Interfacial Deformation of Liquid Drops by Electric Fields at Zero Gravity

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(Received October 1995; accepted November 1995)

Abstract—The dynamic behaviour of a viscous droplet under the influence of applied electric fields in zero gravity is investigated by solving the axisymmetric Navier-Stokes equations numerically using the NASA-VOF2D algorithm and Laplace's equation for the electrostatic potential using a standard finite difference technique. The initial condition of the droplet profile was determined by solving the Young-Laplace equation.

The model is successful in predicting the dynamic deformation process of the droplet including its breakup. Although the numerically observed breakups were too small to be detected experimentally, our computational results were supported by the charge loss measurements of the sessile droplet carried out during the experiment.

Keywords—Drops in electric fields, Zero gravity, Viscous free surface problems.

1. INTRODUCTION

In this paper, we consider a large viscous drop that is subjected to an applied electric field in a zero gravity environment. Due to the applied electric field, an electric charge is induced on the surface of the drop, resulting in an outwardly directed force to the surface. This outward pressure directly counteracts the surface tension, and under suitable conditions, it forces the liquid up into a cone from which small charged droplets are ejected. The emission of the charged droplets depends on the viscosity, the surface tension, the mass density, the permittivity, and the electrical conductivity of the fluid.

In addition to being of theoretical interest, this problem has widespread applications in several areas including electrostatic precipitators for air pollution control, electrostatic painting, insecticide spraying of crops, and ink-jet printing [1,2]. In the case of water drops, the deformation and breakup of the interface is believed to be an important factor in the production of thunder storms [3]. In zero-gravity conditions, drop dynamic problems are important in order to obtain a better understanding of containerless processing technology in space [4].

The interaction of electric fields with fluids and the presence of a cone on the surface of a liquid in an electric field was recorded as early as 1600 by Gilbert. He noted that when an electrified rod...
was brought near a drop of water which was sitting on a dry surface, the drop formed into a conical shape. In 1882, Lord Rayleigh [5] derived the critical amount of the charge (Rayleigh Limit) that was required to destabilize an isolated conductive spherical drop. In 1914, Zeleny photographed and studied the reaction of a meniscus held at the end of an electrified glass capillary tube with a diameter of less than a millimeter [6]. The meniscus oscillated at a lower voltage, and eventually, with increased voltage, the drop disintegrated after forming a conical end and issuing a thin jet. Taylor [7] demonstrated both theoretically and experimentally that electric and surface tension stresses could balance for a liquid in the shape of a cone of a half angle of 49.30° at a particular voltage depending on the surface tension and on the electrode configuration. In recent years, further analytical and experimental investigations of liquid drops in electric fields were carried out by many authors including [8,9]. In 1988, Inculet and Kromann [10] experimentally studied the breakup of large water drops in an electric field in the presence of gravity by suspending an alcohol doped water drop in a dielectric oil. The drop elongated and developed a Taylor cone on one or both sides of the drop and ejected a filament. When the drop was symmetrically positioned between the electrodes, Taylor cones formed at both ends. However, if drops were lightly displaced from the centre, only one cone formed at the end closer to the electrode.

Most of the analytic and numerical work has been concerned with the shape and stability of linearly polarizable dielectric drops that are surrounded by another linearly polarizable dielectric fluid subject to an electric field with the flow being considered inviscid (see [11–14]). One of the main points of interest has been the appearance of hysteresis as the value of the ratio \( \kappa \) of the permeability of the drop to the surrounding fluid varies.

Only a few authors have considered the flow to be viscous. Sherwood [15] supposed that the Reynolds number was low enough that Stokes flow could be assumed. Kaiser [16] considered some two-dimensional problems with a free surface for viscous flows driven by electric fields, but he does not treat the axisymmetric drop problem. He uses a version of the NASA-VOF code for the fluid flow just as we have done. Inculet and his coworkers [17] developed and utilized a numerical simulator based on a finite volume technique and an adaptive grid algorithm to predict the experimentally observed elongation of a large liquid drop by a uniform electric field. The drop was placed between two parallel electrodes and the experiment was carried out in microgravity conditions produced with parabolic KC-135 NASA aircraft flights. Unfortunately, the numerical technique was unable to predict the evolution of the drop beyond the elongation.

In 1990, similar experiments were carried out to investigate the behaviour of sessile drops under the influence of applied electric fields [18]. A large liquid drop generated in microgravity conditions was placed on the bottom electrode of a parallel electrode system. Once the electric field was applied, the evolution of the drop was recorded on a high-speed camera. The experiment shows that when the applied electric field is above a critical value, then the drop elongates into a cone from which small charged droplets are ejected.

The aim of this paper is to develop a computational procedure in order to simulate the elongation and breakup processes of the drops observed during this experiment. As stated above, one of the main difficulties encountered while simulating these types of problems involving interfacial deformations is in tracking the moving interface which is not known a priori. Among the various computational techniques available for solving problems with interfacial dynamics, volume tracking methods have shown great promise for numerical simulations of large surface deformations. We modify a version of the well-known SOLA-VOF procedure, NASA-VOF2D (see [19]), which is based on a volume tracking technique, volume of fluid (see [20]). The NASA-VOF2D algorithm solves two-dimensional transient fluid flow problems with free boundaries. In this algorithm, the location of the interface is identified in terms of a volume fraction parameter \( F \), which represents the fractional volume of the surface cell that is filled with fluid. Therefore, \( F \) is unity in cells filled with fluid, zero in empty cells, and takes intermediate values in the interface cells. A special donor-acceptor method is utilized to advect the volume fraction field and to reconstruct the fluid interface.
In the next section, we discuss the mathematical model describing the dynamical behaviour of the drop. We will suppose that the coupling between the electric quantities and the fluid flow quantities in this drop model occurs at the interface only. There is no bulk force of electrical origin and gravity is neglected. Furthermore, the axisymmetric assumption simplifies the complicated three-dimensional problem. The numerical simulation used to study and predict the behaviour of the drops is discussed in Sections 3, 4 and 5, and a discussion of the numerical results is presented in Section 6. While our results are consistent with the qualitative behaviour of the drop breakup when compared to the experimental data, there is a need for a more realistic three-dimensional model in order to obtain quantitative agreement as well.

2. FORMULATION OF THE PROBLEM

Consider a single spherical incompressible liquid drop of density \( \rho \) and kinematic viscosity \( \nu \) placed on the surface of the lower electrode of a parallel plane electrode system as shown in Figure 1. Gravity is assumed to be zero, and hence, all gravitational effects are assumed to be negligible. We suppose that the problem is two-dimensional and axisymmetrical with respect to the \( z \)-axis from the drop centre normal to the electrodes. Furthermore, we assume that the drop is a perfect conductor, and therefore, that there is no bulk coupling between the electrostatic variables and the fluid flow variables since the free electric charges in a conductor reside on the surface of the conductor. Hence, the electrostatic equations are solved separately and coupled by the appropriate stress condition at the interface.

Governing Equations

Within the bulk of the fluid, the conservation of mass and the conservation of momentum must be satisfied. Thus we have the continuity equation, which for axisymmetric incompressible flow is given by

\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0. \tag{1}
\]

Similarly, the equations expressing the conservation of momentum are the Navier-Stokes equations

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right), \tag{2}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} \right), \tag{3}
\]

where \( r \) and \( z \) are the radial and axial coordinates, respectively, and \( u \) and \( v \) are the velocity components in the \( r \) and \( z \) direction, respectively.
Along with the above equations, we must also solve the relevant electrodynamic equations. Since our model is based on perfect conductivity, the potential $\phi$ is given by
\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0,
\]
which must be solved in the region outside the bulk of the drop where we neglect the mechanical effects of air. The electric field $E$ is then given by $(E_r, E_z) = (-\frac{\partial \phi}{\partial r}, -\frac{\partial \phi}{\partial z})$.

**Boundary Conditions**

The kinematic condition at the surface of the drop can be stated as
\[
[(v - v_s) \cdot n]_{\Gamma_d} = 0,
\]
where $v_s$ is the velocity of the drop’s surface, $\Gamma_d$ represents the surface of the drop, and $n = (n_r, n_z)$ is the unit vector normal to the surface of the drop. This condition represents the conservation of mass, and it can be obtained by integrating the continuity equation across the interface. In terms of the drop profile $\eta(t, r, z) = 0$, this condition can be written as
\[
\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial r} + v \frac{\partial \eta}{\partial z} = 0.
\]

At the interface, the stresses must be continuous and this surface condition can be decomposed into normal and tangential stress conditions. The continuity of the normal stresses is given by
\[
P - P_0 = \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) n_i n_j + \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) - \epsilon_0 \frac{E_n^2}{2},
\]
where $\gamma$ is the constant surface tension, $\mu$ is the viscosity, $E_n$ is the normal component of the electric field, $\epsilon_0$ is the permittivity of free space, and $1/R_1 + 1/R_2$ is the mean radius of curvature where $R_1$ and $R_2$ are the local principal radii of curvature. The first term in equation (4) represents the stress due to viscosity, the second term represents the pressure jump across the interface due to surface tension, and the last term is produced by the electric field. Similarly, the continuity of the tangential stresses is given by
\[
\left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) t_i n_j = 0,
\]
where $t = (t_i, t_j)$ is the unit tangential vector.

We also have wall conditions for the velocities and for the potential $\phi$. The boundary conditions for $v$ result from the no-slip and no-penetration conditions which require the vanishing of the tangential and normal components of the velocity, respectively. A constant contact angle is given at the triple point where the drop interface intersects with the solid electrodes. This condition is usually determined from experiments.

One of the boundary conditions for $\phi$ results from the assumption that the drop is a perfect conductor, so that its surface is equipotential. Therefore, on $\Gamma_b$, which represents all the points on the lower electrode and the drop surface, we require
\[
[\phi]_{\Gamma_b} = \phi_b,
\]
where $\phi_b$ is the constant potential of the lower electrode. Similarly, on the upper electrode, we require
\[
[\phi]_{\Gamma_t} = \phi_t,
\]
where $\phi_t$ is the constant potential at the upper electrode. Finally, the following Neumann boundary condition is imposed along the lines of symmetry:
\[
\nabla \phi \cdot n = 0.
\]

These surface and wall boundary conditions along with the equations of motion given by equations (1)–(3) define the mathematical model. In the next section, we discuss the numerical method used for solving these sets of equations.
3. NUMERICAL PROCEDURE

The Fluid Flow Model

The core of the fluid flow model is solved using the NASA-VOF2D program which solves the fluid flow equations for the velocity and pressure directly with a Eulerian representation of the mesh. It discretizes both the space and time variables of the continuity and momentum equations utilizing finite differences. In this section, a qualitative description of the NASA-VOF2D program is presented. A more detailed examination of the program is provided in [19].

A computational mesh divides the region of interest into rectangular meshes. Each cell has sizes $\delta r_i$ for the $i$th column and $\delta z_j$ for the $j$th row, and the radial velocity components $u_{i-1/2,j}$ and $u_{i+1/2,j}$ are located on the left and right cell faces, respectively, and the axial velocity components $v_{i,j-1/2}$ and $v_{i,j+1/2}$ are located on the bottom and on the top cell faces, respectively. The pressure variable $P$ and the volume of fluid variable $F$ are both located at the centre of the cell. The positioning of the field variables in this manner simplifies boundary condition application and assists in the conservation of mass (1). The variable $F$ assigned to each cell indicates the fractional amount of fluid occupying that cell. A value of one for $F$ corresponds to a cell filled with fluid. A value of zero corresponds to a completely empty cell. A value between zero and one corresponds to a free surface.

The solution of the problem for advancing one time step $\delta t$ proceeds in the following manner. First, by employing the previous values of the velocity field and the pressure, we update the velocities by an explicit finite difference approximation of the momentum equations given by (2) and (3). The updated velocities do not satisfy the continuity equation (1). Therefore, to satisfy the continuity equation, the pressure in each cell is adjusted in an iterative procedure. This iteration is continued until the new velocity components satisfy the continuity equation within a defined convergence criterion. Finally, the volume of fluid function $F$ is updated and the entire procedure is repeated up to any desired interval of time. During each stage, suitable wall and interface boundary conditions must be imposed.

Let us now discuss the calculation of the surface tension effect that must be incorporated into the surface pressure $P_s$. The surface tension term in $P_s$ is given by

$$P_{st}^s = -\gamma H,$$

where the mean curvature $H$ is given by

$$H = \frac{1}{R_{rz}} + \frac{1}{R_{cyl}}.$$

$R_{rz}$ is the principal radius of curvature in the plane and $R_{cyl}$ is the principal radius of curvature associated with the azimuthal direction of the cylindrical coordinates. In order to determine $H$, we must know the exact orientation of the interface. This is done by introducing surface height functions $Z(r)$ and $R(z)$ based on the value of $F$ in the surface cell and its eight neighbours. For example, $Z(r)$ for the $(i, j)$ cell is given by

$$Z_i = Z(r_i) = F_{i,j-1} \delta z_{j-1} + F_{i,j} \delta z_j + F_{i,j+1} \delta z_{j+1}.$$

Therefore, the slope of the fluid interface at cell $(i,j)$ can be approximated by the expression

$$\left( \frac{dZ}{dr} \right)_{i,j} = \frac{1}{d_i} \left( \frac{(Z_{i+1} - Z_i) \delta r_{i-1/2}}{\delta r_{i-1/2}} + \frac{(Z_i - Z_{i-1}) \delta r_{i+1/2}}{\delta r_{i-1/2}} \right),$$

where

$$d_i = \delta r_{i-1/2} + \delta r_{i+1/2}, \quad \delta r_{i+1/2} = \frac{\delta r_i + \delta r_{i+1}}{2},$$
and so forth. By interchanging the roles of $r$ and $z$, a similar equation can be derived for $\frac{dR}{dz}$ at each cell $(i, j)$. If $|\frac{dR}{dz}|$ is smaller than $|\frac{dR}{dz}|$, the boundary is more nearly horizontal than vertical; otherwise it is more nearly vertical. The derivative with the smallest magnitude gives the best approximation of the slope because the corresponding $R$ and $Z$ approximations are more accurate in that case.

The surface tension force $f$ acting across the faces of the computational cell is given by

$$f = \gamma \int_C n \times dL,$$

where $dL$ is the differential element directed along a counterclockwise path on the fluid interface. Then, the surface pressure $P^s_{Rz}$ due to $R_{rz}$ will be

$$P^s_{Rz} = -\gamma \frac{\sin \beta_r + \sin \beta_l}{\delta r_i},$$

where $\beta_r$ is the angle that the normal of the fluid surface makes with the $z$-axis (counterclockwise angle) at the right-cell side and $\beta_l$ is the angle that the normal of the fluid surface makes with the $z$-axis at the left-cell side. The $\beta$ angles are determined from the relations

$$\tan \beta_r = \frac{AVFR - AVFCR}{\delta r_{i+1/2}},$$

and

$$\tan \beta_l = \frac{AVFCR - AVFL}{\delta r_{i-1/2}},$$

where $AVFR$, $AVFCR$, and $AVFL$ are average fluid height values in the right, centre, and left cells, respectively. Then, the $\sin \beta$ required for the surface pressure can be easily determined from the trigonometric identity

$$\sin \beta = \frac{\tan \beta}{\sqrt{1 + \tan^2 \beta}}.$$

The surface pressure contribution $P^s_{sc}$ of $R_{cyl}$ is

$$P^s_{sc} = -\gamma \frac{\cos \theta}{W_i}$$

for a nearly horizontal surface and

$$P^s_{sc} = -\gamma \frac{\cos \theta}{r_{cyl}}$$

for a nearly vertical surface. Here, $\theta$ is the angle between the interface tangent and the radial axis when the interface is mainly horizontal, and $\theta$ is the angle the interface tangent makes with the axial axis if the interface is mainly vertical. $W_i$ is the distance in the $r$-direction from the axis of symmetry to the centre of the cell $(i, j)$, and $r_{cyl}$ is given by $r_{cyl} = -r_{i-1/2} + F_{i,j} \delta r_i$ if the fluid is to the right of the interface and $r_{i+1/2} - F_{i,j} \delta r_i$ if fluid is to the left of the interface.

Therefore, the surface pressure $P^s_s$ due to the surface tension is the sum of the two contributions $P^s_{sc}$ and $P^s_{Rz}$.

$$P^s_s = P^s_{sc} + P^s_{Rz}.$$  

The total surface pressure $P_s$ is then given by

$$P_s = P^s_s + P^s_e,$$

where $P^s_e$ is the surface pressure due to the applied electric fields which are calculated in the next section.
4. THE ELECTROSTATIC MODEL

For the electrostatic field model, we must solve Laplace’s equation for the electric potential $\phi$ within the region of interest. In order to simplify the finite difference approximations near $r = 0$, we use the following change of variable $\sqrt{x} = r/2$ which transforms Laplace’s equation into

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial x} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (6)$$

We solve equation (6) using the finite difference techniques because it is easy to implement and interface with the fluid flow model which is also based on the finite difference method and we use the same rectangular mesh used for the fluid flow problem. The coordinates of the node points are denoted by the mesh $(i,j)$ whose distance from the neighbouring left, right, bottom, and top mesh points are labelled as $h_l$, $h_r$, $h_b$, and $h_t$, respectively. For surface cells, values of the $h$'s are calculated using the VOF variable $F$. The neighbouring mesh point for a surface cell is taken to be a point on the interface. For example, if cell $(i,j)$ is a surface cell where cell $(i,j - 1)$ is a fluid cell, then the $(i,j - 1)$ mesh point will be the point of intersection of the vertical line passing through $(i,j)$ with the interface as depicted in Figure 2. Accordingly, $h_b$ will be the distance between these points. For all the empty cells, the $h$'s are simply the spacing between the neighbouring cells. Because of the perfect conductivity assumption, the value of the potential for all surface cells and full cells is simply the lower electrode potential which is $\phi_t$.

![Figure 2. Definition of $h_b$ for a surface cell.](image)

If the cell $(i,j)$ is a surface cell or a cell on an electrode, then we call the node corresponding to this cell a *metal node*; otherwise, we call it a *nonmetal node*. Using this definition, we obtain the following finite difference approximation of equation (6):

$$a_{i,j} \phi_{i+1,j} + b_{i,j} \phi_{i-1,j} + c_{i,j+1} \phi_{i,j+1} + d_{i,j} \phi_{i,j-1} - e_{i,j} \phi_{i,j} = f_{i,j}, \quad (7)$$

where, for nonmetal nodes,

$$a_{i,j} = \frac{x_i}{h_r(h_r + h_l)} + \frac{1}{2(h_l + h_r)},$$
$$b_{i,j} = \frac{x_i}{h_l(h_r + h_l)} - \frac{1}{2(h_l + h_r)}.$$
If the cell \((i, j)\) or any of its neighbours are metal nodes, then the coefficients \(a_{i,j}\) to \(f_{i,j}\) must be modified. For example, if the bottom neighbour is a metal node, then we set

\[
d_{i,j} = 0,
\]

\[
f_{i,j} = -\frac{\phi_{i,j-1}}{h_b(h_t + h_b)},
\]

Since cell \((i, j - 1)\) is a metal node in this case, \(\phi_{i,j-1}\) is known and has a fixed value. Similarly, a metal node neighbour on the left will modify \(b_{i,j}\) and \(f_{i,j}\), and so on.

Equation (7) represents a linear system of equations which is solved by a standard SOR procedure.

The coupling of the electrostatic model and the fluid flow model occurs at the interface through the electric field \(\mathbf{E} = -\nabla \phi\), so the derivatives of the potential in both the \(r\) and \(z\) directions are required. We have used a third-order polynomial in \(r\) and \(z\) to describe locally the potential at any point. Thus, we can approximate the radial component \(E_r\) of \(\mathbf{E}\) by

\[
E_r(r) = \phi_{12} + \phi_{123}(2r - r_1 - r_2) + \phi_{1234}\left(3r^2 - 2r_1r - 2r_2r - 2r_3r + r_1r_2 + r_1r_3 + r_2r_3\right),
\]

where

\[
\phi_{12} = C(1),
\]

\[
\phi_{123} = \frac{C(2) - C(1)}{r_3 - r_1},
\]

\[
\phi_{1234} = \frac{1}{r_4 - r_1}\left(\frac{C(3) - C(2)}{r_4 - r_2} - \phi_{123}\right),
\]

\[
C(i) = \frac{\phi_{i+1} - \phi_i}{r_{i+1} - r_i},
\]

where \(\phi_i\) is the potential at \(r_i\). This expression can also be used for finding the vertical component of the filed \(E_z(z)\) by replacing the \(r\)'s with the \(z\)'s. If the electric field is required at a nonnodal location, then the potential is determined by interpolating the nearby potentials at the nodal points.

Experimentation has shown that the selection of the four points for the derivative is crucial. It is found that only one metal node must be used to obtain best results. More than one consecutive metal node tends to distort the value of the electric field near metal node boundaries. For example, in Figure 3, we use nodes 2, 3, 4, and 5 instead of 1, 2, 3, and 4.

Once \(E_r\) and \(E_z\) are calculated using the above procedure, we must compute the normal component of the electric field. The pressure discontinuity at the interface due to the electric field as given by equation (4) is

\[
P_s^e = -\epsilon_0 \frac{E_n^2}{2}.
\]

The normal unit vector to the interface can be described by its slope if the surface is nearly horizontal, or by its inverse slope if the surface is nearly vertical. For example, if the neighbouring
interpolation cell is below the surface cell \((i,j)\), then the normal electric field component for a nearly horizontal surface will be

\[ E_n = \frac{mE_r - E_z}{\sqrt{1 + m^2}}, \]

where \(m\) is the slope. We then substitute the computed \(E_n\) into equation (8) and enter it into equation (5) to obtain the total surface pressure \(P_z\).

At a reasonable distance away from the drop, the potential is assumed to be the uniform potential due to two flat parallel electrodes. Experiments showed that this condition could be applied at a distance approximately equal to three times the initial drop radius.

5. THE INITIAL EQUILIBRIUM SHAPE

Since we had some experimental data in the form of photographs taken of the breakup of a drop at different times, we tried to get an initial shape which was very close to the shape used in the experiments. However, it was difficult to measure this shape from a photograph to a reasonable accuracy, so instead we only measured the principal radius of curvature \(b\) at the top of the drop. This value was then used in the derivation of a differential equation for the drop at \(t = 0\).

The equilibrium shape of a drop which is in contact with a surrounding fluid is governed by the Young-Laplace equation. This is the basic equation of capillarity (see [21])

\[ \Delta P = \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right), \]

where \(\gamma\) is the interfacial surface tension. This equation relates the interfacial tension to the pressure difference between the fluids at each point along the interface. The pressure difference \(\Delta P\) may be obtained from the hydrostatic equation

\[ \Delta P = (\rho_1 - \rho_2) gz + \frac{2\gamma}{b}. \]

The principal radii of curvature for the axisymmetric droplet are given

\[ \frac{1}{R_1} = \frac{z''}{(1 + z'^2)^{1/2}}, \]
\[ \frac{1}{R_2} = \frac{z'}{r (1 + z'^2)^{1/2}}, \]

where prime denotes the differentiation with respect to \(r\). In the absence of gravity, the capillarity equation becomes

\[ \frac{z''}{(1 + z'^2)^{3/2}} + \frac{z'}{r (1 + z'^2)^{1/2}} = \frac{2}{b}. \]
This second-order ordinary differential equation which with the boundary conditions \( z(0) = 0 \) and \( z'(0) = 0 \) defines the equilibrium shape of the drop can easily be integrated to give the solution

\[
(c - z)^2 + r^2 = b^2.
\]

The constant \( c \) is determined by the contact angle \( \lambda \), which is generally a material property of the fluids and solids which are present.

6. NUMERICAL RESULTS

The numerical procedure outlined in the previous sections was applied to several different cases. The computations were carried out on a Cyber 962. A series of numerical experiments were carried out to determine the proper values of the different numerical parameters. These experiments showed that reasonably accurate solutions could be obtained if a time step of \( \delta t = 0.1 \) milliseconds and a 100 x 100 mesh for the computational region given by \( 0 \leq x \leq 7.5 \), \( 0 \leq r \leq 3 \times r_o \), where \( r_o \) is the initial drop radius, were used.

We will now present in detail results for the specific setup consisting of a drop with a volume of 3 cm\(^3\) placed on the lower electrode of a parallel electrode system which is kept at a potential difference of 30,000 volts with the electrodes 7.5 cm apart. These parameters are chosen for the purpose of making direct comparisons with the experimental data described below. The typical flow parameters for water at the ambient temperature are \( \rho = 1000 \) kg/m\(^3\), \( \nu = 9.7 \times 10^{-7} \) m\(^2\)/sec, and \( \gamma = 0.72 \times 10^{-2} \) N/m.

In this case, the contact angle is 82 degrees and the drop has an initial maximum horizontal extent of 2.76 cm and a maximum vertical height of 1.14 cm; the initial drop is shown in Figure 4. Figures 5 through 12 depict the various stages in the evolution of the drop from \( t = 0 \) to \( t = 54 \) milliseconds.

![Figure 4. Equilibrium drop profile at t = 0 ms.](image-url)

Once the electric field is applied, the drop elongates at the tip following the direction of the electric field and forms into a cone shape. This is depicted in Figure 5 where \( t = 25.6 \) milliseconds. Figure 6 shows the drop profile at 28.3 milliseconds where the first breakup is about to occur. At
this stage, the drop’s maximum vertical radius has elongated to 1.36 cm. The volume of the drop is conserved, however, since the contact line between the lower electrode and the drop surface is moving inward. The maximum horizontal diameter is 2.66 cm. Less than 0.3 milliseconds later, the first droplet breaks off, as shown in Figure 7, at $t = 28.5$ milliseconds. The emitted droplet occupies only two grid spaces and its estimated volume is approximately 0.3 cubic millimeters. As shown in Figure 8, the emitted droplet travels a distance of approximately 1 cm in less than 8 milliseconds. As the droplet approaches the upper electrode, it accelerates rapidly. Since the
model does not incorporate space charge and other phenomena associated with the breakup, the computations carried out after the first breakup may not be accurate. The second breakup occurs at 40 milliseconds, as shown in Figure 9. This is 11.5 milliseconds after the first breakup. Figure 10 illustrates the subsequent evolution at 51 milliseconds. The conical shape at the tip of the droplet in Figure 11 at $t = 52$ milliseconds indicates the imminent emission of a third droplet. This breakup occurs at 52.5 milliseconds which is 12.5 milliseconds after the second breakup. The profile of the drop after the third breakup is depicted in Figure 12 at $t = 54$ milliseconds. Here, the contact line has moved inward to a maximum horizontal diameter of 2.5 cm.
We have experimented with different mesh sizes, and the results indicate that for reasonably small mesh sizes, the size of the droplet that breaks away is determined solely by the electrical and fluid properties of the fluid and the magnitude of the external field.

7. COMPARISON OF THE MODEL WITH EXPERIMENTAL RESULTS

In 1991, microgravity experiments were carried out jointly by the Applied Electrostatic Research Centre of the University of Western Ontario and the Canadian Space Agency. The
microgravity conditions were produced with NASA KC-135 aircraft flights. The experimental apparatus consisted of a parallel electrode system, a high-speed camera, a charge/current measuring unit, and a data acquisition system. A detailed description of the apparatus and of the experimental methodology is reported in [18].

Large liquid drops were generated and placed between two parallel plane electrodes separated by a distance of 7.5 cm. The maximum potential difference between the electrodes was 30,000 volts. When the electric field was applied, the drop elongated, formed into a cone, and emitted small...
droplets. The evolution of each droplet was recorded with a high-speed camera at a rate of 2000 frames per second. The charge transfer from the sessile drop to the smaller, emitted droplets was recorded using a charge/current measurement unit.

Prior to the application of the electric field, the drop’s surface made a contact angle of 82 degrees with the lower electrode. After the electric field was applied, the contact line, which is the intersection between the drop surface and the lower electrode, moved continuously and the contact angle did not remain constant. The sessile drop then formed into a cone shape and emitted a droplet of 0.3 cm³ at t = 650 milliseconds. The data collected from the charge/current measurement unit shows that there were several intermittent charge losses well before the droplet was emitted. Once this initial breakup occurred, the pictures reveal that the flow was not axisymmetric.

Recall that our numerical results predicted that droplets would be emitted at t = 28.5, 40, and 52.5 milliseconds and that the droplets would be quite small, that is, 0.3 cubic millimeters. The critical breakup time observed during the experiment was considerably longer and the size of the emitted droplet was considerably larger than these predictions. The reason that the experiment was unable to record the existence of droplet emission before t = 650 milliseconds is that the droplets emitted prior to that time were too small to be detected. In reality, these ejected droplets are tiny dots travelling at high speed. This conclusion is supported by the data collected from the charge/current measurement unit. The intermittent charge fluctuations of the sessile droplet observed during the experiment can be attributed to the loss of charge as the small droplets were emitted prior to t = 650 milliseconds. Unfortunately, due to the lack of analyzed charge data, this conclusion cannot yet be quantitatively verified. Therefore, further experimental studies which employ more accurate charge/current instrumentation are suggested in order to evaluate and compare the observed charge losses at the breakup.

8. FUTURE IMPROVEMENTS AND CONSIDERATIONS

Both the experimental and numerical results suggest that further studies are required in a number of areas to successfully model the behaviour of the droplets.

It is clear that the dynamic behaviour of the contact angle and the contact line is an important factor in the deformation process. The dynamic contact line introduces additional modelling difficulties which arise because it describes the intersection of a solid with the interface of the fluids. In our problem, the zero shear-stress boundary condition applies at the air-water interface while the no-slip boundary condition applies at the water-solid interface. Both of these conditions must be satisfied at the contact line. Studies of dynamic contact lines show that this essentially leads to an infinite velocity gradient [22]. This apparent contradiction is due to the fact that, in the immediate neighbourhood of the contact line region, the continuum approach breaks down. Consequently, the molecular activities in that region such as absorption, relaxation, and reorientation become important [23]. Further experimental and numerical investigations are therefore recommended.

Since the experiments were carried out in microgravity conditions, the zero gravity assumption made in our model should be modified to incorporate the effects of residual accelerations present during the flight. The sources of these residual accelerations include the earth’s gravity gradient, the atmospheric drag on the spacecraft, and the spacecraft altitude motions arising from machine vibration and crew movement during the experiment. The effects of these accelerations are not yet completely understood. However, recent studies suggest that they may be important in experiments involving bubbles and drops [24]. Attempts have been made to model microgravity experiments involving bubbles using a sinusoidal function vibration of the gravity environment. However, further investigation is required in order to examine the effect of these residual accelerations on our experiment.
The experimental results show that after a droplet is emitted, the drop becomes asymmetric, and thus a correct model would require the full three-dimensional Navier-Stokes equations. At the moment, it is not really possible to solve successfully the three-dimensional problems with free surfaces. However, it is probable that a good quantitative agreement between numerical results cannot be obtained before the full three-dimensional flow is included in the model. This remains an interesting research problem for future study.

REFERENCES