On the bicommutant for one type of $J$-symmetric nilpotent algebras in Krein spaces\footnote{Research was supported by the project CONICIT (Venezuela) no. 97000668. E-mail address: str@usb.ve (V. Strauss).}

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Abstract

In the present paper we shall consider an operator algebra in a Krein space. One of the interesting questions that arises in this area is a relationship between the algebra and its bicommutant. Here the question will be investigated for a $J$-symmetric weakly closed algebra that is nilpotent up to the identity operator and has an invariant subspace of a special type.

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0. Introduction

A well-known theorem of von Neumann says that the bicommutant of an arbitrary $W^*$-algebra (all definitions can be found below) coincides with the algebra. If we replace a $W^*$-algebra by a $W^J$-algebra, the corresponding result is false even for a finite-dimensional Pontrjagin space with the index of indefiniteness equal one (i.e. for a finite-dimensional space $II_1$). On the other hand, if we consider only commutative $W^J$-algebras, then for the Pontrjagin space $II_1$ (including infinite-dimensional case) an analog of von Neumann’s Theorem is true, but this result cannot be extended for the case of the space $II_2$. The key property of a commutative $W^J$-algebra in
the space $\Pi_1$ is to have a maximal non-negative subspace that is uniformly positive or can be represented as a direct sum of a one-dimensional neutral subspace and a uniformly positive one. In the present paper we study $WJ^*$-algebras in general Krein spaces under the additional hypothesis that an algebra (maybe non-commutative) is nilpotent up to a scalar summand and has an invariant subspace with the same properties as in $\Pi_1$. A principal result is contained in Theorem 2.22.

1. Definitions

The symbols $\mathbb{R}$ and $\mathbb{C}$ denote here the real line and the complex plane respectively. The term “lineal” will mean a vector space over $\mathbb{C}$. If $\mathcal{H}$ is a Hilbert space and $\mathcal{Y} \subset \mathcal{H}$, then the symbol $\text{Lin} \mathcal{Y}$ refers to a linear span of $\mathcal{Y}$ while the symbol $\text{CLin} \mathcal{Y}$ corresponds to the closed linear span of $\mathcal{Y}$. The symbol $\dim \mathcal{X}$ denotes the linear dimension of $\mathcal{X}$.

In what follows the term “Krein space” means a (complex) vector space $H$ with a Hermitian sesquilinear indefinite form $[\cdot, \cdot]$ if for $H$ there is at least one scalar product $(\cdot, \cdot)$ that converts $H$ to a separable Hilbert space and

$$[x, y] = (Jx, y), \quad x, y \in \mathcal{H}, \quad J = J^{-1}. \quad (1.1)$$

The operator $J$ is called a canonical symmetry.

By the definition of the canonical symmetry $J$ we have $J = P_+ - P_-$, where $P_+$ and $P_-$ are ortho-projections $P_+ + P_- = I$ and

$$\mathcal{H}_+ = P_+ \mathcal{H}, \quad \mathcal{H}_- = P_- \mathcal{H}. \quad (1.2)$$

If at least one of the eigen-subspaces of $J$ (corresponding to the eigenvalues $+1$ and $-1$, respectively) has finite dimension, then the Krein space is said to be a Pontrjagin space (a space $\Pi_\kappa$, $\kappa = \min\{\dim \mathcal{H}_+, \dim \mathcal{H}_-\}$). The decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \quad (1.3)$$

is called a canonical decomposition. Here and everywhere below the symbol $\oplus$ means the orthogonal sum with respect to the scalar product $(\cdot, \cdot)$ from (1.1), that will be called the canonical scalar product.

Let us note that there exist different canonical scalar products, canonical symmetries and canonical decompositions on the same Krein space, but if we fix one of these elements then the other two corresponding canonical elements would be uniquely defined via (1.1), (1.2) and (1.3). Let us observe also that all canonical scalar products define the same topology on $\mathcal{H}$.

In this paper we shall use the terminology from [1]. This remark concerns the natural definitions of positive, negative, definite and neutral vectors or lineals, uniformly positive lineals, maximal non-negative subspaces, $J$-orthogonal vectors, $J$-self-adjoint ($J$-s.a.) operators, etc.

In what follows we shall consider algebras and families of bounded linear operators. If $\mathcal{Y}$ is an operator family then the symbol $\mathcal{Y}'$ refers to the commutant of $\mathcal{Y}$, i.e. to the
algebra of all operators $B$ such that $AB = BA$ for every $A \in \mathfrak{B}$. The algebra $\mathfrak{B}'' = (\mathfrak{B}')'$ is said to be a bicommutant of $\mathfrak{B}$. An algebra $\mathfrak{A}$ is called reflexive if $\mathfrak{A}'' = \mathfrak{A}$.

If $A$ is a continuous linear operator, then the symbol $A^\dagger$ refers to its $J$-adjoint operator. An operator family $\mathfrak{B}$ is said to be $J$-symmetric (symmetric) if $A \in \mathfrak{B}$ implies $A^\dagger \in \mathfrak{B}$ ($A^* \in \mathfrak{B}$). The term $WJ^*$-algebra ($W^*$-algebra) means a weakly closed $J$-symmetric (symmetric) algebra with the identity operator.

A $J$-symmetric operator family $\mathfrak{B}$ belongs by definition to the class $D_1^+$ if there is at least one maximal non-negative subspace $\mathfrak{L}_+$ invariant with respect to $\mathfrak{B}$ and such that it is a direct sum of a neutral $\kappa$-dimensional subspace with $\kappa < \infty$ and a uniformly positive subspace.

2. Nilpotent algebras from the class $D_1^+$: a geometrical description

First let us adduce an example of $WJ^*$-algebra $\mathfrak{A} \in D_1^+$, formed by operators acting in finite-dimensional space $\mathfrak{H}$, and such that $\mathfrak{A} \neq \mathfrak{A}''$.

Example 2.1. Let the space $\mathfrak{H}$ be four-dimensional, $\{e_j\}_1^4$ be an orthonormalized basis of the space, the canonical symmetry $J$ be defined by equalities $Je_0 = e_1$, $Je_1 = e_0$, $Je_2 = e_3$, $Je_3 = e_2$ and a $WJ^*$-algebra $\mathfrak{A}$ be formed by the identity operator and the following operators

$$
A_1: \quad A_1e_0 = e_2, \quad A_1e_1 = 0, \quad A_1e_2 = 0, \quad A_1e_3 = e_1; \\
A_2: \quad A_2e_0 = ie_2, \quad A_2e_1 = 0, \quad A_2e_2 = 0, \quad A_2e_3 = -ie_1; \\
S: \quad Se_0 = e_1, \quad Se_1 = Se_2 = Se_3 = 0.
$$

Then the operators $A_1$, $A_2$ and $S$ are $J$-s.a. and $\mathfrak{A} \neq \mathfrak{A}''$.

Proof. First, $A_1^2 = A_2^2 = A_1A_2 = A_1S = A_2S = S^2 = 0$. Next, a direct calculation shows that $\mathfrak{A}'$ is the algebraic span of $\mathfrak{A}$ and an operator $A_3: A_3e_0 = 0, A_3e_1 = 0, A_3e_2 = 0, A_3e_3 = e_2$. So the algebra $\mathfrak{A}'$ is commutative and $\mathfrak{A}' = \mathfrak{A}''$. □

Now let us give an example with the opposite property.

Example 2.2. Let $\hat{\mathfrak{A}}$ be the $WJ^*$-algebra generated by the identity operator and the same operators $A_1$ and $A_2$. Then for $\hat{\mathfrak{A}}$ the equality $\hat{\mathfrak{A}} = \hat{\mathfrak{A}}''$ holds, because the algebra $\hat{\mathfrak{A}}$ is a linear span of $\hat{\mathfrak{A}}$ and three operators $A_3$, $S$ and $Z$, where $Z: Ze_0 = e_0, Ze_1 = -e_1, Ze_2 = e_2, Ze_3 = -e_3$, and, additionally, the operator $Z$ does not commute with $A_3$ and $S$.

These two examples show, first, that for study of the general situation it is useful to investigate some simple $J$-symmetric algebras in finite dimensional spaces and, second, that the operator $S = [\cdot, e_1]e_1$ can play some special role.
Let $\mathfrak{H}$ be such that

1. $\mathfrak{H}$ is a $WJ^+$-algebra (in the general case non-commutative);
2. $\mathfrak{H} \in D^+_1$;
3. every $A \in \mathfrak{H}$ can be represented in the form $A = \alpha I + A_0$, where $A_0$ is a nilpotent operator.

Denote by $\mathfrak{H}^0$ the subset of the algebra $\mathfrak{H}$ that contains all nilpotent operators and only them. Let us assume that

$$\mathfrak{H}^0 \neq \{0\}. \quad (2.5)$$

Let $\mathcal{U}_+$ be a maximal non-negative invariant subspace of the algebra $\mathfrak{H}$, that is a direct sum of a uniformly positive subspace and a one-dimensional neutral subspace. Put $\mathcal{U}_+ = \mathcal{U}_+^{[1]}$, $\mathcal{U}_1 = \mathcal{U}_+ \cap \mathcal{U}_-$. Then the subspace $\mathcal{U}_1$ is invariant with respect to $\mathfrak{H}$, one-dimensional and neutral. This implies $A\mathcal{U}_1 = \{0\}$ for every $A \in \mathfrak{H}^0$. Conversely, if $A \in \mathfrak{H}$ and $A\mathcal{U}_1 = \{0\}$, then $A \in \mathfrak{H}^0$. Thus, $\mathfrak{H}^0$ is a $J$-symmetric lineal. Let $e_1 \in \mathcal{U}_1$ be a fixed vector with unit norm. Put

$$e_0 = Je_1, \quad \mathcal{U}_0 = \mathcal{U}_1, \quad \mathcal{Z} = \mathcal{U}_1^{[1]} \cap \mathcal{U}_1^{[2]} = (\mathcal{U}_0 \oplus \mathcal{U}_1)^{[1]}.$$  

(2.6)

Since $\text{Lin} \{e_0, e_1\}$ is invariant with respect to $J$, the equality $J\mathcal{Z} = \mathcal{Z}$ holds.

Consider a structure of an arbitrary operator $A \in \mathfrak{H}^0$. First, we have $A\mathcal{U}_+ \subset \mathcal{U}_1$. Indeed, without loss of generality one can suppose that $A \in \mathfrak{H}^0$ is $J$-self-adjoint. Then for every $x, y \in \mathcal{U}_+$ and natural $n$ we have $||(Ax,y)|| \leq ||A^{[2]}x,x||^{1/2}$. Since $A$ is nilpotent, $[Ax,y] = 0$, i.e. $Ax \in \mathcal{U}_1$ as required. The relation $A\mathcal{U}_- \subset \mathcal{U}_1$ is proved by the same way. Next, the chain $[Ae_0, e_1] = [e_0, A^he_1] = 0$ shows that $A\mathcal{U}_0 \subset \mathcal{U}_1^{[1]}$. So for the operator $A$ there are vectors $a, a^h \in \mathcal{Z}$ and a number $\alpha$, such that

$$Ae_0 = a + \alpha e_1; \quad Ax = [x, a^h]e_1, \quad \text{where} \ x \in \mathcal{Z}; \quad A e_1 = 0. \quad (2.7)$$

Representation (2.7) implies that $\mathfrak{H}^0$ is a subalgebra of $\mathfrak{H}$. Next, the direct calculations show that

$$A^he_0 = a^h + \alpha e_1; \quad A^h x = [x, a]e_1, \quad \text{where} \ x \in \mathcal{Z}; \quad A^he_1 = 0. \quad (2.8)$$

So, if $A = A^h$, then $a = a^h$ and $\alpha \in \mathbb{R}$.

Note that a choice of the subspaces $\mathcal{U}_0$ and $\mathcal{Z}$ was based on a choice of the canonical symmetry $J$ and therefore we can simplify (if necessary) the operator structure of $\mathfrak{H}^0$ altering $J$.

Introduce the operator $S_0$ getting

$$S_0 x = [x, e_1]e_1, \quad x \in \mathcal{S}. \quad (2.9)$$

In Examples 2.1 and 2.2 $\mathcal{U}_+ = \text{Lin} \{e_1, e_2 + e_3\}$, $\mathcal{U}_- = \text{Lin} \{e_1, e_2 - e_3\}$ and $S_0 = S$. In Example 2.2 $S_0 \in \mathfrak{H}$ but $S_0 \not\in \mathfrak{H}$.

**Proposition 2.3.** If there is at least one definite vector $a; a = Ae_0$, where $A \in \mathfrak{H}^0$, then $S_0 \in \mathfrak{H}$.
Proof. Formulae (2.7) and (2.8) yield $A^* A = [a, a] S_0$. □

Let $\mathcal{A}_J$ be a set of all operators $A_0 \in \mathcal{A}$, such that
\[ A_0 \in \mathcal{A}. \] (2.10)
If $S_0 \in \mathcal{A}$, then $\mathcal{A}_J$ has linear co-dimension with respect to $\mathcal{A}$ equal one, but if $S_0 \notin \mathcal{A}$, then below we shall demonstrate, that there is a canonical symmetry $J$ in $\mathcal{S}$, such that $\mathcal{A}_J = \mathcal{A}$. Let $a \in \mathcal{A}$ be a vector, such that there exists an operator $A = A^* \in \mathcal{A}$ related with $a$ through Representations (2.7) and (2.8). The set of all $a$ under this condition is said to be the shadow of $e_0$ (with respect to $\mathcal{A}$) and is denoted by $sh_{\mathcal{A}}(e_0)$, i.e.
\[ sh_{\mathcal{A}}(e_0) = \{ x : x = A e_0 - [A e_0, e_0] e_1, A = A^* \in \mathcal{A} \}. \] (2.11)
Note that $sh_{\mathcal{A}}(e_0)$ is a closed subset and for all vectors $a, b \in sh_{\mathcal{A}}(e_0)$ and for all numbers $\alpha, \beta \in \mathbb{R}$ the relationship $\alpha a + \beta b \in sh_{\mathcal{A}}(e_0)$ holds.

Recall that $\mathcal{A}$ is a complex Hilbert space. Let $\mathcal{B}$ be its certain subset that is a closed real linear space, i.e. if $x, y \in \mathcal{B}$, $\alpha, \beta \in \mathbb{R}$, then $\alpha x + \beta y \in \mathcal{B}$ and, if $\lim_{j \to \infty} x_j = x, x_j \in \mathcal{B}$, then $x \in \mathcal{B}$. In what follows a subset under this condition is said to be real subspace (with respect to $\mathcal{A}$).

Let us note, that for $x, y \in \mathcal{B}$ the inequality $(x, y) \neq (y, x)$ is possible, i.e. a Hilbert structure, defined on $\mathcal{B}$, may not induce on $\mathcal{B}$ a structure of a real Hilbert space. Indeed, one can define on $\mathcal{B}$ a structure of Euclidean space with the topology equal to the norm topology, generated on $\mathcal{B}$, but, generally speaking, in this case a new scalar product would be defined on $\mathcal{B}$.

If $\mathcal{B}$ is a real subspace, then the subset $i\mathcal{B} = \{ i x \}_{x \in \mathcal{B}}$ is a real subspace too. In general $i\mathcal{B} \neq \mathcal{B}$.

Definition 2.4. A real subspace $\mathcal{E}$ is said to be purely real, if $\mathcal{A} \cap i\mathcal{A} = \{ 0 \}$.

If $\mathcal{E}$ is a purely real subspace, then the direct sum $\mathcal{E} + i\mathcal{E}$ is well defined, in general the same role is played by the lineal Lin $\{ \mathcal{E}, i\mathcal{E} \}$.

Definition 2.5. A real subspace $\mathcal{E}$ is said to be total (with respect to $\mathcal{A}$), if the lineal Lin $\{ \mathcal{E}, i\mathcal{E} \}$ is dense in $\mathcal{A}$.

A simple example of purely real subspace with respect to $\mathcal{A}$ can be given as follows: get in $\mathcal{A}$ some orthonormalized basis and denote $\mathcal{A}_r$ the set of all vectors from $\mathcal{A}$, that have real coordinates with respect to this basis. Then $\mathcal{A}_r$ is a purely real subspace,
\[ \mathcal{A} = \mathcal{A}_r + i\mathcal{A}_r \] (2.12)
and the scalar product of vectors $x = x_1 + i x_2$ and $y = y_1 + i y_2$ has the form
\[ (x, y) = (x_1, y_1) + (x_2, y_2) + i(−(x_1, y_2) + (x_2, y_1)). \] (2.13)
Example 2.6. Let \( \{ e_j \}_{j=0}^{\infty} \) — orthonormalized basis in \( \mathbb{R} \) and a real subspace \( \mathcal{E} \) is spanned on on the following vector system

\[
\{ g_{2j} = e_{2j}; \quad g_{2j+1} = i e_{2j} + \frac{1}{j+1} \cdot e_{2j+1} \}_{j=0}^{\infty}.
\]

Then \( \mathcal{E} + i \mathcal{E} \neq \mathbb{R} \) but \( \mathcal{E} + i \mathcal{E} = \mathbb{R} \).

Proof. In fact, for every finite collection of real numbers \( \{ \alpha_j, \beta_j \}_{j=0}^{n} \) we have

\[
\left\| \sum_{j=0}^{n} (\alpha_j g_{2j} + \beta_j g_{2j+1}) \right\|^2 = \sum_{j=0}^{n} \left( \alpha_j^2 + \beta_j^2 \left( 1 + \frac{1}{(j+1)^2} \right) \right),
\]

therefore \( \mathcal{E} \) consists of vectors that have a form \( x = \sum_{j=0}^{\infty} (\alpha_j g_{2j} + \beta_j g_{2j+1}) \), where \( \sum_{j=0}^{\infty} (|\alpha_j|^2 + |\beta_j|^2) < \infty \). Hence, if for instance \( z = \sum_{j=0}^{\infty} (1/j+1) e_{2j+1} \), then \( z \notin \mathcal{E} + i \mathcal{E} \). □

Remark 2.7. If \( \mathcal{E} \) is a total purely real subspace, but \( \mathcal{E} + i \mathcal{E} \neq \mathbb{R} \), then one can extend \( \mathcal{E} \) preserving the same properties, i.e. there is a total purely real subspace \( \hat{\mathcal{E}} \supset \mathcal{E} \), such that \( \hat{\mathcal{E}} \neq \mathcal{E} \), so there is no sense to introduce a notion of maximality for total purely real subspaces.

Proof. In fact, under the described above conditions there is a vector \( y \in \mathbb{R} \), such that \( y, iy \notin \mathcal{E} + i \mathcal{E} \). Put \( \hat{\mathcal{E}} = \text{R-Lin} \{ y, \mathcal{E} \} \), where R-Lin is a symbol of linear span with real coefficients. Demonstrate that \( \hat{\mathcal{E}} \) is a purely real subspace. Let us suppose the contrary, i.e. let there be a vector \( z \in \mathcal{E} \) such that \( i(y + z) \in \hat{\mathcal{E}} \). Then we have \( i(y + z) = \alpha y + x, \alpha \in \mathbb{R}, x \in \mathcal{E} \), and, as a result, \( y = (1/\alpha)(x - iz) \), i.e. \( y \in \mathcal{E} + i \mathcal{E} \). It is a contradiction. □

Definition 2.8. The subspace \( \mathcal{E}^\times = \mathcal{E} \cap i \mathcal{E} \) is said to be the complex part of a real space \( \mathcal{E} \).

Note that if \( \mathcal{E} \) is a purely real subspace, then \( \mathcal{E}^\times = \{ 0 \} \), in the opposite case \( \mathcal{E}^\times \) is a (complex!) subspace of the initial space.

Proposition 2.9. If a real subspace \( \mathcal{E} \) is not total, then there is a total real subspace \( \hat{\mathcal{E}} \), such that \( \mathcal{E} \subset \hat{\mathcal{E}} \), \( \mathcal{E}^\times = \hat{\mathcal{E}}^\times \).

Proof. It is sufficient to construct a total real subspace with respect to the space \( \text{ClLin}(\mathcal{E}, i \mathcal{E})^\times \), but it is possible to do always using, for instance, the method (2.12). □
Definition 2.10. Let $\mathcal{E} \subset \mathcal{J}$ be a real subset. Denote by $\mathcal{E}^b$ a collection of all vectors $y \in \mathcal{J}$, such that $(x, y) \in \mathbb{R}$ for every $x \in \mathcal{E}$. The collection $\mathcal{E}^b$ is said to be dual real subspace for $\mathcal{E}$ (with respect to $\mathcal{J}$).

It is clear that $\mathcal{E}^b$ is a real subspace. We shall give a simple geometrical method for construction of $\mathcal{E}^b$ on the base of $\mathcal{E}$. Let the space $\mathcal{J}$ be in the form (2.12). Then every vector $x \in \mathcal{E}$ can be represented in the form $x = x_1 + ix_2$, where $x_1 \in \mathcal{J}_r$, $x_2 \in \mathcal{J}_r$. By the same way for $y \in \mathcal{E}^b$ we have $y = y_1 + iy_2$. Let us fix $y$. Since for every $x \in \mathcal{E}$ we have $(x, y) \in \mathbb{R}$, then due to (2.13) we have

$$x_2 y_1 - x_1 y_2 = 0, \quad x \in \mathcal{E}.$$  

(2.14)

The last condition can be treated as a condition of the orthogonality for two vectors $(-x_2 \oplus x_1)$ and $(y_1 \oplus y_2)$ belonged to the paired real Hilbert space $\mathcal{J}_r \oplus \mathcal{J}_r$. On the other hand it is clear that $(x_1 \oplus x_2)$ is a vector from the same space too. Moreover, the map $\mathcal{U}: (x_1 \oplus x_2) \to (-x_2 \oplus x_1)$ is a unitary operator acting in $\mathcal{J}_r \oplus \mathcal{J}_r$. Note also that in view of (2.12) the spaces $\mathcal{J}$ and $\mathcal{J}_r \oplus \mathcal{J}_r$ coincide as real linear spaces under the natural identification between their elements, and from this point of view the operator $\mathcal{U}$ is the multiplication operator by the scalar $i$ acting on $\mathcal{J}$.

Summarizing these arguments we obtain the following result.

Proposition 2.11. Consider the sets $\mathcal{E}$, $\mathcal{E}^b$ and $\mathcal{E}^b$ as subspaces naturally embedded according with to (2.12) into the real Hilbert space $\mathcal{J}_r \oplus \mathcal{J}_r$. Then the subspace $\mathcal{E}^b$ coincides with the orthogonal complement for the subspace $\mathcal{E}$.

Corollary 2.12. For every real subspace $\mathcal{E}$ the equality $(\mathcal{E}^b)^{\perp} = \mathcal{E}$ holds.

Corollary 2.13. For every real subspace $\mathcal{E}$ the equality $\mathcal{E}^\perp = (\mathcal{E}^b)^\perp$ holds (the orthogonal complement is treated here in the sense of the complex Hilbert space $\mathcal{J}$).

Corollary 2.14. If $\mathcal{E}$ is a real subspace, then $\mathcal{E}^b$ is total and, in particular, if $\mathcal{E}$ is simultaneously real and total, then $\mathcal{E}^b$ has the same properties.

Now let $\mathcal{J}$ be simultaneously a Hilbert space and a Krein space.

Definition 2.15. Let $\mathcal{E}$ be a real subspace with respect to $J$-space $\mathcal{J}$. Let us denote as $\mathcal{E}^{(b)}$ a real subspace, that is formed by all vectors $y \in \mathcal{J}$ such that $[x, y] \in \mathbb{R}$ for every $x \in \mathcal{E}$. Then $\mathcal{E}^{(b)}$ is said to be the $J$-dual subspace to $\mathcal{E}$.

Proposition 2.16. For every real subspace $\mathcal{E}$ the equality $(\mathcal{E}^{(b)})^{\perp} = \mathcal{E}$ holds.

This proposition follows directly from Corollary 2.12.

Let us return to the investigation of the structure of $\mathcal{J}^0$. It is clear that the subspace $sh_{\mathcal{J}^0}(e_0)$ is real (with respect to $\mathcal{J}$).
Proposition 2.17. If $S_0 \not\in \mathfrak{U}^0$, then there is a choice of a fundamental symmetry $J$ such that $\mathfrak{U}^0 e_0 \subset \mathfrak{J}$.

Proof. Let $\alpha = \text{dim } \mathfrak{J}$ ($\alpha = \infty$ is not excluded). Let $\{a_j\}^\alpha_{j=1}$ be a Riesz basis in $sh_M(e_0)$ such that the subsystem $\{a_j\}^\alpha_{j=1} \cap sh_M(e_0)^x$ is a basis in $sh_M(e_0)^x$. By the definition of $sh_M(e_0)$ in $\mathfrak{U}^0$ there is an operator system $\{A_j\}^\alpha_{j=1}$, such that $A_j e_0 = a_j + \alpha j$, $\alpha \in \mathbb{R}$, $j = 1, 2, \ldots, \alpha$, and for $a_j \in sh_M(e_0)^x$ there is an operator $B_j \in \mathfrak{U}^0$ such that $B_j e_0 = i a_j + \beta j$, $\beta \in \mathbb{R}$. Next, note that due to the condition $S_0 \not\in \mathfrak{U}^0$ the real subspace $sh_M(e_0)$ is neutral (see Proposition 2.3). Introduce a new system $\{b_j\}^\alpha_{j=1}$ that is $J$-conjugate to $\{a_j\}^\alpha_{j=1}$ in the following sense: $[a_j, b_1] = \delta j$, where $\delta j$ is Kronecker’s symbol. It is clear that the system $\{b_j\}^\alpha_{j=1}$ is a Riesz basis too and $A_j b_1 = \delta j e_1$ and, if $a_j \in sh_M(e_0)^x$, then $B_j b_1 = - i \delta j e_1$. For simplification put $\beta_j = 0$, $B_j = 0$, if $a_j \not\in sh_M(e_0)^x$. Note that $\sum_{j=1}^\alpha |a_j + i \beta_j|^2 < \infty$. In fact, if $\sum_{j=1}^\alpha |a_j + i \beta_j|^2 = \infty$, then there exists a sequence of coefficients $\{y_j^{(m)}\}^\alpha_{j=1}$, $m = 1, 2, \ldots, \alpha$, such that for $m \to \infty$ the conditions $\sum_{j=1}^\alpha y_j^{(m)}(A_j + i B_j)e_0 \to e_1$, $\sum_{j=1}^\alpha |y_j^{(m)}|^2 \to 0$ hold, that implies $S_0 \in \mathfrak{U}^0$. It contradicts to the hypothesis. Now put $\hat{e}_0 = e_0 - \sum_{j=1}^\alpha ((a_j + i \beta_j)/2)b_j$, $\hat{a}_j = a_j + ((a_j - i \beta_j)/2)e_1$, $j = 1, 2, \ldots, \alpha$. Then $A_j \hat{e}_0 = a_j + \alpha j e_1 - ((a_j + i \beta_j)/2)e_1 = \hat{a}_j$ and, if $a_j \in sh_M(e_0)^x$ then $B_j \hat{e}_0 = i a_j + \beta j e_1 + i((a_j + i \beta_j)/2)e_1 = \hat{b}_j$, $j = 1, 2, \ldots, \alpha$. For finishing put $\hat{T}a_j = b_j$, $\hat{T}b_j = \hat{a}_j$, $j = 1, 2, \ldots, \alpha$. $\hat{T}e_1 = \hat{e}_0$, $\hat{T}\hat{e}_0 = e_1$ and define a canonical symmetry $\hat{J}$ arbitrarily on $\mathfrak{J} \cap (\text{CLin}[a_j, b_j])^{[\perp]}$. $\square$

Remark 2.18. Although Representation (2.7) for operators $A \in \mathfrak{U}^0$ and the real linear subspace $sh_M(e_0)$ depend on the choice of a canonical symmetry $J$, most important properties of $sh_M(e_0)$, connected with the form $\cdot \cdot$, will be the same for different ways to define $J$. However, while these properties are not picked out yet, we suppose that if $S_0 \not\in sh_M(e_0)$, then (due to Proposition 2.17) the choice of $J$ is put into effect in such a way that $\mathfrak{U}^0 e_0 \subset \mathfrak{J}$.

Lemma 2.19. Let $B = B^\#$ and $B e_1 = 0$. Put $b = B e_0 - [B e_0, e_0]e_1$. Then $B \in \mathfrak{U}$ if and only if the following conditions

(a) there is a $J$-self-adjoint operator $B_{\mathfrak{J}} : \mathfrak{J} \to \mathfrak{J}$, such that $B x = [x, b]e_1 + B_{\mathfrak{J}} x$ for all $x \in \mathfrak{J}$;

(b) $b (sh_M(e_0)) \in \mathfrak{U}$;

(c) $B_{\mathfrak{J}} (sh_M(e_0)) = [0, \mathfrak{J}]$;

hold.

Proof. Let $B \in \mathfrak{U}$. Since $B \mathfrak{J} \subset \mathfrak{U}^{[\perp]}$, $\beta = [B e_0, e_0] \in \mathbb{R}$ and $S_0 \in \mathfrak{U}$, one can suppose $b = B e_0 \in \mathfrak{J}$. Next, if $x \in \mathfrak{J}$, $A = A^\# \in \mathfrak{U}$, $A e_0 = a + a e_1$, then $[B x, e_0] = [x, b]$, $[a, b] = [A e_0, B e_0] = [B e_0, A e_0] = [b, a]$, i.e. $b \in sh_M(e_0)^{[b]}$ and the opera-
Ator \( B_0 \): \( B_0 e_0 = b, \ B_0 x = [x, b] e_1 \) for \( x \in \mathcal{D} \), \( B_0 e_1 = 0 \), is J-s.a., so without loss of the generality one can assume \( B e_0 = 0 \) and \( B \not\subset \mathcal{D} \). Since \( a \in sh_{\mathcal{H}}(e_0) \), then \( B a = B A e_0 = A B e_0 = 0 \) and the condition \( B \subset sh_{\mathcal{H}}(e_0) \) follows from the J-self-adjointing of \( B \). Conversely, if for operator \( B \) Conditions (2.15) hold, then the direct verification shows that \( B \in \mathcal{A} \).

Corollary 2.20. The algebra \( \mathcal{A} \) is commutative if and only if for all \( x, y \in sh_{\mathcal{H}}(e_0) \) the equality \( [x, y] = [y, x] \) holds.

Let us denote by symbol \((\mathcal{A}')_0\) the subset of all operators \( B \in \mathcal{A}' \), such that \( B e_1 = 0 \).

Proposition 2.21. If \( S_0 \not\in \mathcal{A} \), then the linear codimension of \((\mathcal{A}')_0\) with respect to \( \mathcal{A}' \) is equal two, and if \( S_0 \in \mathcal{A} \), then the same codimension is equal one.

Proof. Let \( S_0 \not\in \mathcal{A} \). Then due to Proposition 2.3 the real linear subspace \( sh_{\mathcal{H}}(e_0) \) is neutral, so the (complex) Hilbert subspace \( csh_{\mathcal{H}}(e_0) = \text{CLin}\{sh_{\mathcal{H}}(e_0), ish_{\mathcal{H}}(e_0)\} \) is neutral too. Next, since \( csh_{\mathcal{H}}(e_0) \) is neutral, we have \( csh_{\mathcal{H}}(e_0) \subset \text{Ker}\ 0 \), therefore the subspaces \( \mathcal{L}_0 \oplus csh_{\mathcal{H}}(e_0) \) and \( J(csh_{\mathcal{H}}(e_0)) \oplus \mathcal{L}_1 \) are invariant with respect to the algebra \( \mathcal{A} \). Thus for an \( C \), described by the conditions
\[
\begin{align*}
Cx &= -ix \text{ for } x \in \mathcal{L}_0 \oplus csh_{\mathcal{H}}(e_0); \\
Cx &= ix \text{ for } x \in J(csh_{\mathcal{H}}(e_0)) \oplus \mathcal{L}_1; \\
Cx &= 0 \text{ for } x[\mathcal{L}_0 \oplus csh_{\mathcal{H}}(e_0) \oplus J(csh_{\mathcal{H}}(e_0)) \oplus \mathcal{L}_1].
\end{align*}
\]

we have \( C \in \mathcal{A}' \). Now let \( B = B^* \in \mathcal{A}' \) and \( B e_1 = (\alpha + \beta) e_1 \). Then \( B - \alpha I = -\beta C \in (\mathcal{A}')_0 \). This proves what we wanted for the first part.

Let us go to the second one. If \( B = B^* \in \mathcal{A}' \), \( B e_1 = (\alpha + \beta) e_1 \), then \( B e_0 = (\alpha - \beta) e_0 + z \), where \( z \in \mathcal{L}_1 \). Since now \( S_0 \in \mathcal{A} \), then \( S_0 B = BS_0 \); therefore, \( S_0 B e_0 = B S_0 e_0 \), i.e. \( \beta = 0 \). Thus, \( B - \alpha I \in (\mathcal{A}')_0 \).

Theorem 2.22. Let an algebra \( \mathcal{A} \) satisfy Conditions (2.4), \( \mathcal{L}_+ \) be the corresponding invariant subspace of \( \mathcal{A} \) and let \( 0 \not\in e_1 \in \mathcal{L}_+ \cap \mathcal{L}_1 \) be an arbitrary fixed vector. Let \( e_0 \) be a arbitrary fixed neutral vector such that \( [e_1, e_0] = 1 \), and let the operator \( S_0 \) and the set \( sh_{\mathcal{H}}(e_0) \) correspond to Formulae (2.6), (2.9) and (2.11). If \( S_0 \not\in \mathcal{A} \), then \( \mathcal{A} = \mathcal{A}' \). If \( S_0 \in \mathcal{A} \), then \( \mathcal{A} = \mathcal{A}'' \) if and only if the set \( sh_{\mathcal{H}}(e_0) \) is a purely real subspace.

Proof. Without loss of generality we can suppose that \( e_1 \) has the unit norm and \( J e_1 = e_0 \). Let \( S_0 \not\in \mathcal{A} \). Then \( sh_{\mathcal{H}}(e_0) \) is neutral end (see Corollary 2.20) the algebra \( \mathcal{A} \) is commutative, so \( \mathcal{A}'' \subset \mathcal{A}' \). Thus, we need to extract all operators from \( \mathcal{A}' \), such that they belong to \( \mathcal{A}'' \). First, note that \( S_0, C \in \mathcal{A}' \), where the operator \( C \) is
described by the conditions from (2.16), but \( CS_0 \neq S_0 C \), so \( C \notin \mathfrak{W} \), \( S_0 \notin \mathfrak{W} \). Now, let \( B = B^k \in \mathfrak{W}' \) and \( B \epsilon_1 = 0 \). In virtue of the structure of \( C \) and Lemma 2.19 it is clear, that \( Bx = 0 \) for every \( x \in \mathfrak{L} \) \( \oplus csh_{\mathfrak{H}}(e_0) \oplus J(csh_{\mathfrak{H}}(e_0)) \oplus \mathfrak{L}' \). Thus, without loss of generality one can assume that \( \mathfrak{J} = csh_{\mathfrak{H}}(e_0) \oplus J(csh_{\mathfrak{H}}(e_0)) \). Next, Lemma 2.19 implies \( b = B \epsilon_0 \in \mathfrak{J} [b] [b] \). Then due to Proposition 2.16 we obtain \( b \in sh_{\mathfrak{H}}(e_0) \). As a next step note that for the operator \( B \) the corresponding operator \( B \) from (2.15) is equal zero, because in the opposite case \( BC \neq CB \). It proves the equality \( \mathfrak{H} = \mathfrak{W}'' \) as stated.

Let us pass to the second part. Let \( S_0 \in \mathfrak{H} \) and \( \mathfrak{W}' = \mathfrak{H} \). We need to show that \( sh_{\mathfrak{H}}(e_0) \cap \mathfrak{W}'' \neq \{ 0 \} \). Let \( z \in sh_{\mathfrak{H}}(e_0) \cap \mathfrak{W}'' \). Put \( Z: Zx = [x, z]e \). Then \( Z = Z^\phi, Ze_0 = Ze_1 = 0 \) and \( ZB = 0 \) for every \( B = B^k \in (\mathfrak{W}' \). In fact, due to Corollary 2.13 and the conditions from (2.15) we have that \( B \epsilon_0 = [B \epsilon_0, e_0] e_1 \in sh_{\mathfrak{H}}(e_0) \setminus \{ 0 \} \), i.e. \( Bz = 0, Bx \in sh_{\mathfrak{H}}(e_0) \setminus 1 = \mathfrak{L}_1 \) for arbitrary \( x \in \mathfrak{J} \). Thus, \( Z \in \mathfrak{W}' \). It is a contradiction! The necessity has been justified.

Let us go to the sufficiency. Let \( sh_{\mathfrak{H}}(e_0) \cap \mathfrak{W}'' \) is total with respect to \( \mathfrak{J} \) and one can extract from \( \mathfrak{W} \) a minimal subalgebra \( \mathfrak{L} \) defined by the conditions
\[ (a) \ I \in \mathfrak{L}, S_0 \in \mathfrak{L}; \]
\[ (b) \ for \ every \ q \in sh_{\mathfrak{H}}(e_0) [b] \ the \ corresponding \ J-s.a. \ operator \ Q: Qe_0 = q, Qe_1 = 0, \]
\[ Qx = [x, q]e_1 \ for \ x \in \mathfrak{J}. \]

Note that \( sh_{\mathfrak{H}}(e_0) \cap \mathfrak{W}'' \) is total in \( \mathfrak{J} \), so \( sh_{\mathfrak{H}}(e_0) [1] = \{ 0 \} \). Thus, replacing \( \mathfrak{W} \) by \( \mathfrak{L} \), from Lemma 2.19 we have \( \mathfrak{L}' = \mathfrak{W} \). From the other hand by the construction \( \mathfrak{L} \subset \mathfrak{W} \), so \( \mathfrak{L}' \supset \mathfrak{W}'' \). Thus, \( \mathfrak{W}'' = \mathfrak{W} \).

**Remark 2.23.** Now we can describe the characteristic of \( sh_{\mathfrak{H}}(e_0) \), that defines the structure of \( \mathfrak{W}'' \): it is a property of \( sh_{\mathfrak{H}}(e_0) \) to be or not to be a purely real linear subspace.

**Remark 2.24.** Example 2.1 was constructed using exactly some reasons related with Lemma 2.22. In this one \( \mathfrak{W} \in D_1^+ \), codimension of the nilpotent subalgebra \( \mathfrak{W}_0 \) is equal to one, the real subspace \( sh_{\mathfrak{H}}(e_0) \), spanned on vectors \( e_2 \) and \( ie_2 \), is not purely real. Note also that in this case the algebra \( \mathfrak{W}'' \) does not belong to the class \( D_1^+ \), but enters the class \( D_1^+ \). If we slightly modify this example taking instead of the four-dimensional space \( \mathfrak{S} \) an infinite-dimensional one, spanning on an orthonormalized basis \( \{ e_j \} \), and taking \( e_{2j} = e_{2j+1} \), \( e_{2j+1} = e_{2j}, j = 0, 1, 2, \ldots; A_{2j-1} = e_{2j}, A_{2j-1} = e_{2j+1} = e_{2j}, A_{2j} = e_{2j+1} = e_{2j} = e_{2j+1} = 0, m \neq 0, 2j+1; A_{2j}: A_{2j} = e_{2j+1} \), \( A_{2j} = e_{2j} \), \( A_{2j} \) then we get a such \( WJ^* \)-algebra \( \mathfrak{W} \), that \( \mathfrak{W}'' \) does not belong to \( D_1^+ \) for any \( \kappa \) (let us recall that \( \kappa < \infty \)).

Theorem 2.22 shows not only a criteria for the reflexivity of the corresponding algebra but a possibility of an extension of the initial algebra within the same class.
**Definition 2.25.** Let $\mathcal{A}$ be an operator algebra with a collection of properties $\mathcal{A}$. An operator algebra $\mathcal{B} \supset \mathcal{A}$, $\mathcal{B} \neq \mathcal{A}$, is said to be a state preserving extension of $\mathcal{A}$ (with respect to $\mathcal{A}$) if it has the properties $\mathcal{A}$ too. A state preserving extension $\mathcal{B}$ (or the same algebra $\mathcal{A}$) is called maximal if it has no proper state preserving extensions.

**Theorem 2.26.** Let $\mathcal{A}$ satisfy Conditions (2.4), be reflexive and assume that $S_0 \notin \mathcal{A}$. Then $\mathcal{A}$ has at least one maximal state preserving extension with respect to the collection of (2.4) united with the reflexivity. Among these maximal extensions there is at least one that does not contain $S_0$.

**Proof.** Under the conditions of the theorem $sh_\mathcal{A}(e_0)$ is a neutral set, so there is a maximal neutral (complex) subspace $\mathcal{G}$: $sh_\mathcal{A}(e_0) \subset \mathcal{G} \subset \mathcal{B}$. Now the demanded algebra $\mathcal{B}$ can be constructed as the minimal $\mathcal{WJ}^*$-algebra, generated by all operators $B$: $B e_0 = b \in \mathcal{G}$, $B x = [x, b] e_1$, $x \in \mathcal{B}$, $B e_1 = 0$. □

**Theorem 2.27.** Let $\mathcal{A}$ satisfy Conditions (2.4), be reflexive and assume that $S_0 \in \mathcal{A}$. Then $\mathcal{A}$ has a maximal state preserving extension with respect to the collection of (2.4) united with the reflexivity if and only if the direct sum $sh_\mathcal{A}(e_0) + i sh_\mathcal{A}(e_0)$ is closed.

**Proof.** The necessity follows from Note 2.7 and the sufficiency can be obtained from Proposition 2.9. □

3. Nilpotent algebras from the class $D_1^+$: an operator approach

In the above we investigated the properties of the algebra $\mathcal{A}$ subject to Condition (2.4), based on a geometrical object, i.e. the real linear subspace $sh_\mathcal{A}(e_0)$. Now we pass to another approach to describe properties of $\mathcal{A}$. In this case $\mathcal{A}$ is subjected simultaneously to (2.4) and to the following condition (vector $e_0$ is the same as before)

if $A \in \mathcal{A}$ and $A e_0 = 0$, then $A = 0$. (3.17)

Note the connection between (3.17) and concepts introduced above.

**Proposition 3.28.** Condition (3.17) holds for the algebra $\mathcal{A}$ if and only if $sh_\mathcal{A}(e_0)$ is a purely real subspace.

**Proof.** If $A \in \mathcal{A}$, $A = B + i C$, $B = B^\#, C = C^\#$, then, on the one hand, $A \neq 0$ if and only if $B \neq 0$ or $C \neq 0$, and, on the other hand, $A e_0 = 0$ if and only if $B e_0 = -i C e_0$. □

Let the subspaces $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{B}$ and vectors $e_0$, $e_1$ be the same as above, and let $\mathcal{Z}$ be an operator set, defined by the following conditions ($A \in \mathcal{Z}$):
Let \( \{g_j\}_1^\infty \) be a fixed orthonormalized basis of Riesz in \( \mathcal{B} \). For \( x, y \in \mathcal{B}, x = \sum_{j=1}^{\infty} \alpha_j g_j, y = \sum_{j=1}^{\infty} \beta_j g_j \) put \( \langle x, y \rangle = \sum_{j=1}^{\infty} \alpha_j \beta_j \). By the construction \( \langle \cdot, \cdot \rangle \) it is a symmetric form, that is linear, in contrast to \( [\cdot, \cdot] \), not only for the first argument, but also for the second one. Thus, \( \langle x, y \rangle = [x, G y] \), where \( G : \mathcal{B} \rightarrow \mathcal{B} \) is a continuous anti-linear operator, \( G^2 = I \). Note also, that
\[
(\Gamma x, \Gamma y) = \overline{\langle x, y \rangle}, [\Gamma x, \Gamma y] = [y, x].
\]
Let us associate with an arbitrary vector \( x = A e_0, A \in \mathfrak{I} \) and every vector \( y \in \mathcal{B} \) a number \( [Ay, e_0] \). This relationship is form for \( x \) and \( y \). The mention form is linear for both arguments and it is continuous with respect to \( y \), so due to classical Riesz's Theorem we can write \( [Ay, e_0] = \langle Gx, y \rangle \), where \( G \) is a linear operator with the domain \( \mathfrak{D}(G) = \{x : x = A e_0, A \in \mathfrak{I}\} \) and the range being a subset of \( \mathcal{B} \). Operator \( G \) is well defined due to one-to-one correspondence between vectors from \( \mathfrak{D}(G) \) and operators from \( \mathfrak{I} \) (see (3.18)). Moreover, if vectors \( e_0, e_1 \) and a form \( \langle \cdot, \cdot \rangle \) on \( \mathcal{B} \) are fixed, then the linear set \( \mathfrak{I} \) is reconstructed uniquely through \( G \). So properties of \( \mathfrak{I} \) can be reformulated as properties of \( G \) and vice versa. Operator \( G \) related with \( \mathfrak{I} \) we shall denote as \( \text{cod}(\mathfrak{I}) \).

**Proposition 3.29.** The operator set \( \mathfrak{I} \) is closed if and only if the operator \( G = \text{cod}(\mathfrak{I}) \) is closed.

**Proof.** Let \( B \in \mathfrak{I} \), where \( \mathfrak{I} \) is a \( u \)-closure of \( \mathfrak{I} \). It is clear that for operator \( B \) the conditions \( a) - c) \) from (3.18) are fulfilled, therefore there is a pair of vectors \( x_B, y_B \in \mathcal{B} \), such that for all \( \alpha, \beta \in \mathfrak{C} \) and \( z \in \mathcal{B} \) the equality
\[
B(\alpha e_0 \oplus z \oplus \beta e_1) = \alpha x_B + \langle y_B, z \rangle e_1
\]
holds. Moreover, if \( x_B, y_B \in \mathcal{B} \) is an ordered pair of vectors, one can associate the pair with at least one operator \( B \in \mathfrak{I} \) if and only if the pair in question belongs to the \( u \)-closure of the graph of \( G \). Since the graph lies in the paired Hilbert space \( \mathcal{B} \oplus \mathcal{B} \) and in a Hilbert space a weak closure of a lineal coincides with a strong one, the rest is plain. \( \square \)

**Proposition 3.30.** The operator family \( \mathfrak{I} \) is \( J \)-symmetric if and only if the following condition
\[
\Gamma G \Gamma |_{\mathfrak{D}(G)} = I |_{\mathfrak{D}(G)}.
\]
holds for operator \( G = \text{cod}(\mathfrak{I}) \).

**Proof.** Let \( \mathfrak{I} \) be a \( J \)-symmetric set, \( x \in \mathfrak{D}(G) \), \( A e_0 = x \). Then for every \( y \in \mathcal{B} \) due to (3.19) \( [A^y e_0, y] = [e_0, Ay] = [Ay, e_0] = (Gx, y) = (\Gamma G x, \Gamma y) = [\Gamma G x, y] \), so \( A^y e_0 = \Gamma G x \). On the other hand \( [A^y y, e_0] = [y, A e_0] = [y, x] = (Fx, y) \). Finally, since \( A^y \in \mathfrak{I} \), then \( \Gamma G x \in \mathfrak{D}(G) \) and the representation \( [A^y y, e_0] = (G A^y e_0, y) = \).
\( (G^2 x, y) \) holds. Thus, \( G^2 x = I x \) for every \( x \in \mathfrak{D}(G) \), that is equivalent \( (3.20) \). The converse reasoning follows the same path, in particular, the condition \( IG x \in \mathfrak{D}(G) \) for \( x \in \mathfrak{D}(G) \) follows directly from \( (3.20) \). \( \square \)

**Proposition 3.31.** Let \( \mathcal{T}_1 \) be a lineal of operators acting in \( \mathfrak{S} \) and having properties like \( (3.18) \), but in general different from \( \mathcal{T} \), let the form \( \langle \cdot, \cdot \rangle \) be the same as above. Then the relationship \( \mathcal{T}_1 \subset \mathcal{T} \) is true if and only if the equality \( \langle G x, y \rangle = \langle x, G^2 y \rangle \) holds for all \( x \in \mathfrak{D}(G) \), \( y \in \mathfrak{D}(G^2) \), \( G = \operatorname{cod}(\mathcal{T}) \), \( G^2 = \operatorname{cod}(\mathcal{T}_1) \).

**Proof.** By virtue of the conditions from \( (3.18) \) a commutativity of operators \( A \in \mathcal{T} \) and \( B \in \mathcal{T}_1 \) takes place if and only if \( AB e_0 = BA e_0 \). Let \( A e_0 = x \) and \( B e_0 = y \). Then \( AB e_0 = \langle G x, y \rangle \) and \( BA e_0 = \langle G^2 y, x \rangle \). \( \square \)

**Theorem 3.32.** If \( J \)-symmetric operator family \( \mathcal{S} \) with properties \( (3.18) \) is commutative, then \( G = \operatorname{cod}(\mathcal{S}) \) is an inverse \( J \)-isometric operator.

**Proof.** In the present case the invertibility of \( G \) follows directly \( (3.20) \). Next, let \( x, y \in \mathfrak{D}(G) \). Then \( \langle G x, y \rangle = \langle x, G^2 y \rangle \) and, therefore, \( [G x, G y] = \langle G x, IG y \rangle = \langle x, GIG y \rangle = \langle x, G^2 y \rangle = [x, y] \). \( \square \)

Return to the algebra \( \mathfrak{A} \). Agree that if \( S_0 \notin \mathfrak{A} \) then a choice of a scalar product on \( \mathfrak{S} \) corresponds to Proposition 2.17, and if \( S_0 \in \mathfrak{A} \), then it is arbitrary. Let us use the above notation \( \mathfrak{A}_J \) (see \( (2.10) \)). If for \( \mathfrak{A} \) there is \( (3.17) \), then for \( \mathfrak{A}_J \) the conditions from \( (3.18) \) hold (i.e. \( \mathcal{S} = \mathfrak{A}_J \)), so the operator \( G = \operatorname{cod}(\mathfrak{A}_J) \) is well defined.

**Proposition 3.33.** If for commutative \( WJ^* \)-algebra \( \mathfrak{A} \) the Condition \( (3.17) \) holds, then the operator \( G = \operatorname{cod}(\mathcal{A}_J) \) is closed and \( J \)-isometric.

This proposition follows directly from Propositions 3.29–3.31.

**Corollary 3.34.** If under the conditions of Proposition 3.33 the space \( \mathfrak{S} \) is a Pontrjagin space, then the lineal \( \mathfrak{D}(G) \) is closed and the operator \( G \) is bounded.

**Remark 3.35.** Theorem 3.32 shows that the existence problem of state preserving extension for a commutative reflexive algebra \( \mathfrak{A}, S_0 \in \mathfrak{A} \), subject to Conditions \( (2.4) \), can be reduced to the extension problem for \( J \)-isometric operators in Krein spaces or Pontrjagin spaces.

4. Closing remarks

These results had a long story and finally were presented in the author’s talk in IWOTA (Faro, Portugal) in 2000. A theorem on the equality \( \mathfrak{A}'' = \mathfrak{A} \) for an algebra
generated by a single $J$-s.a. operator in a space $\Pi_1$ was announced by author during IX School on Operator Theory in Functional Spaces (Ternopol, Ukraine, 1984), the same result with a complete proof was published in [2]. A generalization of the theorem for a case of an algebra generated by a single $J$-s.a. operator of the class $D_1^+$ contains in [3]. Next, Litvinov [4] and Benderskii et al. [5] proved the corresponding theorem for an arbitrary commutative $WJ^*$-algebra in $\Pi_1$. Theorem 2.22 was announced by author in 1990 during XV School on Operator Theory in Functional Spaces (Ulyanovsk, Russia). The operator $\text{cod}(\mathcal{Z})$ was appeared at the first time in the Naymark’s papers (see [6]), but in his case it was an isometric operator with respect to a Hilbert space scalar product. In the present form the construction of $\text{cod}(\mathcal{Z})$ can be found in [7].

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