Discrete Mathematics 310 (2010) 2091–2099

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Discrepancy and signed domination in graphs and hypergraphs

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ARTICLE INFO

Article history:
Received 22 September 2009
Received in revised form 5 February 2010
Accepted 27 March 2010
Available online 20 April 2010

Keywords:
Graphs
Signed domination function
Signed domination number

ABSTRACT

For a graph G, a signed domination function of G is a two-colouring of the vertices of G with colours $+1$ and -1 such that the closed neighbourhood of every vertex contains more $+1$'s than -1's. This concept is closely related to combinatorial discrepancy theory as shown by Füredi and Mubayi [Z. Füredi, D. Mubayi, Signed domination in regular graphs and set-systems, J. Combin. Theory Ser. B 76 (1999) 223–239]. The signed domination number of G is the minimum of the sum of colours for all vertices, taken over all signed domination functions of G. In this paper, we present new upper and lower bounds for the signed domination number. These new bounds improve a number of known results.

1. Discrepancy theory and signed domination

Originated from number theory, discrepancy theory is, generally speaking, the study of irregularities of distributions in various settings. The classical combinatorial discrepancy theory is devoted to the problem of partitioning the vertex set of a hypergraph into two classes in such a way that all hyperedges are split into approximately equal parts by the classes, i.e. we are interested in measuring the deviation of an optimal partition from perfect, when all hyperedges are split into equal parts. It may be pointed out that many classical results in various areas of mathematics, e.g. geometry and number theory, can be formulated in these terms. The combinatorial discrepancy theory was introduced and studied by Beck in [2]. Also, studies on discrepancy theory have been conducted in [3–5,15].

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with the vertex set V and the hyperedge set $\mathcal{E} = \{E_1, \ldots, E_m\}$. One of the main problems in classical combinatorial discrepancy theory is to colour the elements of V by two colours in such a way that all of the hyperedges have almost the same number of elements of each colour. Such a partition of V into two classes can be represented by a function

$$f : V \to \{+1, -1\}.$$

For a set $E \subseteq V$, let us define the *imbalance* of E as follows:

$$f(E) = \sum_{v \in E} f(v).$$

First defined by Beck [2], the *discrepancy of* $\mathcal{H}$ *with respect to* f is

$$\mathcal{D}(\mathcal{H}, f) = \max_{E_i \in \mathcal{E}} |f(E_i)|$$

and the *discrepancy* of $\mathcal{H}$ is

$$\mathcal{D}(\mathcal{H}) = \min_{f : V \to \{+1, -1\}} \mathcal{D}(\mathcal{H}, f).$$

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doi:10.1016/j.disc.2010.03.030

Thus, the discrepancy of a hypergraph tells us how well all its hyperedges can be partitioned. Spencer [16] proved a fundamental "six-standard-deviation" result that for any hypergraph $\mathcal{H}$ with n vertices and n hyperedges,

$$\mathcal{D}(\mathcal{H}) \le 6\sqrt{n}.$$

As shown in [1], this bound is best possible up to a constant factor. More precisely, if a Hadamard matrix of order $n > 1$ exists, then there is a hypergraph $\mathcal{H}$ with n vertices and n hyperedges such that

$$\mathcal{D}(\mathcal{H}) \ge 0.5\sqrt{n}.$$

It is well known that a Hadamard matrix of order between n and $(1-\epsilon)n$ does exist for any ϵ and sufficiently large n. The following important result, due to Beck and Fiala [4], is valid for a hypergraph with any number of hyperedges:

$$\mathcal{D}(\mathcal{H}) \le 2\Delta - 1,$$

where Δ is the maximum degree of vertices of $\mathcal{H}$. They also posed the discrepancy conjecture that for some constant K

$$\mathcal{D}(\mathcal{H}) < K\sqrt{\Delta}.$$

Another interesting aspect of discrepancy was discussed by Füredi and Mubayi in their fundamental paper [9]. A function $g : V \to \{+1, -1\}$ is called a *signed domination function* (SDF) of the hypergraph $\mathcal{H}$ if

$$g(E_i) = \sum_{v \in E_i} g(v) \ge 1$$

for every hyperedge $E_i \in \mathcal{E}$, i.e. each hyperedge has a positive imbalance. The *signed discrepancy* of $\mathcal{H}$, denoted by $\mathcal{SD}(\mathcal{H})$, is defined in the following way:

$$\mathcal{SD}(\mathcal{H}) = \min_{SDFg} g(V),$$

where the minimum is taken over all signed domination functions of $\mathcal{H}$. Thus, in this version of discrepancy, the success is measured by minimizing the imbalance of the vertex set V, while keeping the imbalance of every hyperedge $E_i \in \mathcal{E}$ positive.

One of the main results in this context, formulated in terms of hypergraphs, is due to Füredi and Mubayi [9]:

Theorem 1 ([9]). *Let $\mathcal{H}$ be an n-vertex hypergraph with hyperedge set $\mathcal{E} = \{E_1, \ldots, E_m\}$, and suppose that every hyperedge has at least k vertices, where $k \ge 100$. Then*

$$\mathcal{SD}(\mathcal{H}) \le 4\sqrt{\frac{\ln k}{k}}n + \frac{1}{k}m.$$

This theorem can be easily re-formulated in terms of graphs by considering the neighbourhood hypergraph of a given graph. A signed domination function of a graph G is a two-colouring of the vertices of G with colours $+1$ and -1 such that the closed neighbourhood of every vertex contains more $+1$'s than -1's. The signed domination number of G, denoted $\gamma_s(G)$, is the minimum of the sum of colours for all vertices, taken over all signed domination functions of G.

Theorem 2 ([9]). *If G has n vertices and minimum degree $\delta \ge 99$, then*

$$\gamma_s(G) \le \left(4\sqrt{\frac{\ln(\delta+1)}{\delta+1}} + \frac{1}{\delta+1}\right)n.$$

Moreover, Füredi and Mubayi [9] found quite good upper bounds for very small values of δ and, using Hadamard matrices, constructed a δ-regular graph G of order 4δ with

$$\gamma_s(G) \ge 0.5\sqrt{\delta} - O(1).$$

This construction shows that the upper bound in Theorem 2 is off from optimal by at most the factor of $\sqrt{\ln \delta}$. They posed an interesting conjecture that, for some constant C,

$$\gamma_s(G) \le \frac{C}{\sqrt{\delta}}n,$$

and proved that the above discrepancy conjecture, if true, would imply this upper bound for δ-regular graphs. A strong result of Matoušek [14] shows that the bound is true, but the constant C in his proof is big making the result of rather theoretical interest.

The lower bound for the signed domination number given in the theorem below is formulated in terms of the degree sequence of a graph. Other lower bounds are also known, see Corollaries 4–6.

Theorem 3 (*[7]*). *Let G be a graph with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. If k is the smallest integer for which*

$$\sum_{i=0}^{k-1} d_{n-i} \geq 2(n-k) + \sum_{i=1}^{n-k} d_i,$$

then

$$\gamma_s(G) \geq 2k - n.$$

In this paper, we present new upper and lower bounds for the signed domination number, which improve the above theorems and also generalise three known results formulated in Corollaries 4–6. Note that our results can be easily reformulated in terms of hypergraphs. Moreover, we refine Füredi–Mubayi's conjecture formulated above as follows: for some $C \leq 10$ and α, $0.18 \leq \alpha < 0.21$,

$$\gamma_s(G) \leq \min\left\{\frac{n}{\delta^\alpha}, \frac{Cn}{\sqrt{\delta}}\right\}.$$

2. Notation and technical results

All graphs will be finite and undirected without loops and multiple edges. If G is a graph of order n, then $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices in G and d_i denotes the degree of v_i. Let $N(x)$ denote the neighbourhood of a vertex x. Also, let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of vertices of G, respectively. Put $\delta = \delta(G)$ and $\Delta = \Delta(G)$.

A set X is called a *dominating set* if every vertex not in X is adjacent to a vertex in X. The minimum cardinality of a dominating set of G is called the *domination number* $\gamma(G)$. The domination number can be defined equivalently by means of a *domination function*, which can be considered as a characteristic function of a dominating set in G. A function $f : V(G) \to \{0, 1\}$ is a domination function on a graph G if for each vertex $v \in V(G)$,

$$\sum_{x \in N[v]} f(x) \geq 1. \tag{1}$$

The value $\sum_{v \in V(G)} f(v)$ is called the weight $f(V(G))$ of the function f. It is obvious that the minimum of weights, taken over all domination functions on G, is the domination number $\gamma(G)$ of G.

It is easy to obtain different variations of the domination number by replacing the set $\{0, 1\}$ by another set of numbers. If $\{0, 1\}$ is exchanged by $\{-1, 1\}$, then we obtain *the signed domination number*. A signed domination function of a graph G was defined in [7] as a function $f : V(G) \to \{-1, 1\}$ such that for each $v \in V(G)$, the expression (1) is true. The signed domination number of a graph G, denoted $\gamma_s(G)$, is the minimum of weights $f(V(G))$, taken over all signed domination functions f on G. Signed domination has been studied in [7–14].

Let $d \geq 2$ be an integer and $0 \leq p \leq 1$. Let us denote

$$f(d, p) = \sum_{m=0}^{\lceil 0.5d \rceil} (\lceil 0.5d \rceil - m + 1) \binom{d+1}{m} p^m (1-p)^{d+1-m}.$$

We will need the following technical results:

Lemma 1 (*[9]*). *If d is odd, then*

$$f(d+1, p) < 2(1-p) f(d, p).$$

If d is even, then

$$f(d+1, p) < \left(2p + (1-p)\frac{d+4}{d+2}\right) f(d, p).$$

In particular, if

$$2(1-p)\left(2p + (1-p)\frac{d+4}{d+2}\right) < 1,$$

then

$$\max_{d \geq \delta} f(d, p) \in \{f(\delta, p), f(\delta+1, p)\}.$$

Lemma 2 (*[6]*). *Let $p \in [0, 1]$ and $X_1, \ldots, X_k$ be mutually independent random variables with*

$$\mathbf{P}[X_i = 1 - p] = p,$$
$$\mathbf{P}[X_i = -p] = 1 - p.$$

If $X = X_1 + \cdots + X_k$ *and* $c > 0$, *then*

$$\mathbf{P}[X < -c] < \mathrm{e}^{-\frac{c^2}{2pk}}.$$

Let us also denote

$$\widetilde{d}_{0.5} = \binom{\delta' + 1}{\lceil 0.5\delta' \rceil},$$

where

$$\delta' = \begin{cases} \delta & \text{if } \delta \text{ is odd;} \\ \delta + 1 & \text{if } \delta \text{ is even.} \end{cases}$$

3. Upper bounds for the signed domination number

The following theorem provides an upper bound for the signed domination number, which is better than the bound of Theorem 2 for 'relatively small' values of δ. For example, if $\delta(G) = 99$, then, by Theorem 2, $\gamma_s(G) \leq 0.869n$, while Theorem 4 yields $\gamma_s(G) \leq 0.537n$. For larger values of δ, the latter result is improved in Corollaries 1–3.

Theorem 4. *For any graph* G *with* $\delta > 1$,

$$\gamma_s(G) \leq \left(1 - \frac{2\widehat{\delta}}{(1+\widehat{\delta})^{1+1/\widehat{\delta}}\, \widetilde{d}_{0.5}^{1/\widehat{\delta}}}\right) n, \tag{2}$$

where $\widehat{\delta} = \lfloor 0.5\delta \rfloor$.

Proof. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with the probability

$$p = 1 - \frac{1}{(1+\widehat{\delta})^{1/\widehat{\delta}}\, \widetilde{d}_{0.5}^{1/\widehat{\delta}}}.$$

For $m \geq 0$, let us denote by B_m the set of vertices $v \in V(G)$ dominated by exactly m vertices of A and such that $|N[v] \cap A| < \lceil 0.5 d_v \rceil + 1$, i.e.

$$|N[v] \cap A| = m \leq \lceil 0.5 d_v \rceil.$$

Note that each vertex $v \in V(G)$ is in at most one of the sets B_m and $0 \leq m \leq \lceil 0.5 d_v \rceil$. Then we form a set B by selecting $\lceil 0.5 d_v \rceil - m + 1$ vertices from $N[v]$ that are not in A for each vertex $v \in B_m$ and adding them to B. We construct the set D as follows: $D = A \cup B$. Let us assume that f is a function $f : V(G) \to \{-1, 1\}$ such that all vertices in D are labelled by 1 and all other vertices by -1. It is obvious that $f(V(G)) = |D| - (n - |D|)$ and f is a signed domination function.

The expectation of $f(V(G))$ is

$$\begin{aligned} \mathbf{E}[f(V(G))] &= 2\mathbf{E}[|D|] - n \\ &= 2(\mathbf{E}[|A|] + \mathbf{E}[|B|]) - n \\ &\leq 2\sum_{i=1}^{n} \mathbf{P}(v_i \in A) + 2\sum_{i=1}^{n} \sum_{m=0}^{\lceil 0.5d_i \rceil} (\lceil 0.5 d_i \rceil - m + 1)\mathbf{P}(v_i \in B_m) - n \\ &= 2pn + 2\sum_{i=1}^{n} \sum_{m=0}^{\lceil 0.5d_i \rceil} (\lceil 0.5 d_i \rceil - m + 1) \binom{d_i + 1}{m} p^m (1-p)^{d_i + 1 - m} - n \\ &\leq 2pn + 2\sum_{i=1}^{n} \max_{d_i \geq \delta} f(d_i, p) - n. \end{aligned}$$

It is not difficult to check that $2(1-p)(2p + (1-p)(d+4)/(d+2)) < 1$ for any $d \geq \delta \geq 2$. As noted by the referee, the simplest way to check this inequality is to observe first that $p > 3/4$. By Lemma 1,

$$\max_{d \geq \delta} f(d, p) \in \{f(\delta, p), f(\delta + 1, p)\}.$$

The last inequality implies $2(1-p) < 1$ because $2p > 1$. Therefore, by Lemma 1,

$$\max_{d \geq \delta} f(d, p) = f(\delta, p)$$

if δ is odd. If δ is even, then we can prove that

$$\max_{d\ge\delta} f(d,p) = f(\delta+1,p).$$

Thus,

$$\max_{d\ge\delta} f(d,p) = f(\delta',p).$$

Therefore,

$$\mathbf{E}[f(V(G))] \le 2pn + 2n\sum_{m=0}^{\lceil 0.5\delta'\rceil}(\lceil 0.5\delta'\rceil - m + 1)\binom{\delta'+1}{m}p^m(1-p)^{\delta'+1-m} - n.$$

Since

$$(\lceil 0.5\delta'\rceil - m + 1)\binom{\delta'+1}{m} \le \binom{\delta'+1}{\lceil 0.5\delta'\rceil}\binom{\lceil 0.5\delta'\rceil}{m},$$

we obtain

$$\begin{aligned}\mathbf{E}[f(V(G))] &\le 2pn + 2n\sum_{m=0}^{\lceil 0.5\delta'\rceil}\binom{\delta'+1}{\lceil 0.5\delta'\rceil}\binom{\lceil 0.5\delta'\rceil}{m}p^m(1-p)^{\delta'+1-m} - n\\ &= 2pn + 2n\binom{\delta'+1}{\lceil 0.5\delta'\rceil}(1-p)^{\delta'-\lceil 0.5\delta'\rceil+1}\sum_{m=0}^{\lceil 0.5\delta'\rceil}\binom{\lceil 0.5\delta'\rceil}{m}p^m(1-p)^{\lceil 0.5\delta'\rceil-m} - n\\ &= 2pn + 2n\widetilde{d}_{0.5}(1-p)^{\delta'-\lceil 0.5\delta'\rceil+1} - n.\end{aligned}$$

Taking into account that $\delta' - \lceil 0.5\delta'\rceil = \lfloor 0.5\delta'\rfloor = \lfloor 0.5\delta\rfloor = \widehat{\delta}$, we have

$$\begin{aligned}\mathbf{E}[f(V(G))] &\le 2pn + 2n\widetilde{d}_{0.5}(1-p)^{\widehat{\delta}+1} - n\\ &\le \left(1 - \frac{2\widehat{\delta}}{(1+\widehat{\delta})^{1+1/\widehat{\delta}}\widetilde{d}_{0.5}^{1/\widehat{\delta}}}\right)n,\end{aligned}$$

as required. The proof of Theorem 4 is complete. □

Our next result and its corollaries give a modest improvement of Theorem 2. More precisely, the upper bound of Theorem 5 is asymptotically 1.63 times better than the bound of Theorem 2, and for $\delta = 10^6$ the improvement is 1.44 times.

Theorem 5. *If* $\delta(G) \ge 10^6$, *then*

$$\gamma_s(G) \le \frac{\sqrt{6\ln(\delta+1)} + 1.21}{\sqrt{\delta+1}}n.$$

Proof. Denote $\delta^+ = \delta + 1$, $N_v = N[v]$ and $n_v = |N_v|$. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with the probability

$$p = 0.5 + \sqrt{1.5\ln\delta^+/\delta^+}.$$

Let us construct two sets Q and U in the following way: for each vertex $v \in V(G)$, if $|N_v \cap A| \le 0.5n_v$, then we put $v \in U$ and add $\lfloor 0.5n_v + 1\rfloor$ vertices of N_v to Q. Furthermore, we assign "+" to $A \cup Q$, and "–" to all other vertices. The resulting function $g : V(G) \to \{-1, 1\}$ is a signed domination function, and

$$g(V(G)) = 2|A\cup Q| - n \le 2|A| + 2|Q| - n.$$

The expectation of $g(V(G))$ is

$$\begin{aligned}\mathbf{E}[g(V(G))] &\le 2\mathbf{E}[|A|] + 2\mathbf{E}[|Q|] - n\\ &= 2pn - n + 2\mathbf{E}[|Q|]. \end{aligned}\qquad (3)$$

It is easy to see that $|Q| \le \sum_{v\in U}\lfloor 0.5n_v + 1\rfloor$, hence

$$\mathbf{E}[|Q|] \le \sum_{v\in V(G)}\lfloor 0.5n_v + 1\rfloor\mathbf{P}[v\in U], \qquad (4)$$

where

$$\mathbf{P}[v \in U] = \mathbf{P}[|N_v \cap A| \le 0.5n_v].$$

Let us define the following random variables

$$X_w = \begin{cases} 1-p & \text{if } w \in A \\ -p & \text{if } w \notin A \end{cases}$$

and let $X_v^* = \sum_{w \in N_v} X_w$. We have

$$|N_v \cap A| \le 0.5n_v \quad \text{if and only if} \quad X_v^* \le (1-p)0.5n_v + (-p)0.5n_v.$$

Thus,

$$\mathbf{P}[|N_v \cap A| \le 0.5n_v] = \mathbf{P}[X_v^* \le (0.5-p)n_v].$$

By Lemma 2,

$$\mathbf{P}[X_v^* \le (0.5-p)\,n_v] < \mathrm{e}^{-\frac{1.5n_v \ln\delta^+/\delta^+}{1+\sqrt{6\ln\delta^+/\delta^+}}}.$$

For $n_v \ge \delta^+ > 10^6$, let us define

$$y(n_v, \delta^+) = \frac{1.5n_v \ln\delta^+/\delta^+}{1+\sqrt{6\ln\delta^+/\delta^+}} - \ln(2.25n_v^{1.5}) + 1.$$

The function $y(n_v, \delta^+)$ is an increasing function of n_v and $y(\delta^+, \delta^+) > 0$ for $\delta^+ > 10^6$. Hence $y(n_v, \delta^+) \ge y(\delta^+, \delta^+) > 0$ and

$$\frac{1.5n_v \ln\delta^+/\delta^+}{1+\sqrt{6\ln\delta^+/\delta^+}} > \ln(2.25n_v^{1.5}) - 1.$$

We obtain

$$\mathbf{P}[|N_v \cap A| \le 0.5n_v] < \mathrm{e}^{1-\ln(2.25n_v^{1.5})} = \frac{e}{2.25n_v^{1.5}},$$

and, using inequality (4),

$$2\mathbf{E}[|Q|] \le 2\sum_{v \in V(G)} \frac{e(0.5n_v+1)}{2.25n_v^{1.5}} \le \frac{e(\delta+3)n}{2.25(\delta+1)^{1.5}} \le \frac{1.21}{\sqrt{\delta+1}}n.$$

Thus, (3) yields

$$\begin{aligned} \mathbf{E}[g(V(G))] &\le 2pn - n + \frac{1.21n}{\sqrt{\delta+1}} \\ &= \frac{\sqrt{6\ln(\delta+1)} + 1.21}{\sqrt{\delta+1}}n, \end{aligned}$$

as required. The proof of Theorem 5 is complete. □

Corollary 1. *If* $24{,}000 \le \delta$, *then*

$$\gamma_s(G) \le \frac{\sqrt{6.8\ln(\delta+1)} + 0.32}{\sqrt{\delta+1}}n.$$

Proof. Putting $p = 0.5 + \sqrt{1.7\ln\delta^+/\delta^+}$ in the proof of Theorem 5, we obtain by Lemma 2,

$$\mathbf{P}[X_v^* \le (0.5-p)\,n_v] < \mathrm{e}^{-\frac{1.7n_v \ln\delta^+/\delta^+}{1+\sqrt{6.8\ln\delta^+/\delta^+}}}.$$

Let us define the following function:

$$y(n_v, \delta^+) = \frac{1.7n_v \ln\delta^+/\delta^+}{1+\sqrt{6.8\ln\delta^+/\delta^+}} - \ln(3.14n_v^{1.5})$$

for $n_v \geq \delta^+ > 24{,}000$. The function $y(n_v, \delta^+)$ is an increasing function of n_v and $y(\delta^+, \delta^+) > 0$ for $\delta^+ > 24{,}000$. Hence $y(n_v, \delta^+) \geq y(\delta^+, \delta^+) > 0$ and

$$\frac{1.7 n_v \ln \delta^+/\delta^+}{1 + \sqrt{6.8 \ln \delta^+/\delta^+}} > \ln(3.14 n_v^{1.5}).$$

We obtain

$$2\mathbf{E}[|Q|] \leq 2 \sum_{v \in V(G)} \frac{0.5 n_v + 1}{3.14 n_v^{1.5}} \leq \frac{(\delta + 3)n}{3.14(\delta + 1)^{1.5}} \leq \frac{0.32}{\sqrt{\delta + 1}} n.$$

Thus, (3) yields

$$\begin{aligned} \mathbf{E}[g(V(G))] &\leq 2pn - n + \frac{0.32n}{\sqrt{\delta + 1}} \\ &= \frac{\sqrt{6.8 \ln(\delta + 1)} + 0.32}{\sqrt{\delta + 1}} n, \end{aligned}$$

as required. The proof is complete. □

Corollary 2. *If* $1{,}000 \leq \delta \leq 24{,}000$, *then*

$$\gamma_s(G) \leq \frac{\sqrt{\ln(\delta + 1)(11.8 - 0.48 \ln \delta)} + 0.25}{\sqrt{\delta + 1}} n.$$

Proof. It is similar to the proof of Corollary 1 if we put $p = 0.5 + \sqrt{(2.95 - 0.12 \ln \delta) \ln \delta^+/\delta^+}$ and consider the the following function for $1{,}000 \leq \delta \leq 24{,}000$:

$$y(n_v, \delta^+) = \frac{(2.95 - 0.12 \ln \delta) n_v \ln \delta^+/\delta^+}{1 + \sqrt{(11.8 - 0.48 \ln \delta) \ln \delta^+/\delta^+}} - \ln(4.01 n_v^{1.5}). \quad \square$$

Corollary 3. *If* $230 \leq \delta \leq 1{,}000$, *then*

$$\gamma_s(G) \leq \frac{\sqrt{\ln(\delta + 1)(18.16 - 1.4 \ln \delta)} + 0.25}{\sqrt{\delta + 1}} n.$$

Proof. It is similar to the proof of Corollary 1 if we put $p = 0.5 + \sqrt{(4.54 - 0.35 \ln \delta) \ln \delta^+/\delta^+}$ and consider the following function for $230 \leq \delta \leq 1{,}000$:

$$y(n_v, \delta^+) = \frac{(4.54 - 0.35 \ln \delta) n_v \ln \delta^+/\delta^+}{1 + \sqrt{(18.16 - 1.4 \ln \delta) \ln \delta^+/\delta^+}} - \ln(4.04 n_v^{1.5}). \quad \square$$

We believe that Füredi–Mubayi's conjecture, saying that $\gamma_s(G) \leq \frac{Cn}{\sqrt{\delta}}$, is true for some small constant C. However, as the Peterson graph shows, $C > 1$, i.e. the behaviour of the conjecture is not good for relatively small values of δ. Therefore, we propose the following refined conjecture, which, roughly speaking, consists of two functions for 'small' and 'large' values of δ.

Conjecture 1. *For some* $C \leq 10$ *and* α, $0.18 \leq \alpha < 0.21$,

$$\gamma_s(G) \leq \min\left\{\frac{n}{\delta^\alpha}, \frac{Cn}{\sqrt{\delta}}\right\}.$$

The above results imply that if $C = 10$ and $\alpha = 0.13$, then this upper bound is true for all graphs with $\delta \leq 96 \times 10^4$.

4. A lower bound for the signed domination number

The following theorem provides a lower bound for the signed domination number of a graph G depending on its order and a parameter λ, which is determined on the basis of the degree sequence of G (note that λ may be equal to 0, in this case we put $\sum_{i=1}^{\lambda} = 0$). This result improves the bound of Theorem 3 and, in some cases, it provides a much better lower bound. For example, let us consider a graph G consisting of two vertices of degree 5 and $n-2$ vertices of degree 3. Then, by Theorem 3,

$$\gamma_s(G) \geq 0.25n - 1,$$

while Theorem 6 yields

$$\gamma_s(G) \geq 0.5n - 1.$$

Theorem 6. *Let G be a graph with n vertices and degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. Then*

$$\gamma_s(G) \geq n - 2\lambda,$$

where $\lambda \geq 0$ is the largest integer such that

$$\sum_{i=1}^{\lambda} \left\lceil \frac{d_i}{2} + 1 \right\rceil \leq \sum_{i=\lambda+1}^{n} \left\lfloor \frac{d_i}{2} \right\rfloor.$$

Proof. Let f be a signed domination function of minimum weight of the graph G. Let us denote

$$X = \{v \in V(G) : f(v) = -1\},$$

and

$$Y = \{v \in V(G) : f(v) = 1\}.$$

We have

$$\gamma_s(G) = f(V(G)) = |Y| - |X| = n - 2|X|.$$

By definition, for any vertex v,

$$f[v] = \sum_{u \in N[v]} f(u) \geq 1.$$

Therefore, for all $v \in V(G)$,

$$|N[v] \cap Y| - |N[v] \cap X| \geq 1.$$

Using this inequality, we obtain

$$|N[v]| = \deg(v) + 1 = |N[v] \cap Y| + |N[v] \cap X| \leq 2|N[v] \cap Y| - 1.$$

Hence

$$|N[v] \cap Y| \geq \frac{\deg(v)}{2} + 1.$$

Since $|N[v] \cap Y|$ is an integer, we conclude

$$|N[v] \cap Y| \geq \left\lceil \frac{\deg(v)}{2} \right\rceil + 1$$

and

$$|N[v] \cap X| = \deg(v) + 1 - |N[v] \cap Y| \leq \left\lfloor \frac{\deg(v)}{2} \right\rfloor.$$

Denote by $e_{X,Y}$ the number of edges between the parts X and Y. We have

$$e_{X,Y} = \sum_{v \in X} |N[v] \cap Y| \geq \sum_{v \in X} \left(\left\lceil \frac{\deg(v)}{2} \right\rceil + 1 \right) \geq \sum_{i=1}^{|X|} \left(\left\lceil \frac{d_i}{2} \right\rceil + 1 \right).$$

Note that if $X = \emptyset$, then we put $\sum_{i=1}^{0} g(i) = 0$. On the other hand,

$$e_{X,Y} = \sum_{v \in Y} |N[v] \cap X| \leq \sum_{v \in Y} \left\lfloor \frac{\deg(v)}{2} \right\rfloor \leq \sum_{i=n-|Y|+1}^{n} \lfloor d_i/2 \rfloor = \sum_{i=|X|+1}^{n} \lfloor d_i/2 \rfloor.$$

Therefore, the following inequality holds:

$$\sum_{i=1}^{|X|}\left(\left\lceil\frac{d_i}{2}\right\rceil+1\right)\leq\sum_{i=|X|+1}^{n}\left\lfloor\frac{d_i}{2}\right\rfloor.$$

Since $\lambda \geq 0$ is the largest integer such that

$$\sum_{i=1}^{\lambda}\left(\left\lceil\frac{d_i}{2}\right\rceil+1\right)\leq\sum_{i=\lambda+1}^{n}\left\lfloor\frac{d_i}{2}\right\rfloor,$$

we conclude that

$$|X| \leq \lambda.$$

Thus,

$$\gamma_s(G) = n - 2|X| \geq n - 2\lambda.$$

The proof is complete. □

Theorem 6 immediately implies the following known results:

Corollary 4 ([10,17]). *For any graph G,*

$$\gamma_s(G) \geq \left(\frac{\lceil 0.5\delta\rceil - \lfloor 0.5\Delta\rfloor + 1}{\lceil 0.5\delta\rceil + \lfloor 0.5\Delta\rfloor + 1}\right)n.$$

Note that Haas and Wexler [10] formulated the above bound only for graphs with $\delta \geq 2$, while Zhang et al. [17] proved a weaker version without the ceiling and floor functions.

Corollary 5 ([13]). *If δ is odd and G is δ-regular, then*

$$\gamma_s(G) \geq \frac{2n}{\delta + 1}.$$

Corollary 6 ([7]). *If δ is even and G is δ-regular, then*

$$\gamma_s(G) \geq \frac{n}{\delta + 1}.$$

Disjoint unions of complete graphs show that these lower bounds are sharp whenever $n/(\delta + 1)$ is an integer, and therefore the bound of Theorem 6 is sharp for regular graphs.

Acknowledgement

The authors thank the referee for helpful comments.

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