Optimal ternary formally self-dual codes

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Dedicated to the Memory of Professor Edward F. Assmus Jr.

Abstract

In this paper, we study ternary optimal formally self-dual codes. Bounds for the highest minimum weight are given for length up to 30 and examples of optimal formally self-dual codes are constructed. For some lengths, we have found formally self-dual codes which have a higher minimum weight than any self-dual code. It is also shown that any optimal formally self-dual [10,5,5] code is related to the ternary Golay code of length 12. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

A ternary linear \([n,k]\) code \(C\) is a \(k\)-dimensional vector subspace of \(\mathbb{F}_3^n\), where \(\mathbb{F}_3\) is the field of 3 elements. We shall take the elements of \(\mathbb{F}_3\) to be either \(\{0,1,2\}\) or \(\{0,1,-1\}\), using whichever form is more convenient. The elements of \(C\) are called codewords and the weight \(wt(c)\) of a codeword \(c\) is the number of non-zero coordinates. An \([n,k,d]\) code is an \([n,k]\) code with minimum (non-zero) weight \(d\). The ambient space is equipped with the standard inner product, i.e. \([v,w] = \sum v_iw_i\). The dual code is \(C^\perp = \{v \in \mathbb{F}_3^n | [v,w] = 0 \text{ for all } w \in C\}\). The Hamming weight enumerator of a code \(C\) is given by \(W_C(x,y) = \sum A_ix^{n-i}y^i\) where there are \(A_i\) codewords of weight \(i\) in \(C\). Two codes are said to be equivalent if one can be obtained from the other by permuting and changing signs of coordinates.

We shall say that a code is formally self-dual if \(W_C(x,y) = W_{C^\perp}(x,y)\). Formally self-dual codes come in pairs, i.e. \(C\) and \(C^\perp\), and if \(C = C^\perp\) the code is self-dual.
Further, a formally self-dual code is optimal if the code has the highest minimum weight for that length.

In this paper, bounds for the highest minimum weight are given for length up to 30 and examples of optimal formally self-dual codes are constructed. For some lengths, we have found formally self-dual codes which have a higher minimum weight than any self-dual code. It is also shown that any optimal formally self-dual $[10,5,5]$ code is related to the ternary Golay code of length 12.

2. Preliminaries

The following three results are well known.

**Fact 2.1.** If $C$ is a code such that $C$ and $C^\perp$ are equivalent, then $C$ is formally self-dual.

**Fact 2.2.** If $C$ is a formally self-dual code which has all weights divisible by 3, then $C$ is self-dual.

Formally self-dual codes are divided into the following three classes:

(1) $C$ is self-dual,
(2) $C$ and $C^\perp$ are equivalent,
(3) $C$ and $C^\perp$ are not equivalent.

The second class is often called *isodual*.

The following is useful for eliminating possible weight enumerators with the highest minimum weight in Section 6.

**Fact 2.3.** Each nontrivial coefficient in the weight enumerator of a ternary code is even.

**Theorem 2.4** (MacWilliams et al. [9]). The weight enumerator of a formally self-dual code over a field $\mathbb{F}_q$ of order $q$ is a polynomial in $\phi_3$ and $\phi_4$ where

$$\phi_3 = x^2 + (q - 1)xy \quad \text{and} \quad \phi_4 = x^2 + (q - 1)y^2.$$  

Note also, as stated in [9], that $\phi_3$ corresponds to the code with generator matrix $(01)$ and $\phi_4$ corresponds to the code with generator matrix $(11)$.

3. Constructions

3.1. Double circulant codes

Here we describe some basic constructions of formally self-dual codes.
Proposition 3.1. Let $C$ be a ternary code with generator matrix $(I, A)$ where $I$ is the identity matrix. If there are monomial matrices $P$ and $Q$ over $\mathbb{F}_3$ such that $A^T = P \cdot A \cdot Q$ where $A^T$ denotes the transpose of $A$, then $C$ is a formally self-dual code.

Proof. Since $A^T = P \cdot A \cdot Q$, $(I, A)$ and $(I, A^T)$ generate equivalent codes. By Fact 2.1, $C$ is formally self-dual. \( \square \)

By the above proposition, codes with generator matrix $(I, A)$, where $A$ is a symmetric or skew-symmetric matrix, are a family of formally self-dual codes.

We now present generator matrices of double circulant codes. A pure double circulant code of length $2n$ has a generator matrix of the form

$$(I, R),$$

where $R$ is an $n$ by $n$ circulant matrix. A code with generator matrix of the form

$$
\begin{pmatrix}
\alpha & \beta & \cdots & \beta \\
\gamma & & & \\
\vdots & \ddots & \ddots & \\
\gamma & & & \alpha
\end{pmatrix},
$$

where $R'$ is an $n - 1$ by $n - 1$ circulant matrix, is called a bordered double circulant code of length $2n$. These two families of codes are collectively called double circulant codes.

Corollary 3.2. A double circulant code is formally self-dual.

Double circulant codes are used to construct optimal formally self-dual codes. These codes are a remarkable class of formally self-dual codes.

We now give a classification of optimal pure double circulant formally self-dual codes of length up to 14 (see Section 6 for the highest minimum weight). We shall show that any optimal formally self-dual [4,2,3] code is self-dual in Section 6.

There are three distinct pure double circulant [6,3,3] codes, with first rows listed in Table 1. It is easy to show that $P_{6,1}$ and $P_{6,3}$ are equivalent. In addition, $P_{6,1}$ and $P_{6,2}$ have distinct weight enumerators $W_{6,2}$ and $W_{6,1}$, respectively, as given in Section 6.

For lengths 8 to 14, we complete the classification of optimal pure double circulant codes by listing the first rows of all inequivalent codes in Table 2 where $A_i$ denotes

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Pure double circulant formally self-dual codes of length 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Code</td>
<td>First row</td>
</tr>
<tr>
<td>$P_{6,1}$</td>
<td>110</td>
</tr>
</tbody>
</table>
Table 2
Optimal pure double circulant formally self-dual codes of lengths 8 to 14

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Code</th>
<th>First row</th>
<th>$A_0$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
<th>$A_7$</th>
<th>$A_8$</th>
<th>$A_{10}$</th>
<th>$A_{11}$</th>
<th>$A_{12}$</th>
<th>$A_{13}$</th>
<th>$A_{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[8,4,4]</td>
<td>$P_{51}$</td>
<td>2110 1</td>
<td>20</td>
<td>32</td>
<td>8</td>
<td>16</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$P_{52}$</td>
<td>1110 1</td>
<td>22</td>
<td>24</td>
<td>20</td>
<td>8</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$P_{53}$</td>
<td>2111 1</td>
<td>24</td>
<td>16</td>
<td>32</td>
<td>0</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[10,5,5]</td>
<td>$P_{10,1}$</td>
<td>12210 1</td>
<td>1</td>
<td>0</td>
<td>72</td>
<td>60</td>
<td>0</td>
<td>90</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[14,7,6]</td>
<td>$P_{14}$</td>
<td>1121100 1</td>
<td>0</td>
<td>0</td>
<td>182</td>
<td>156</td>
<td>364</td>
<td>364</td>
<td>546</td>
<td>364</td>
<td>182</td>
<td>0</td>
<td>28</td>
</tr>
</tbody>
</table>

the number of codewords of weight $i$. Note that there is no optimal double circulant formally self-dual [12,6,6] code which is not self-dual.

**Proposition 3.3.** All optimal pure double circulant formally self-dual codes are classified for length up to 14.

For larger lengths, we shall give optimal double circulant codes in Section 6.

3.2. Codes from weighing matrices and Hadamard matrices

Here we describe a construction of ternary formally self-dual codes using weighing matrices. A weighing matrix $W(n,k)$ of order $n$ and weight $k$ is an $n$ by $n$ $(0,1,-1)$-matrix such that $W(n,k) \cdot W(n,k)^\top = kI_n$, $k \leq n$. A weighing matrix $W(n,n)$ is also called a Hadamard matrix of order $n$. Two weighing matrices $W_1$ and $W_2$ of order $n$ and weight $k$ are equivalent if there exist monomial matrices $P$ and $Q$ of $0$'s, $1$'s and $-1$'s such that $W_1 = P \cdot W_2 \cdot Q$. Here we say that a weighing matrix $W$ is **self-dual** if $W$ is equivalent to $W^\top$.

3.2.1. Codes from weighing matrices

**Corollary 3.4.** Let $W$ be a self-dual weighing matrix of order $n$ and weight $k$. Then the matrix $(I,W)$ generates a formally self-dual code $C(W)$ of length $2n$. Moreover if $k - 1$ is divisible by three then the code is self-dual.

**Proof.** Since $W^\top$ is equivalent to $W$, there are monomial matrices $P$ and $Q$ such that $W = P \cdot W^\top \cdot Q$. □

**Remark.** If there is a unique weighing matrix $W$ of order $n$ and weight $k$ for given $n$ and $k$, then $W$ must be self-dual. Thus the code constructed from $W$ is formally self-dual.
**Lemma 3.5.** Let W and W' be two equivalent weighing matrices of order n and weight k. Then the codes constructed from W and W' are equivalent.

**Proof.** Since W is equivalent to W', $W' = P \cdot W \cdot Q$, where $P$ and $Q$ are $n \times n$ monomial matrices of 0's, 1's and -1's. Thus, we have

$$(I, W') = (I, P \cdot W \cdot Q) = P(I, W)R,$$

where

$$R = \begin{pmatrix} P^{-1} & 0 \\ 0 & Q \end{pmatrix}$$

is a $2n \times 2n$ monomial matrix. Here $O$ denotes the $n \times n$ zero matrix. Therefore the two codes are equivalent. \(\square\)

**Lemma 3.6** (Chan et al. [1]). Any $W(n,2)$ is equivalent to $\bigoplus_{n/2} W_{2,2}$ and any $W(n,3)$ is equivalent to $\bigoplus_{n/4} W_{4,3}$, where

$$W_{2,2} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad W_{4,3} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix},$$

and \(-1\) is denoted by \(-\) in the above matrices.

**Remark.** The code constructed from $W_{2,2}$ is self-dual.

We consider the code $C(W_{4,3})$ with generator matrix $(I, W_{4,3})$. The weight enumerator of $C(W_{4,3})$ is

$$W_5 = 1 + 8y^3 + 8y^4 + 24y^5 + 24y^6 + 16y^7.$$ 

Thus, we have the characterization of formally self-dual codes constructed from weighing matrices of weight 3.

**Proposition 3.7.** Let C be a code with generator matrix $(I, W)$ where W is a weighing matrix of order n and weight 3. Then C is a formally self-dual code of length $2n$ with weight enumerator $W_5^{n/4}$.

**Proof.** By Lemmas 3.5 and 3.6, C is a formally self-dual code which is equivalent to the code obtained by a direct sum of $C(W_{4,3})$. \(\square\)

Now, let us investigate weighing matrices of weight 4.

**Lemma 3.8** (Chan et al. [1]). Any $W(n,4)$ is equivalent to

$$W(4,4) \oplus B(8,4) \oplus W(7,4) \oplus C(6 + 2i,4),$$

where the matrices $W(4,4), B(8,4), W(7,4)$ and $C(6 + 2i,4)$ are given in [1].
For weight $\leq 3$, there is a unique matrix for each order. For weight 4 there are inequivalent matrices for some orders [1].

**Corollary 3.9.** Let $C$ be a code with generator matrix $(I, W)$ where $W$ is a weighing matrix of order $n$ and weight 4. Then $C$ is a formally self-dual code of length $2n$.

**Proof.** All of the matrices $W(4,4), B(8,4), W(7,4)$ and $C(6+2i,4)$ are symmetric (see [1]). By Fact 3.1, $C$ is formally self-dual. □

We study formally self-dual codes from weighing matrices of weight 4 for order up to 10. If $w(n)$ is the number of inequivalent weighing matrices of order $n$ and weight 4 then

$$
w(4) = 1, \ w(5) = 0, \ w(6) = 1, \ w(7) = 1, \ w(8) = 3, \ w(9) = 0 \text{ and } w(10) = 2,
$$

(cf. [1]). We obtained the weight enumerators of codes constructed from weighing matrices of order up to 10. We denote the weight enumerator of $C(M)$ by $W_M(y)$:

$$
W_{W(4,4)}(y) = 1 + 24y^4 + 16y^5 + 32y^6 + 8y^8,
$$

$$
W_{C(6,4)}(y) = 1 + 12y^4 + 24y^5 + 112y^6 + 96y^7 + 228y^8 + 96y^9 + 144y^{10} + 16y^{12},
$$

$$
W_{W(7,4)}(y) = 1 + 28y^5 + 112y^6 + 168y^7 + 434y^8 + 336y^9 + 588y^{10} + 224y^{11} + 280y^{12} + 16y^{14},
$$

$$
W_{W(8,4)@W(4,4)}(y) = 1 + 48y^4 + 64y^5 + 592y^6 + 768y^7 + 1792y^8 + 1024y^{11} + 1408y^{12} + 256y^{13} + 512y^{14} + 64y^{16},
$$

$$
W_{C(8,4)}(y) = 1 + 16y^4 + 32y^5 + 64y^6 + 128y^7 + 592y^8 + 768y^9 + 1600y^{10} + 1024y^{11} + 1408y^{12} + 384y^{13} + 512y^{14} + 32y^{16},
$$

$$
W_{B(8,4)}(y) = 1 + 32y^5 + 64y^6 + 192y^7 + 592y^8 + 768y^9 + 1504y^{10} + 1024y^{11} + 1408y^{12} + 448y^{13} + 512y^{14} + 16y^{16},
$$

$$
W_{W(6,4)@W(4,4)}(y) = 1 + 36y^4 + 40y^5 + 144y^6 + 96y^7 + 524y^8 + 864y^9 + 3600y^{10} + 4864y^{11} + 10704y^{12} + 9216y^{13} + 13184y^{14} + 6144y^{15} + 6816y^{16} + 1024y^{17} + 1664y^{18} + 128y^{20},
$$

$$
W_{C(10,4)}(y) = 1 + 20y^4 + 40y^5 + 80y^6 + 160y^7 + 460y^8 + 1120y^9 + 3504y^{10} + 5120y^{11} + 10320y^{12} + 9280y^{13} + 12800y^{14} + 6400y^{15} + 6800y^{16} + 1280y^{17} + 1600y^{18} + 64y^{20}.
$$
The weight enumerators yield the classification of formally self-dual codes constructed from weighing matrices of weight 4 and order up to 10. \( C(W(4,4)) \) is an optimal code of length 8 (cf. Section 6).

3.2.2. Codes from Hadamard matrices

We now consider formally self-dual codes constructed from Hadamard matrices. There is a unique Hadamard matrix of order up to 12. A Hadamard matrix of order 4 is the unique weighing matrix of order 4 and weight 4. The weight enumerators of the formally self-dual codes constructed from Hadamard matrices of orders 8 and 12 are

\[
1 + 224y^6 + 2720y^9 + 3360y^{12} + 256y^{15}
\]

and

\[
1 + 264y^6 + 264y^8 + 440y^9 + 3960y^{10} + 7920y^{11} + 24752y^{12} + 38832y^{13} + 63360y^{14} + 73920y^{15} + 88704y^{16} + 85272y^{17} + 71808y^{18} + 42768y^{19} + 19800y^{20} + 6160y^{21} + 2640y^{22} + 288y^{23} + 288y^{24},
\]

respectively.

There are exactly five inequivalent Hadamard matrices of order 16, three of which are self-dual [5]. We denote the three self-dual matrices by \( H_{16,1}, H_{16,2} \) and \( H_{16,3} \) and the remaining two matrices by \( H_{16,4} \) and \( H_{16,5} \) where \( H_{16,5} = H_{16,4}^t \). By Proposition 3.1, \( C(H_{16,i}) \) is formally self-dual for \( i = 1, 2 \) and 3. The weight enumerators \( W_{C(H_{16,1})}(y) \), \( W_{C(H_{16,2})}(y) \) and \( W_{C(H_{16,3})}(y) \) of the three codes are

\[
W_{C(H_{16,1})}(y) = 1 + 1120y^8 + 960y^{10} + 27776y^{12} + 53760y^{13} + 197120y^{14} + 439040y^{15} + 962592y^{16} + 1630784y^{17} + 2865920y^{18} + 4139520y^{19} + 5742016y^{20} + 6157312y^{21} + 6448128y^{22} + 5168640y^{23} + 4307200y^{24} + \cdots,
\]

\[
W_{C(H_{16,2})}(y) = 1 + 608y^8 + 960y^{10} + 4096y^{11} + 27776y^{12} + 53760y^{13} + 182784y^{14} + 439040y^{15} + 962592y^{16} + 1659456y^{17} + 2865920y^{18} + 4139520y^{19} + 5706176y^{20} + 6157312y^{21} + 6448128y^{22} + 5197312y^{23} + \cdots,
\]

\[
W_{C(H_{16,3})}(y) = 1 + 352y^8 + 960y^{10} + 6144y^{11} + 27776y^{12} + 53760y^{13} + 175616y^{14} + 439040y^{15} + 962592y^{16} + 1673792y^{17} + 2865920y^{18} + 4139520y^{19} + 5688256y^{20} + 6157312y^{21} + 6448128y^{22} + 5211648y^{23} + \cdots.
\]

Moreover, we checked by computer that \( C(H_{16,4}) \) is formally self-dual. Since \( C(H_{16,5}) \) is equivalent to the dual code of \( C(H_{16,4}) \), \( C(H_{16,5}) \) is also formally self-dual.
They have identical weight enumerators:

\[ 1 + 224y^8 + 960y^{10} + 7168y^{11} + 27776y^{12} + 53760y^{13} \\
+ 172032y^{14} + 439040y^{15} + 962592y^{16} + 1680960y^{17} + 2865920y^{18} \\
+ 4139520y^{19} + 5679296y^{20} + 6157312y^{21} + 6448128y^{22} \\
+ 5218816y^{23} + \cdots. \]

The next order is 20, and there are exactly three inequivalent Hadamard matrices [6]. In this case, the codes are self-dual and it was shown in [7] that there are exactly three inequivalent self-dual codes constructed from the three Hadamard matrices.

4. Shadow codes

4.1. Shadow construction

Let \( C \) be a self-dual code over a finite field \( F \) and \( C_0 \) any subcode of codimension 1 in \( C \). Let \( t \) and \( s \) be vectors such that \( C = \langle C_0, t \rangle \) and \( C_0^\perp = \langle C, s \rangle \). These vectors can be chosen so that \( [s, t] = 1 \), that is if \( [s, t] = \eta \) then replace \( t \) with \( \eta^{-1}t \).

Define

\[ C_{x,\beta} = C_0 + \alpha t + \beta s \]

for \( \alpha, \beta \in F \), so that \( C_0^\perp = \bigcup C_{x,\beta} \). Let

\[ D_{x,\beta} = (v_{x,\beta}, C_{x,\beta}) \]

i.e. to the beginning of each codeword in \( C_{x,\beta} \) concatenate the vector \( v_{x,\beta} = v_{1,0} + \beta v_{0,1} \) so that \( D \) is linear.

Let

\[ E_{x,\beta} = (w_{x,\beta}, C_{x,\beta}) \]

and as before to make \( E = \bigcup E_{x,\beta} \) linear we specify \( w_{1,0} \) and \( w_{0,1} \) and then the vector \( w_{x,\beta} = \alpha v_{1,0} + \beta w_{0,1} \). The vectors \( v_{x,\beta} \) and \( w_{x',\beta'} \) are chosen so that \( wt(v_{x,\beta}) = wt(w_{x',\beta'}) \)

if \( \alpha = \alpha' \) and \( \beta = \beta' \) and \( [v_{x,\beta}, w_{x',\beta'}] = -[C_{x,\beta}, C_{x',\beta'}] \), where \( [C_{x,\beta}, C_{x',\beta'}] \) is the inner-product of any two vectors in these cosets.

**Theorem 4.1.** If \( D \) and \( E \) can be constructed as above, with the length of \( v_{x,\beta} \) and \( w_{x',\beta'} \) equal to 2, then \( D \) and \( E \) are duals of each other and are formally self-dual.

**Remark.** Unlike the binary case there is not a single shadow, but instead \( |F| - 1 \) shadows of the code, i.e. the cosets of \( C \) in \( C_0^\perp \). For a complete discussion see [3]. In [4], this technique is shown for formally self-dual binary codes.
Given this construction we shall say that $D$ and $E$ are formed by the shadow construction. If $C$ is a ternary code and $s$ is a vector not in $C$ with $[s,s]=\gamma$ then we need

$$[v_{1,0},w_{1,0}] = 0, \quad [v_{0,1},w_{0,1}] = -\gamma, \quad [v_{1,0},w_{0,1}] = [v_{0,1},w_{1,0}] = 2.$$ 

This can be achieved by choosing the following vectors for a given $\gamma$:

- if $\gamma = 0$ then choose $v_{1,0} = (11)$, $v_{0,1} = (01)$, $w_{1,0} = (12)$, and $w_{0,1} = (20)$,
- if $\gamma = 1$ then choose $v_{1,0} = (11)$, $v_{0,1} = (01)$, $w_{1,0} = (12)$, and $w_{0,1} = (02)$,
- if $\gamma = 2$ then choose $v_{1,0} = (11)$, $v_{0,1} = (20)$, $w_{1,0} = (12)$, and $w_{0,1} = (20)$.

Given a generator matrix $M_0$ of the code $C_0$ and with vectors $s$ and $t$ described above, a generator matrix for the code $D$ is given by (for $\gamma = 0, 1$ and $2$, respectively):

$$\begin{pmatrix}
00 \\
00 \\
11 \\
01 \\
\vdots & \vdots & \vdots & \vdots \\
M_0 \\
00 \\
11 \\
01 \\
\vdots & \vdots & \vdots & \vdots \\
00 \\
\end{pmatrix}, \begin{pmatrix}
00 \\
00 \\
11 \\
01 \\
\vdots & \vdots & \vdots & \vdots \\
M_0 \\
00 \\
11 \\
01 \\
\vdots & \vdots & \vdots & \vdots \\
00 \\
\end{pmatrix}, \begin{pmatrix}
00 \\
00 \\
11 \\
01 \\
\vdots & \vdots & \vdots & \vdots \\
M_0 \\
00 \\
11 \\
01 \\
\vdots & \vdots & \vdots & \vdots \\
00 \\
\end{pmatrix}.$$

In particular, note that $[s,t]=1$ and $t$ is orthogonal to every row of $M_0$. The matrix

$$\begin{pmatrix}
M_0 \\
t \\
\end{pmatrix},$$

generates the self-dual code $D$.

4.2. A ternary shadow

We now describe a non-linear code that resembles the shadow of a binary code.

**Lemma 4.2.** Let $C$ be a ternary code of length $n$ such that $|C|=|C^\perp|$ and the weights of all codewords of $C$ are congruent to either 0 or 2 (mod 3) such that the number of codewords that have weights $\equiv 0$ (mod 3) is $|C|/3$. Then the subcode $C_0$ generated by self-orthogonal codewords is codimension 1 in $C$ and the codewords that are not self-orthogonal are contained in $C_0^\perp$.

**Proof.** Either $C_0$ is all of $C$ or it is codimension 1. Assume there are two self-orthogonal codewords $v,w$ such that $v+w$ is not self-orthogonal. We know $[v+w,v+w] \equiv 2$ (mod 3) since there are no weights that are congruent to 1 (mod 3).

From

$$[v+w,v+w] = [v,v] + [w,w] + 2[v,w] = 2[v,w],$$

we conclude that $v+w$ is not self-orthogonal.
it follows that $[v, w] = 1$ and hence we have

$$[v + 2w, v + 2w] = [v, v] + [2w, 2w] + 2[v, 2w] = [v, w] = 1,$$

which is a contradiction. Therefore $C_0$ is codimension 1 in $C$.

Let $w$ be a codeword in $C$ that is not self-orthogonal. If there exists a self-orthogonal codeword $v$ in $C$ with $[t, v] \neq 0$ then without loss of generality we may assume that $[t, v] = 1$, (otherwise replace $v$ with $2v$). Then it must be that

$$[t + v, t + v] = [t, t] + 2[v, t] + [v, v] = 1$$

which is a contradiction. Therefore $C \subset C_0^\perp$. □

Let $C$ be as described in the previous lemma. Then since $C \subset C_0^\perp$ we have

$$C = \langle C_0, t \rangle$$

and $C_0^\perp = \langle C, s \rangle$ for some vectors $s$ and $t$. The weight enumerator of $C_0$ is easily determined from the weight enumerator of $C$, i.e.

$$W_{C_0}(x, y) = (\frac{1}{3})(W_C(x, y) + W_C(x, \xi y) + W_C(x, \xi^2 y)),$$

where $\xi$ is a complex third root of unity. Again it is easy to compute $W_{C_0^\perp}$ using the MacWilliams relations, and finally to determine the weight enumerator of the non-linear code $C_0^\perp - C$ which we shall call the ternary shadow.

The weight enumerator of the ternary shadow is given by

$$W_S(x, y) = W_C \left( \frac{x + 2y}{\sqrt{3}}, \frac{\xi(x - y)}{\sqrt{3}} \right) + W_C \left( \frac{x + 2y}{\sqrt{3}}, \frac{\xi^2(x - y)}{\sqrt{3}} \right).$$

Note that unlike the binary case the shadow is not defined for all self-dual ternary codes, but only for those with no codewords of weight congruent to 1 (mod 3).

Let $C$ be a self-dual ternary code such that the codewords of each weight contain a 2-design. The weight enumerator of the code $C'$ formed by subtracting (i.e. taking all codewords beginning with 00, 10, 20 and deleting these coordinates) is easily computed. Notice that the code $C'$ satisfies the conditions of the first lemma. Given a weight enumerator for a putative code which would hold a 2-design one could compute these weight enumerators and make sure that all coefficients are non-negative integers. This computation was done on all open cases of extremal ternary self-dual codes, i.e. those with minimum weight equal to $3\lfloor n/12 \rfloor + 3$, which hold 2-designs. However, none of these produce a shadow with an inadmissible weight enumerator.

5. A formally self-dual code related to the Golay code

In this section, we show that any optimal formally self-dual code of length 10 is related to the Golay $[12,6,6]$ code. We also classify all optimal formally self-dual codes of length 10.
5.1. Classification of optimal formally self-dual codes of length 10

If C is an optimal formally self-dual [10,5,5] code then we shall show in Section 6 that it must have weight enumerator

\[ 1 + 72y^5 + 60y^6 + 90y^8 + 20y^9. \]

Let \( C_0 \) be the subcode generated by the self-orthogonal codewords. By Lemma 4.2, \( C_0 \) is codimension 1 in \( C \) and \( C \subseteq C_0^\perp \). As before we have \( C = \langle C_0, t \rangle \) and \( C_0^\perp = \langle C, s \rangle \) for some vectors \( t \) and \( s \). Of course \( [t, t] = 2 \) since the only other weights in \( C \) are 5 and 8. Let \( C_{a,b} = C_0 + at + bs \) and adjoin an initial vector of length 2, \( v_{a,b} \) with \( v_{a,b} = av_{1,0} + bv_{0,1} \). Taking \( v_{1,0} = (10) \), we can assume that \( [s, t] = 1 \). Then if \( [s, s] = \gamma \), \( \gamma \) cannot be 0 since \( \langle C_0, s \rangle \) is not self-dual. Now, take

\[
\begin{cases}
  v_{0,1} = (21) & \text{if } \gamma = 1, \\
  v_{0,1} = (20) & \text{if } \gamma = 2.
\end{cases}
\]

The weight enumerator of \( C_0^\perp \) is

\[ 1 + 60y^4 + 144y^5 + 60y^6 + 240y^7 + 180y^8 + 20y^9 + 24y^{10}, \]

and the weight enumerator of the ternary shadow is

\[ 60y^4 + 72y^5 + 240y^7 + 90y^8 + 24y^{10}. \]

Define

\[ C_{x,\beta} = C_0 + \alpha t + \beta s \]

for \( \alpha, \beta \in \mathbb{F}_3 \) so that \( C_0^\perp = \bigcup C_{x,\beta} \). The code

\[ D_{x,\beta} = (v_{x,\beta}, C_{x,\beta}) \]

has weight enumerator

\[
1 + y^2 \times 60y^4 + y \times 144y^5 + 60y^6 + y^2 \times 240y^7 + y \times 180y^8 \\
+ 20y^9 + y^2 \times 24y^{10}, \\
= 1 + 264y^6 + 440y^9 + 24y^{12},
\]

which is the weight enumerator of the Golay [12,6,6] code.

Therefore, we have the following lemma.

**Lemma 5.1.** Any optimal formally self-dual [10,5,5] code can be extended to the Golay [12,6,6] code.

Now, we consider the converse assertions of the above lemma. In [2], the *subtracting method* was defined in order to construct a self-dual code of length \( n - n' \) from two self-dual codes of lengths \( n \) and \( n' \). Here this method is used to construct formally
self-dual codes. An \([n - 2, n/2 - 1]\) code \(D\) formed by subtracting a \([2,1]\) code \(C_2\) with generator matrix \((01)\) from a self-dual \([n, n/2]\) code \(C\), consists all vectors \(v \in \mathbb{F}_q^{n-2}\) such that \((u, v) \in C\) for some \(u \in C_2\).

**Lemma 5.2.** Let \(C_{10}\) be a code of length 10 formed by subtracting from the Golay \([12,6,6]\) code \(G_{12}\). Then \(C_{10}\) is an optimal formally self-dual \([10,5,5]\) code. Moreover, all formally self-dual \([10,5,5]\) codes constructed from the Golay code by subtracting all pairs of coordinates are equivalent.

**Proof.** First \(C_{10}\) is the same as the code formed from the codewords in \(G_{12}\) with \((00),(10)\) or \((20)\) as the first two coordinates by deleting these coordinates. Thus \(C_{10}\) is a linear \([10,5]\) code. By the Assmus-Mattson theorem, the supports of codewords of weight 6 (resp. 9) in \(G_{12}\) form a \(2-(12,6,30)\) design (resp. \(2-(12,9,120)\) design). Therefore \(C_{10}\) has weight enumerator \(1 + 72y^5 + 60y^6 + 90y^8 + 20y^9\), and so \(C_{10}\) is formally self-dual.

The second assertion follows from the fact that \(\text{Aut}(G_{12})/\{\pm I\}\) is the Mathieu group \(M_{12}\) where \(\text{Aut}(G_{12})\) is the automorphism group of \(G_{12}\). \(\square\)

By Lemmas 5.1 and 5.2, we have the classification of all optimal formally self-dual codes of length 10.

**Theorem 5.3.** All formally self-dual \([10,5,5]\) codes are equivalent.

5.2. Other lengths

We now apply this idea to codes of lengths 22 and 24. There are exactly two inequivalent extremal self-dual \([24,12,9]\) codes, namely the Pless symmetry code \(P_{24}\) and the extended quadratic residue code \(Q_{24}\) [8]. By the Assmus-Mattson theorem, the supports of the codewords of weights 9, 12, 15 and 18 in \(Q_{24}\) and \(P_{24}\) forms a \(5\)-design. It is known that the support of the codewords of weight 21 in \(Q_{24}\) also forms a \(2-(24,21,9240)\) design. The weight enumerator of the code \(C_{22}\) obtained by subtracting from \(Q_{24}\) is

\[
1 + 990y^8 + 1540y^9 + 16128y^{11} + 14784y^{12} + 59400y^{14} + 31680y^{15} + 38808y^{17} + 10780y^{18} + 2772y^{20} + 264y^{21}.
\]

Hence \(C_{22}\) is an optimal formally self-dual code of length 22. The weight enumerator of its ternary shadow is:

\[
528y^7 + 990y^8 + 14784y^{10} + 16128y^{11} + 92400y^{13} + 59400y^{14} + 109956y^{16} + 38808y^{17} + 18480y^{19} + 2772y^{20} + 48y^{22}.
\]
Table 3
The highest minimum Hamming weights

<table>
<thead>
<tr>
<th>Length $n$</th>
<th>$d_F(n)$</th>
<th>$N(n)$</th>
<th>$d_L(n)$</th>
<th>$d_S(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>1,</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1, $E_4$ in [10]</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>$\geq 3$, $P_{6,1}, P_{6,2}, B_6$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>$\geq 3$, $P_{8,1}, P_{8,2}, B_8$</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>1, Section 5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>1, the Golay code</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>$\geq 1$, $P_{14}$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>6</td>
<td>$\geq 12$, $P_{6,1},\ldots,P_{6,11}, 2f_8$ in [2]</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>18</td>
<td>6</td>
<td>$\geq 52$, $P_{18,1},\ldots,P_{18,52}$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>7</td>
<td>$\geq 8$, $P_{20,1},\ldots,P_{20,8}$</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>22</td>
<td>8</td>
<td>$\geq 1$, $P_{22}, C_{22}$</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>9</td>
<td>$\geq 2$, [8]</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>26</td>
<td>8 or 9</td>
<td>$\geq 9$, $P_{26}$ ($d = 8$)</td>
<td>8 or 9</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>9 or 10</td>
<td>$\geq 9$, $P_{28}$ ($d = 9$)</td>
<td>9 or 10</td>
<td>9</td>
</tr>
<tr>
<td>30</td>
<td>9, 10 or 11</td>
<td>$\geq 9$, $P_{30}$ ($d = 9$)</td>
<td>9, 10 or 11</td>
<td></td>
</tr>
</tbody>
</table>

6. Optimal formally self-dual codes

The highest possible minimum weight can be obtained by Fact 2.3 and Theorem 2.4. Upper bounds for minimum weights of ternary linear codes were used for some lengths. By constructing formally self-dual codes with the desired minimum weight, we determined the (exact) highest minimum weight $d_F(n)$ for length up to 30, where $d_F(n)$ is listed in Table 3. In this table the third column gives the number $N(n)$ of known inequivalent optimal formally self-dual codes together with optimal codes. The fourth (resp. fifth) column gives the highest minimum weight $d_L(n)$ (resp. $d_S(n)$) among all linear $[n,n/2]$ codes (resp. self-dual codes of length $n$). We also give possible weight enumerators $W_n$ with minimum weight $d_F(n)$ for length $n$ and some optimal formally self-dual codes for $n \leq 30$. Note that we list only the first few terms in the possible weight enumerators for large lengths.

6.1. Possible weight enumerators

- $n = 2$: $W_2 = 1 + 2y^2$. Any code is formally self-dual, the code with generator matrix (12) is a unique formally self-dual code with this weight enumerator. By the Assmus-Mattson theorem, codewords of a fixed weight hold a $1$-design.
- $n = 4$: $W_4 = 1 + 8y^3$. By Fact 2.2, any formally self-dual code with $W_4$ must be self-dual. The code $E_4$ given in [10] has this weight enumerator.
- $n = 6$: By Theorem 2.4, we obtain the following possible weight enumerators for $d_F(6) = 3$:

$$W_{6,1} = 1 + 8y^3 + 6y^4 + 12y^5,$$
$$W_{6,2} = 1 + 6y^3 + 12y^4 + 6y^5 + 2y^6,$$
\[ W_{6,3} = 1 + 4y^3 + 18y^4 + 4y^6, \]
\[ W_{6,4} = 1 + 7y^3 + 9y^4 + 9y^5 + y^6. \]

However, Fact 2.3 eliminates \( W_{6,4} \), so there are three possible weight enumerators. \( P_{6,1} \) and \( P_{6,2} \) have weight enumerators \( W_{6,2} \) and \( W_{6,1} \), respectively.

Let \( B_6 \) be the code with generator matrix
\[
\begin{pmatrix}
100 & 011 \\
010 & 121 \\
001 & 112
\end{pmatrix},
\]

\( B_6 \) is a formally self-dual code with weight enumerator \( W_{6,3} \). By the Assmus-Mattson theorem, codewords of a fixed weight in a code with \( W_{6,3} \) hold a 1-design.

\( \bullet \) \( n = 8 \): There are three possible weight enumerators for \( d_F(8) = 4 \):
\[ W_{8,1} = 1 + 20y^4 + 32y^5 + 8y^6 + 16y^7 + 4y^8, \]
\[ W_{8,2} = 1 + 22y^4 + 24y^5 + 20y^6 + 8y^7 + 6y^8, \]
\[ W_{8,3} = 1 + 24y^4 + 16y^5 + 32y^6 + 8y^8. \]

Note that the highest attainable minimum weight for self-dual codes of length 8 is 3. \( P_{8,1} \), \( P_{8,2} \) and \( P_{8,3} \) have weight enumerators \( W_{8,2} \), \( W_{8,1} \) and \( W_{8,3} \), respectively. \( C(W(4,4)) \) in Section 3 also has weight enumerator \( W_{8,3} \). Of course, \( C(W(4,4)) \) and \( P_{8,3} \) are equivalent. Thus, there are formally self-dual codes which have a higher minimum weight than any self-dual code of length 8. By the Assmus-Mattson theorem, the support of the codewords of a fixed weight in a code with \( W_{8,3} \) holds a 1-design.

\( \bullet \) \( n = 10 \): \( W_{10} = 1 + 72y^5 + 60y^6 + 90y^7 + 20y^9 \). There is a unique optimal formally self-dual \([10,5,5]\) code, up to equivalence (cf. Theorem 5.3). By the Assmus-Mattson theorem, the support of the codewords of a fixed weight in a code holds a 3-design.

This also follows from Lemma 5.2 and Theorem 5.3.

\( \bullet \) \( n = 12 \): \( W_{12} = 1 + 264y^6 + 440y^9 + 24y^{12} \). By Fact 2.2, a formally self-dual code with \( W_{12} \) must be self-dual. The Golay code is a unique self-dual code with this weight enumerator. Thus there is no optimal formally self-dual code of length 12 which is not self-dual.

\( \bullet \) \( n = 14 \):
\[
W_{14}(\alpha, \beta) = 1 + (252 + 64\beta + 448\alpha)y^6 + (-384\beta - 2560\alpha - 224)y^7
+ (6720\alpha + 1274 + 1088\beta)y^8 + (-1036 - 11648\alpha - 2048\beta)y^9
+ (2296 + 2880\beta + 15680\alpha)y^{10}
+ (-16128\alpha - 1456 - 2944\beta)y^{11} + \cdots.
\]

The weight enumerator of \( P_{14} \) is \( W_{14}(\frac{-5}{16}, \frac{35}{32}) \).
Table 4
Optimal pure double circulant formally self-dual codes of lengths 16 to 22

<table>
<thead>
<tr>
<th>Length</th>
<th>Code</th>
<th>First row</th>
<th>Code</th>
<th>First row</th>
<th>Code</th>
<th>First row</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>( P_{16,1} )</td>
<td>122111000</td>
<td>( P_{16,2} )</td>
<td>211201000</td>
<td>( P_{16,3} )</td>
<td>121111000</td>
</tr>
<tr>
<td></td>
<td>( P_{16,4} )</td>
<td>222111000</td>
<td>( P_{16,5} )</td>
<td>211010100</td>
<td>( P_{16,6} )</td>
<td>112110100</td>
</tr>
<tr>
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<td>( P_{18,1} )</td>
<td>121110000</td>
<td>( P_{18,2} )</td>
<td>112110000</td>
<td>( P_{18,3} )</td>
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<td>( P_{18,4} )</td>
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<td>( P_{18,5} )</td>
<td>111101000</td>
<td>( P_{18,6} )</td>
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<tr>
<td></td>
<td>( P_{18,7} )</td>
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<td>( P_{18,8} )</td>
<td>122101000</td>
<td>( P_{18,9} )</td>
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<tr>
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<td>( P_{18,10} )</td>
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<td>( P_{18,11} )</td>
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<td>( P_{18,12} )</td>
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<tr>
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</tr>
<tr>
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<td>( P_{18,17} )</td>
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<td>( P_{18,18} )</td>
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<td>( P_{18,19} )</td>
<td>122110100</td>
<td>( P_{18,20} )</td>
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<td>( P_{18,21} )</td>
<td>222210100</td>
</tr>
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<td>( P_{18,22} )</td>
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<td>( P_{18,23} )</td>
<td>121111000</td>
<td>( P_{18,24} )</td>
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<tr>
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<td>( P_{20,3} )</td>
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<tr>
<td>22</td>
<td>( P_{22} )</td>
<td>212111000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- \( n = 16 \):

\[
W_{16}(\alpha, \beta, \gamma) = 1 + (128 + 64\gamma + 448\alpha + 64\beta)\gamma^6 \\
+ (-1664\alpha - 384\gamma + 192 - 256\beta)\gamma^7 \\
+ (480 + 320\beta + 1600\alpha + 1216\gamma)\gamma^8 \\
+ (1184 + 128\beta + 1792\alpha - 2816\gamma)\gamma^9 \\
+ (5056\gamma - 1216\beta + 480 - 7616\alpha)\gamma^{10} + \cdots.
\]

\( P_{16,1}, \ldots, P_{16,11} \) listed in Tables 4 and 5 are optimal formally self-dual codes. Note that \( P_{16,11} \) is equivalent to code 2f8 given in [2], which is the unique self-dual [16,8,6] code.

- \( n = 18 \):

\[
W_{18}(\alpha, \beta, \gamma, \delta) = 1 + (68 + 1792\alpha + 448\beta + 64\gamma + 64\delta)\gamma^6 \\
+ (-1664\beta + 240 - 256\gamma - 6144\alpha - 384\delta)\gamma^7 \\
+ (2496\beta + 7680\alpha + 448\gamma + 1344\delta + 474)\gamma^8 \\
+ (1532 - 3072\alpha - 1536\beta - 384\gamma - 3584\delta)\gamma^9 \\
+ (-576\gamma - 4416\beta - 16128\alpha + 7488\delta + 1944)\gamma^{10} + \cdots.
\]
Table 5
Optimal bordered double circulant formally self-dual codes of lengths 16–20

<table>
<thead>
<tr>
<th>Length</th>
<th>Code</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
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<th>$\alpha$</th>
<th>$\beta$</th>
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</tr>
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</table>

$P_{18,1}, \ldots, P_{18,52}$ listed in Tables 4 and 5 are optimal formally self-dual codes of length 18. They have distinct weight distributions (given in Table 6) and therefore are inequivalent.

- **$n = 20$:**

$$W_{20}(\alpha, \beta, \gamma, \delta) = 1 + (368 + 128\gamma + 4608\alpha - 128\delta + 1024\beta)\gamma^7$$

$$+ (24 - 20736\alpha + 896\delta - 4864\beta - 640\gamma)\gamma^8$$

$$+ (2376 - 3456\delta + 37376\alpha + 1408\gamma + 9728\beta)\gamma^9$$

$$+ (2684 - 1664\gamma + 9856\delta - 31232\alpha - 9728\beta)\gamma^{10}$$

$$+ (-384\gamma + 4464 - 22144\delta - 32256\alpha - 6144\beta)\gamma^{11} + \cdots.$$  

$P_{20,1}, \ldots, P_{20,8}$ are inequivalent optimal formally self-dual double circulant codes of length 20. Their first rows are listed in Tables 4 and 5, and their weight enumerators are listed in Table 6.

- **$n = 22$:**

$$W_{22}(\alpha, \beta, \gamma, \delta) = 1 + (1350 + 256\beta + 256\gamma + 11520\alpha + 2304\delta)\gamma^8$$

$$+ (-64000\alpha - 1252 - 2048\gamma - 13312\delta - 1536\beta)\gamma^9$$

$$+ (33280\delta + 8704\gamma + 4096\beta + 149504\alpha + 9204)\gamma^{10}$$

$$+ (-45056\delta + 2688 - 182272\alpha - 6144\beta - 26624\gamma)\gamma^{11}$$

$$+ (2560\beta + 9728\delta + 10316 + 7680\alpha + 64000\gamma)\gamma^{12} + \cdots.$$
Table 6

<table>
<thead>
<tr>
<th>Code</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
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<td>$7/128$</td>
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<td>$-45/128$</td>
<td>$63/64$</td>
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</table>

$C_{22}$ constructed from the extended quadratic residue code by subtracting is an optimal formally self-dual code. $P_{22}$ has the same parameters.

- $n = 24$:

\[ W_{24}(\alpha, \beta, \gamma, \delta) = 1 + (-512\alpha + 5120\gamma + 28160\beta + 512\delta + 4784)\gamma^9 \]

\[ + (-3584\delta - 34816\gamma + 4608\alpha - 8112 - 185856\beta)\gamma^{10} \]

\[ + (104448\gamma + 36288 - 21504\alpha + 11264\delta + 531456\beta)\gamma^{11} \]
There are exactly two inequivalent extremal self-dual \([24,12,9]\) codes [8]. The weight enumerator of these codes is \(W_{24}(3/256,451/256,-33/2,8535/128)\).

- \(n = 26, 28, 30\): For these lengths, it is not known if there is a formally self-dual code with the highest possible minimum weight. If such a code exists then the code has higher minimum weight than any known linear code of that length and dimension. We list below the possible weight enumerators for these lengths.

\[
W_{26}(x, \beta, \gamma, \delta, \epsilon) = 1 + (28160\beta + 512\delta - 512\epsilon + 112640x + 5120\gamma + 3320)\epsilon^y + (-185856\beta - 3584\delta + 4608\epsilon - 720896x - 34816\gamma - 5804)\gamma^y + (587776\beta + 12288\delta - 22528\epsilon + 2187264x + 114688\gamma + 35424)\gamma^y + (-1212416\beta - 27648\delta + 79872\epsilon - 4339712x - 22360 - 246784)\gamma^y + (1569792\beta + 39936\delta - 224256\epsilon + 5414912x + 103200 + 335872\gamma)\gamma^y + \cdots.
\]

\[
W_{28}(x, \beta, \gamma, \delta, \epsilon) = 1 + (11408 + 1024\delta + 11264\beta + 292864x + 1024\gamma + 67584\epsilon)\gamma^y + (-25936 - 2183168\epsilon - 8192\gamma - 516096\epsilon - 88064\beta - 10240\delta)\gamma^y + (7694336\alpha + 330752\beta + 1878016\epsilon + 125488 + 31744\gamma + 54272\delta)\gamma^y + (-132768 - 1745712\alpha - 79872\gamma - 802816\beta - 4403200\epsilon - 204800\delta)\gamma^y + (1300480\beta + 6864896\epsilon + 135168\gamma + 608256\delta + 375360 + 26374144x)\gamma^y + \cdots.
\]

\[
W_{30}(x, \beta, \gamma, \delta, \epsilon) = 1 + (24576\gamma + 42896 + 2048\delta - 2048\epsilon + 159744\beta + 745472\alpha)\gamma^y + (-217088\gamma - 138916 - 6336512\alpha + 22528\epsilon - 18432\delta - 1384448\beta)\gamma^y + (25460736\alpha - 129024\epsilon + 917504\gamma + 568248 + 5705728\beta + 79872\delta)\gamma^y + (-2490368\epsilon - 223232\delta + 518144\gamma - 65355776\alpha - 924848 - 15052800\beta)\gamma^y + (-110702592\alpha - 27496448\beta - 187566 - 4968448\gamma - 491520\delta + 4190208\epsilon)\gamma^y + \cdots.
\]

**Problem.** Determine the exact highest minimum weights for lengths 26, 28 and 30.
6.2. Optimal double circulant codes

All optimal pure double circulant formally self-dual codes were classified in Proposition 3.3 for length up to 14. By exhaustive search, we have found all optimal formally self-dual double circulant codes of lengths 16 to 24. We here list only codes with different weight enumerators. In Table 4 (resp. 5) we list the first rows for optimal pure (resp. bordered) double circulant codes of lengths 16 to 22. Note that there are no pure double circulant formally self-dual [24,12,9] codes which are not self-dual. All optimal bordered codes of length 22 have the same weight enumerator as one of the pure double circulant codes, and so these codes are omitted. The weight enumerators for these codes are listed in Table 6.

We have found pure double circulant codes with parameters [26,13,8], [28,14,9] and [30,15,9]. These codes attain the lower bound on minimum weight for ternary linear \([n,n/2]\) codes. However, it is not known if these codes are optimal formally self-dual codes. Here we list only one example for these parameters. The first rows of \(R\) are

\[
1112110100000, \quad 21112111100100 \quad \text{and} \quad 221012111000000,
\]

respectively.

References