B \cap C)$, where $T(x) = G(x) \cup H(x)$ for all $x \in B \cap C \neq \emptyset$. And assume that $(F, A) \Cap (T, B \cap C) = (W, A \cap (B \cap C))$, where $W(x) = F(x) \cap T(x) = F(x) \cap (G(x) \cup H(x))$ for all $x \in A \cap (B \cap C)$.

Now consider the right-hand side of the equality. Assume that $(F, A) \Cap (G, B) = (M, A \cap B)$, where $M(x) = F(x) \cap G(x)$ for all $x \in A \cap B$. And let $(F, A) \Cap (H, C) = (N, A \cap C)$, where $N(x) = F(x) \cap H(x)$ for all $x \in A \cap C$. Suppose that $(M, A \cap B) \cup G(x) \cup H(x)(N, A \cap C) = (K, (A \cap B) \cap (A \cap C)) = (K, (A \cap B) \cap C)$, where $K(x) = M(x) \cup N(x) = (F(x) \cap G(x)) \cup F(x) \cap H(x)$ for all $x \in (A \cap B) \cap (A \cap C)$. Since W and K are the same set-valued mapping for all $x \in (A \cap B) \cap (A \cap C) = A \cap (B \cap C)$, the proof is completed.

(e) By using similar techniques used to prove (d), (e) can be illustrated, too. So we omit it.

(h) First of all, we look through the left-hand side of the equality. Suppose that $(G, B) \smile_{\mathcal{R}} (H, C) = (T, B \cap C)$, where $T(x) = G(x) \setminus H(x)$ for all $x \in B \cap C \neq \emptyset$. And assume $(F, A) \Cap (T, B \cap C) = (W, A \cap (B \cap C))$, where $W(x) = F(x) \cap T(x) = F(x) \cap (G(x) \setminus H(x)) = (F(x) \cap G(x)) \setminus (F(x) \cap H(x))$ for all $x \in A \cap (B \cap C) \neq \emptyset$.

Now, consider the right-hand side of the equality. Assume that $(F, A) \Cap (G, B) = (M, A \cap B)$, where $M(x) = F(x) \cap G(x)$ for all $x \in A \cap B \neq \emptyset$. And let $(F, A) \Cap (H, C) = (N, A \cap C)$, where $N(x) = F(x) \cap H(x)$ for all $x \in A \cap C$. Suppose that $(M, A \cap B) \smile_{\mathcal{R}} (N, A \cap C) = (K, (A \cap B) \cap (A \cap C)) = (K, A \cap (B \cap C))$, where $K(x) = M(x) \setminus N(x) = (F(x) \cap G(x)) \setminus (F(x) \cap H(x))$ for all $x \in (A \cap B) \cap (A \cap C)$. Since W and K are the same set-valued mapping for all $x \in (A \cap B) \cap (A \cap C) = A \cap (B \cap C)$, this completes the proof.

(i) By using similar techniques used to prove (h), (i) can be shown, too, therefore we skip the proof. □

Now, we give a corresponding example of part (d) of Theorem 5.

Example 4. Consider the soft sets (F, A), (G, B) and (H, C) in Example 2. First we handle the left-hand side of the equality. Let $(G, B) \smile_{\mathcal{R}} (H, C) = (M, B \cap C)$, where $M(x) = F(x) \setminus G(x)$ for all $x \in B \cap C = \{e_1, e_4\}$. Then,

$$(M, B \cap C) = \{(e_1, \{h_1, h_7\}), (e_4, \{h_1, h_2, h_3, h_5, h_6, h_8\})\}.$$

And let $(F, A) \Cap (M, B \cap C) = (N, A \cap (B \cap C))$, where $N(x) = F(x) \cap M(x)$ for all $x \in A \cap (B \cap C) = \{e_1\}$. Then,

$$(N, A \cap (B \cap C)) = (F, A) \Cap ((G, B) \smile_{\mathcal{R}} (H, C)) = \{(e_1, \{h_1\})\}.$$

Now, we look through the right-hand side of the equality. Let $(F, A) \Cap (G, B) = (T, A \cap B)$, where $T(x) = F(x) \cap G(x)$ for all $x \in A \cap B = \{e_1, e_5\}$. Then,

$$(T, A \cap B) = \{(e_1, \{h_1, h_8\}), (e_5, \emptyset)\}.$$

And let $(F, A) \Cap (H, C) = (W, A \cap C)$, where $W(x) = F(x) \cap H(x)$ for all $x \in A \cap C = \{e_1\}$. Then,

$$(W, A \cap C) = \{(e_1, \{h_3, h_8\})\}.$$

Let $(T, A \cap B) \smile_{\mathcal{R}} (W, A \cap C) = (P, A \cap (B \cap C))$, where $P(x) = T(x) \setminus W(x)$ for all $x \in A \cap (B \cap C) = \{e_1\}$. Then,

$$(P, A \cap (B \cap C)) = (F, A) \Cap (G, B) \smile_{\mathcal{R}} ((F, A) \Cap (H, C)) = \{(e_1, \{h_1\})\}.$$

Since N and P are the same set-valued mapping, $(F, A) \Cap ((G, B) \smile_{\mathcal{R}} (H, C)) = ((F, A) \Cap (G, B)) \smile_{\mathcal{R}} ((F, A) \Cap (H, C))$ is satisfied.

Proposition 4. *Let (F, A) be a soft set over U. Then we have the following;*

(i) $(F, A) \widetilde{\cup} (F, A)^r = (F, A) \cup_{\mathcal{R}} (F, A)^r = \mathcal{U}_A$.
(ii) $(F, A) \sqcap_{\varepsilon} (F, A)^r = (F, A) \Cap (F, A)^r = \Phi_A$.
(iii) $(\mathcal{U}_E)^r = \Phi_E$, $(\mathcal{U}_A)^r = \Phi_A$.

Proof. It is obvious, therefore omitted. □

Theorem 6. *Properties of restricted difference ($\smile_{\mathcal{R}}$) operation and its interrelations between other operations on soft sets*

(a) $(F, A) \smile_{\mathcal{R}} \Phi_A = (F, A) \smile_{\mathcal{R}} \Phi_E = (F, A)$.
(b) $(F, A) \smile_{\mathcal{R}} (F, A) = \Phi_A$.
(c) $\mathcal{U}_A \smile_{\mathcal{R}} (F, A) = (F, A)^r$.
(d) $\mathcal{U}_E \smile_{\mathcal{R}} (F, A) = (F, A)^r$ [10].
(e) *Restricted difference holds a right distribution law over intersection, restricted union, extended intersection and union.*
(f) $(F, A) \smile_{\mathcal{R}} ((G, B)) \Cap (H, C) = ((F, A) \smile_{\mathcal{R}} (G, B)) \cup_{\mathcal{R}} ((F, A) \smile_{\mathcal{R}} (H, C))$.
(g) $(F, A) \smile_{\mathcal{R}} ((G, B)) \cup_{\mathcal{R}} (H, C) = ((F, A) \smile_{\mathcal{R}} (G, B)) \Cap ((F, A) \smile_{\mathcal{R}} (H, C))$.
(h) $(F, A) \smile_{\mathcal{R}} ((G, B) \sqcap_{\varepsilon} (H, C)) = ((F, A) \smile_{\mathcal{R}} (G, B)) \widetilde{\cup} ((F, A) \smile_{\mathcal{R}} (H, C))$.
(i) $(F, A) \smile_{\mathcal{R}} ((G, B)) \widetilde{\cup} (H, C) = ((F, A) \smile_{\mathcal{R}} (G, B)) \sqcap_{\varepsilon} ((F, A) \smile_{\mathcal{R}} (H, C))$.

Proof. (a) Let $\Phi_A = (M, A)$ and $(F, A) \smile_{\mathcal{R}} \Phi_A = (F, A) \smile_{\mathcal{R}} (M, A) = (H, A)$, where $H(e) = F(e) \setminus M(e)$ for all $e \in A$. Since $M(e) = \emptyset$ for all $e \in A$, it follows that $H(e) = F(e) \setminus \emptyset = F(e)$. This means that F and H are the same set-valued mapping, which completes the proof. The following can be shown similarly.

(b) It is obvious, hence omitted.

(c) Let $\mathcal{U}_A = (G, A)$ and $\mathcal{U}_A \smile_{\mathcal{R}} (F, A) = (G, A) \smile_{\mathcal{R}} (F, A) = (W, A)$, where $W(e) = G(e) \setminus F(e)$ for all $e \in A$. Since $G(e) = U$ for all $e \in A$, it follows that $W(e) = U \setminus F(e) = F^r(e)$, which completes the proof.

(e) We show that restricted difference holds a right distribution law over restricted union and extended intersection, respectively. The others can be shown similarly. First we handle the left-hand side of the equality of $((F, A) \cup_{\mathcal{R}} (G, B)) \smile_{\mathcal{R}} (H, C) = ((F, A) \smile_{\mathcal{R}} (H, C)) \cup_{\mathcal{R}} ((G, B) \smile_{\mathcal{R}} (H, C))$. Suppose that $(F, A) \cup_{\mathcal{R}} (G, B) = (T, A \cap B)$, where $T(x) = F(x) \cup G(x)$ for all $x \in A \cap B \neq \emptyset$. And assume $(T, A \cap B) \smile_{\mathcal{R}} (H, C) = (P, (A \cap B) \cap C)$, where $P(x) = T(x) \setminus H(x) = (F(x) \cup G(x)) \setminus H(x) = (F(x) \setminus H(x)) \cup (G(x) \setminus H(x))$ for all $x \in (A \cap B) \cap C \neq \emptyset$.

Now we handle the right-hand side of the equality. Assume that $(F, A) \smile_{\mathcal{R}} (H, C) = (M, A \cap C)$, where $M(x) = F(x) \setminus H(x)$ for all $x \in A \cap C \neq \emptyset$. And let $(G, B) \smile_{\mathcal{R}} (H, C) = (N, B \cap C)$, where $N(x) = G(x) \setminus H(x)$ for all $x \in B \cap C \neq \emptyset$. Suppose that $(M, A \cap C) \cup_{\mathcal{R}} (N, B \cap C) = (Q, (A \cap C) \cap (B \cap C)) = (Q, (A \cap B) \cap C)$, where $Q(x) = M(x) \cup N(x) = (F(x) \setminus H(x)) \cup (G(x) \setminus H(x))$ for all $x \in (A \cap C) \cap (B \cap C)$. Since P and Q are the same set-valued mapping for all $x \in (A \cap C) \cap (B \cap C) = (A \cap B) \cap C$, the proof is completed.

Now we show that $((F, A) \sqcap_{\varepsilon} (G, B)) \smile_{\mathcal{R}} (H, C) = ((F, A) \smile_{\mathcal{R}} (H, C)) \sqcap_{\varepsilon} ((G, B) \smile_{\mathcal{R}} (H, C))$. First we investigate the left-hand side of the equality. Suppose that $(F, A) \sqcap_{\varepsilon} (G, B) = (X, A \cup B)$, where

$$X(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in A \cup B$. Assume that $(X, A \cup B) \smile_{\mathcal{R}} (H, C) = (Y, (A \cup B) \cap C)$, where $Y(e) = X(e) \setminus H(e)$ for all $e \in (A \cup B) \cap C$. By taking into account the properties of operations in set theory and the definitions of Y along with X and considering that X

is a piecewise function, we can write below the equalities for Y:

$$Y(e)=\begin{cases}F(e)\setminus H(e) & \text{if } e\in(A\setminus B)\cap C=(A\cap C)\setminus(B\cap C),\\ G(e)\setminus H(e) & \text{if } e\in(B\setminus A)\cap C=(B\cap C)\setminus(A\cap C),\\ (F(e)\cap G(e))\setminus H(e) & \text{if } e\in(A\cap B)\cap C\end{cases}$$

for all $e\in(A\cup B)\cap C$.

Now we investigate the right-hand side of the equality. Assume that $(F,A)\smile_{\mathcal{R}}(H,C)=(K,A\cap C)$, where $K(x)=F(x)\setminus H(x)$, for all $x\in A\cap C$. Assume $(G,B)\smile_{\mathcal{R}}(H,C)=(L,B\cap C)$, where $L(x)=G(x)\setminus H(x)$, for all $x\in B\cap C$. Let $(K,A\cap C)\sqcap_{\varepsilon}(L,B\cup C)=(V,(A\cap C)\cup(B\cap C))$, where,

$$V(e)=\begin{cases}K(e) & \text{if } e\in(A\cap C)\setminus(B\cap C),\\ L(e) & \text{if } e\in(B\cap C)\setminus(A\cap C),\\ K(e)\cap L(e) & \text{if } e\in(A\cap C)\cap(B\cap C)\end{cases}$$

for all $e\in(A\cap C)\cup(B\cap C)$. By taking into account the definitions of K and L, we can rewrite V as below:

$$V(e)=\begin{cases}F(e)\setminus H(e) & \text{if } e\in(A\cap C)\setminus(B\cap C),\\ G(e)\setminus H(e) & \text{if } e\in(B\cap C)\setminus(A\cap C),\\ (F(e)\setminus H(e))\cap(G(e)\setminus H(e)) & \text{if } e\in(A\cap C)\cap(B\cap C)\end{cases}$$

for all $e\in(A\cap B)\cup(A\cap C)$. This follows that Y and V are the same set-valued mapping. Therefore we complete the proof.

(f)–(i) The proofs can be illustrated similar to (e), therefore omitted. □

Now, we give a corresponding example of part (h) of Theorem 6.

Example 5. Consider the soft sets (F,A), (G,B) and (H,C) in Example 2. First we investigate the left-hand side of the equality. Let $(G,B)\sqcap_{\varepsilon}(H,C)=(M,B\cup C)$. In Example 3, it has been shown that

$$(M,B\cup C)=\{(e_1,\{h_4,h_8\}),(e_4,\{h_4,h_7\}),(e_5,\{h_1,h_3\}),(e_6,\{h_1,h_6\})\}.$$

And let $(F,A)\smile_{\mathcal{R}}(M,B\cup C)=(N,A\cap(B\cup C))$, where $N(x)=F(x)\setminus M(x)$ for all $x\in A\cap(B\cup C)=\{e_1,e_5\}$. Then,

$$(N,A\cap(B\cup C))=(F,A)\smile_{\mathcal{R}}((G,B)\sqcap_{\varepsilon}(H,C))=\{(e_1,\{h_1,h_3\}),(e_5,\{h_2,h_4,h_7,h_8\})\}.$$

Now, we handle the right-hand side of the equality. Let $(F,A)\smile_{\mathcal{R}}(G,B)=(T,A\cap B)$, where $T(x)=F(x)\setminus G(x)$ for all $x\in A\cap B=\{e_1,e_5\}$. Then,

$$(T,A\cap B)=\{(e_1,\{h_3\}),(e_5,\{h_2,h_4,h_7,h_8\})\}.$$

Let $(F,A)\smile_{\mathcal{R}}(H,C)=(W,A\cap C)$, where $W(x)=F(x)\setminus H(x)$ for all $x\in A\cap C=\{e_1\}$. Then,

$$(W,A\cap C)=\{(e_1,\{h_1\})\}.$$

Let $(T,A\cap B)\widetilde{\cup}(W,A\cap C)=(P,(A\cap B)\cup(A\cap C))=(P,A\cap(B\cup C))$, then

$$(P,A\cap(B\cup C))=((F,A)\smile_{\mathcal{R}}(G,B))\widetilde{\cup}((F,A)\smile_{\mathcal{R}}(H,C))=\{(e_1,\{h_1,h_3\}),(e_5,\{h_2,h_4,h_7,h_8\})\}.$$

Since N and P are the same set-valued mapping, $(F,A)\smile_{\mathcal{R}}((G,B)\sqcap_{\varepsilon}(H,C))=((F,A)\smile_{\mathcal{R}}(G,B))\widetilde{\cup}((F,A)\smile_{\mathcal{R}}(H,C))$ is satisfied.

Proposition 5. *Let (F,A), (G,B) be two soft sets over a common universe U. Then $(F,A)\smile_{\mathcal{R}}(G,B)=(F,A)\Cap(G,B)^r$.*

Proof. Let $(F,A)\smile_{\mathcal{R}}(G,B)=(H,C)$, where $H(c)=F(c)\setminus G(c)$ for all $c\in C=A\cap B\neq\emptyset$. By Definition 4, $(G,B)^r=(G^r,B)$, where $G^r:B\to P(U)$ is a mapping given by $G^r(\alpha)=U\setminus G(\alpha)$ for all $\alpha\in B$.

Suppose that $(F,A)\Cap(G,B)^r=(F,A)\Cap(G^r,B)=(T,C)$, where $T(y)=F(y)\cap G^r(y)$ for all $y\in C=A\cap B$. To illustrate (H,C) is soft equal to (T,C), let $x\in C$ and $h\in H(x)$, then

$$\begin{aligned}h\in H(x)&\Leftrightarrow h\in F(x)\wedge h\notin G(x)\\&\Leftrightarrow h\in F(x)\wedge h\in G^r(x)\\&\Leftrightarrow h\in F(x)\cap G^r(x)\\&\Leftrightarrow h\in T(x).\end{aligned}$$

This completes the proof. □

Now we are ready to give the definition of *restricted symmetric difference* and its basic properties.

Definition 15. The *restricted symmetric difference* of two soft sets (F,A) and (G,B) over a common universe U is defined by $(F,A)\widetilde{\triangle}(G,B)=((F,A)\cup_{\mathcal{R}}(G,B))\smile_{\mathcal{R}}((F,A)\Cap(G,B))$.

Theorem 7. *Let (F, A) and (G, B) be two soft sets over a common universe U. Then we have the following:*

(i) $(F, A)\widetilde{\triangle}(F, A) = \Phi_A$.
(ii) $(F, A)\widetilde{\triangle}\Phi_A = (F, A)$.
(iii) $(F, A)\widetilde{\triangle}(G, B) = (G, B)\widetilde{\triangle}(F, A)$.
(iv) $(F, A)\widetilde{\triangle}(G, B) = ((F, A) \smile_{\mathcal{R}} (G, B)) \cup_{\mathcal{R}} ((G, B) \smile_{\mathcal{R}} (F, A))$.

Proof. (i) By Definition 15, $(F, A)\widetilde{\triangle}(F, A) = ((F, A) \cup_{\mathcal{R}} (F, A)) \smile_{\mathcal{R}} ((F, A) \Cap (F, A))$. It follows from Theorem 6(b) that $(F, A)\widetilde{\triangle}(F, A) = (F, A) \smile_{\mathcal{R}} (F, A) = \Phi_A$ as required.

(ii) By Definition 15, $(F, A)\widetilde{\triangle}\Phi_A = ((F, A) \cup_{\mathcal{R}} \Phi_A) \smile_{\mathcal{R}} ((F, A) \Cap \Phi_A)$. It follows from Theorem 3(b), Theorem 5(b) that $(F, A)\widetilde{\triangle}\Phi_A = (F, A) \smile_{\mathcal{R}} \Phi_A = (F, A)$ by Theorem 6(a).

(iii) It follows from Definitions 13 and 14.

(vi) By Definition 15, $(F, A)\widetilde{\triangle}(G, B) = ((F, A) \cup_{\mathcal{R}} (G, B)) \smile_{\mathcal{R}} ((F, A) \Cap (G, B))$. Let $(F, A) \cup_{\mathcal{R}} (G, B) = (H, C)$, where $C = A \cap B \neq \emptyset$ and $H(x) = F(x) \cup G(x)$ for all $x \in C$. And suppose that $(F, A) \Cap (G, B) = (Q, C)$, where $C = A \cap B$ and $Q(x) = F(x) \cap G(x)$ for all $x \in C$. And assume that $(H, C) \smile_{\mathcal{R}} (Q, C) = (L, C \cap C = C)$, where $L(x) = H(x) \setminus Q(x)$ for all $x \in C$. Then,

$$\begin{aligned} L(x) &= [(F(x)) \cup G(x)] \setminus [(F(x)) \cap G(x)] \\ &= [(F(x)) \cup G(x)] \cap [(F(x)) \cap (G(x))]' \\ &= [(F(x)) \cup G(x)] \cap [((F(x))' \cup (G(x))'] \\ &= [(F(x)) \cup G(x)) \cap (F(x))'] \cup [(F(x)) \cup (G(x)) \cap (G(x))'] \\ &= [F(x) \cap (F(x))'] \cup [G(x) \cap (F(x))'] \cup [F(x) \cap (G(x))'] \cup [G(x) \cap (G(x))'] \\ &= [G(x) \cap (F(x))'] \cup [(F(x)) \cap (G(x))'] \\ &= [G(x) \setminus F(x)] \cup [F(x) \setminus G(x)]. \end{aligned}$$

Now consider $((F, A) \smile_{\mathcal{R}} (G, B)) \cup_{\mathcal{R}} ((G, B) \smile_{\mathcal{R}} (F, A))$. Suppose that $(F, A) \smile_{\mathcal{R}} (G, B) = (M, C)$, where $M(x) = F(x) \setminus G(x)$ for all $x \in C = A \cap B$, and let $(G, B) \smile_{\mathcal{R}} (F, A) = (N, C)$, where $N(x) = G(x) \setminus F(x)$ for all $x \in C = A \cap B$. And suppose that $(M, C) \cup_{\mathcal{R}} (N, C) = (T, C)$, where $T(x) = M(x) \cup N(x)$ for all $x \in C$. Then $T(x) = (F(x) \setminus G(x)) \cup (G(x) \setminus F(x))$ for all $x \in C$. Since T and H are indeed the same set-valued mapping, $(F, A)\widetilde{\triangle}(G, B) = ((F, A) \smile_{\mathcal{R}} (G, B)) \cup_{\mathcal{R}} ((G, B) \smile_{\mathcal{R}} (F, A))$ is satisfied, as required. □

Now, we give a corresponding example of part (iv) of Theorem 7.

Example 6. Let E be the universe set of parameters, A and B be the subsets of E such that

$$E = \{e_1, e_2, e_3, e_4, e_5\}, \qquad A = \{e_1, e_2, e_3\}, \qquad B = \{e_2, e_3, e_5\}.$$

Let (F, A) and (G, B) be two soft sets over the same universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ such that

$$(F, A) = \{(e_1, \{h_1, h_2, h_6\}), (e_2, \{h_3, h_4, h_5\}), (e_3, \{h_2, h_3, h_6\})\},$$
$$(G, B) = \{(e_2, \{h_5\}), (e_3, \{h_6\}), (e_5, \{h_3, h_5, h_6\})\}.$$

By Definition 15, $(F, A)\widetilde{\triangle}(G, B) = ((F, A) \cup_{\mathcal{R}} (G, B)) \smile_{\mathcal{R}} ((F, A) \Cap (G, B))$. Let $(F, A) \cup_{\mathcal{R}} (G, B) = (H, C)$, where $H(x) = F(x) \cup G(x)$ for all $x \in C = A \cap B = \{e_2, e_3\}$. Then,

$$(H, C) = \{(e_2, \{h_3, h_4, h_5\}), (e_3, \{h_2, h_3, h_6\})\}.$$

And let $(F, A) \Cap (G, B) = (L, C)$, where $L(x) = F(x) \cap G(x)$ for all $x \in C$. Then,

$$(L, C) = \{(e_2, \{h_5\}), (e_3, \{h_6\})\}.$$

Assume that $((F, A) \cup_{\mathcal{R}} (G, B)) \smile_{\mathcal{R}} ((F, A) \Cap (G, B)) = (H, C) \smile_{\mathcal{R}} (L, C) = (W, C)$, where $W(x) = H(x) \cap L(x)$ for all $x \in C$. Then,

$$(W, C) = (F, A)\widetilde{\triangle}(G, B) = \{(e_2, \{h_3, h_4\}), (e_3, \{h_2, h_3\})\}.$$

Now consider $((F, A) \smile_{\mathcal{R}} (G, B)) \cup_{\mathcal{R}} ((G, B) \smile_{\mathcal{R}} (F, A))$. Suppose that $(F, A) \smile_{\mathcal{R}} (G, B) = (M, C)$, where $M(x) = F(x) \setminus G(x)$ for all $x \in C = A \cap B = \{e_2, e_3\}$. Then,

$$(M, C) = \{(e_2, \{h_3, h_4\}), (e_3, \{h_2, h_3\})\},$$

and let $(G, B) \smile_{\mathcal{R}} (F, A) = (N, C)$, where $N(x) = G(x) \setminus F(x)$ for all $x \in C$. Then

$$(N, C) = \{(e_2, \emptyset), (e_3, \emptyset)\}.$$

Assume that $((F, A) \smile_{\mathcal{R}} (G, B)) \cup_{\mathcal{R}} ((G, B) \smile_{\mathcal{R}} (F, A)) = (M, C) \cup_{\mathcal{R}} (N, C) = (T, C)$, where $T(x) = M(x) \cup N(x)$ for all $x \in C$. Then,

$$(T, C) = ((F, A) \smile_{\mathcal{R}} (G, B)) \cup_{\mathcal{R}} ((G, B) \smile_{\mathcal{R}} (F, A)) = \{(e_2, \{h_3, h_4\}), (e_3, \{h_2, h_3\})\}.$$

Since T and W are the same set-valued mapping, $(F,A)\widetilde{\triangle}(G,B) = ((F,A) \smile_{\mathcal{R}} (G,B)) \cup_{\mathcal{R}} ((G,B) \smile_{\mathcal{R}} (F,A))$ is satisfied, as required.

5. Conclusion

In this paper, we have presented a detailed theoretical study of operations on soft sets. We have investigated the algebraic properties of them and looked thorough their interconnections between each other. We have proved that a certain De Morgan's law holds in soft set theory with respect to different operations on soft set theory. We have also defined the restricted symmetric difference and investigated its properties with an illustrative example.

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