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LOCALIZATION AND COMPLETION

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§ 0. Introduction

Throughout this paper, R will be an associative ring with unity and I will be an injective right R-module.

With I we associate a torsion theory and a linear topology. The former gives rise to the process of localization, the latter to the process of completion. The reader should bear in mind the classical example $I = I(R/P)$, the injective hull of the R-module R/P, when R is commutative and P is a prime ideal.

It is a maxim of commutative algebra that one should first localize and then complete. We shall examine the simple functor which arises when one does first one and then the other, under fairly general circumstances.

In § 1, we briefly review the relevant concepts of the I-torsion theory and the I-adic topology. In § 2, we discuss the quotient functor Q associated with a torsion theory. Our main purpose is to obtain many examples in which Q is an exact endofunctor of Mod R. Among other things, we show that every I-divisible module is injective if and only if I has zero singular submodule. In § 3, we study the triple (S, η, μ) on Mod R which arises from $S(M) = \mathrm{Hom}_E(\mathrm{Hom}_R(M, I), I)$, where $E = \mathrm{Hom}_R(I, I)$. We show that $Q(M) \subseteq S(M)$, with equality holding when $\mathrm{Hom}_R(M, I)$ is a finitely generated left E-module. In § 4, we introduce the finite topology on $S(M)$ and show that, when Q is exact, $S(M)$ is the I-adic completion of $Q(M)$. In § 5, we describe in detail the algebras of the triple (S, η, μ) when Q is exact. They are the I-torsionfree and I-divisible R-modules which have been equipped with a certain limit operation that assigns a limit to each I-adic Cauchy net. For any right R-module M, $I(M)$ will denote its injective hull.

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§ 1. The I-torsion theory and the I-adic topology

The torsion theory associated with an injective module was first investigated by Findlay and the present author in 1958, but it was only called "torsion theory" by the latter in 1966. Equivalent concepts are the idempotent filters of right ideals by Bourbaki (1961) and Gabriel (1962), the torsion radicals of Maranda (1964), the modular closure operations of Chew (1965), the hereditary torsion theories of Dickson (1966), and the idempotent kernel functors of Goldman (1969). An exposition of torsion theories with references to the literature may be found in [5] or [11]. Here we shall only give a brief review of what is most relevant to our present purpose.

The following concepts depend on the injective right R-module I; however, we shall write "torsion" in place of "I-torsion", "divisible" in place of "I-divisible", etc.

A right R-module M is called *torsion* if $\mathrm{Hom}_R(M, I) = 0$, *torsionfree* if M is isomorphic to a submodule of some power of I, and *divisible* if $I(M)/M$ is torsionfree. It follows that M is divisible if and only if $\mathrm{Ext}_R(T, M) = 0$ for every torsion module T, or even only for every cyclic torsion module $T = R/D$.

Every module M has a *torsion submodule*

$$T(M) = \{m \in M \mid mR \text{ is torsion}\},$$

and a *divisible hull* $D(M)$ defined by

$$D(M)/M = T(I(M)/M).$$

One also defines the *quotient module*

$$Q(M) = D(M/T(M)),$$

sometimes called the *localization* of M at I. D is not a functor, but T and Q are.

1.1. Proposition. *A homomorphism $f: M \to N$ is isomorphic to the canonical homomorphism $M \to Q(M)$ if and only if*

Q1. *Ker f and $N/f(M)$ are torsion,*

Q2. *N and $I(N)/N$ are torsionfree.*

Proof. The result is clear if we add the following condition:

Q3. N is an essential extension of $f(M)$.

We shall derive Q3 from Q1 and Q2.

Suppose P is a submodule of N such that $f(M) \cap P = 0$. Then

$$P \cong (f(M) + P)/f(M) \subseteq N/f(M),$$

and this is torsion by Q1. On the other hand, P is torsionfree by Q2, hence $P = 0$. Therefore Q3 holds[1].

There are other ways of obtaining Q. For example, let

$$L(M) = \varinjlim \{\mathrm{Hom}_R(D, M) \mid R/D \text{ is torsion}\},$$

then

$$Q(M) = L(M/T(M)) = L^2(M).$$

We shall call two injectives *similar* if they give rise to the same torsion theory. Clearly, this is the case if and only if each is isomorphic to a submodule of a power of the other. It is easily seen that every injective is similar to an injective of the form

$$\Pi \{I(R/A) \mid R/A \text{ is torsionfree}\}.$$

The following facts are known from "additive semantics" (see e.g. [5, § 1]).

1.2. Proposition.

(1) *$Q(R)$ is a ring.*

(2) *The canonical mapping $R \to Q(R)$ is a ring homomorphism.*

(3) *Every torsionfree divisible module is a $Q(R)$-module.*

(4) *Every R-homomorphism between torsionfree divisible modules is a $Q(R)$-homomorphism.*

(5) *The functor from the category of torsionfree divisible R-modules to* Mod $Q(R)$ *implied by* (3) *and* (4) *is a best approximation of the former category by a module category* Mod R' *with $R \to R'$.*

The functor from Mod-R to the full subcategory of torsionfree divisible modules determined by Q is exact. In fact, every full reflective subcategory of Mod R for which the reflector preserves regular monomorphisms is obtainable in this way [10].

[1] In view of this observation, condition Q3 is redundant in [5, p. 37, Proposition 2.5 and Corollary 1; similarly on p. 39, Proposition 2.6]. That Q3 follows from Q1 and Q2 is known (see [4, Lemma 3.8]).

Let P be a class of R-modules closed under isomorphic images, submodules and finite direct products. Then we know (see e.g. [5, §3]) how to define a topology on each R-module M by taking as a fundamental system of open neighborhoods of 0 all kernels of homomorphisms $M \to P$ where $P \in \mathsf{P}$.

We shall write

$$M_0 = \bigcap \{\operatorname{Ker} f \mid f : M \to P \in \mathsf{P}\},$$

$$\hat{M} = \varprojlim \{\operatorname{Im} f \mid f : M \to P \in \mathsf{P}\}.$$

1.3. Proposition. *Let M and R be endowed with the topology determined by P, $\hat{M}$ and $\hat{R}$ with the topology of inverse limits of discrete modules. Then:*

(1) R is a topological ring;

(2) M_R is a topological R-module;

(3) every R-homomorphism $M \to N$ is continuous;

(4) M/M_0 is Hausdorff;

(5) $\hat{M}$ is the completion of M/M_0.

(6) $R \to \hat{R}$ is a continuous ring homomorphism;

(7) $\hat{M}$ is a topological $\hat{R}$-module.

The composite mapping $M \to M/M_0 \to \hat{M}$ is called the *Hausdorff completion* of M.

Given an injective I_R, we could take P to be the class of all I-torsion modules. This is not the topology we are interested in here. Instead, we take P to be the class of all modules isomorphic to submodules of I^n for some natural number n. The resulting topology is called the I-adic topology on M, a fundamental system of open neighborhoods of 0 consists of all kernels of homomorphisms $M \to I^n$ for some finite cardinal n.

To see the difference between the two topologies, take $R = \mathbf{Z}$, the ring of integers and $I = I(\mathbf{Z}/p\mathbf{Z})$, the Prüfer group associated with the prime number p. A fundamental system of open neighborhoods of 0 on $\mathbf{Z}$ in the I-torsion topology consists of all ideals $m\mathbf{Z}$, where p does not divide m, while in the I-adic topology it consists of all ideals $p^n\mathbf{Z}$, where n is any natural number.

The I-adic topology behaves nicely with respect to submodules, as the following observation shows.

1.4. Proposition. *If N is a submodule of M and M has the I-adic topology, then the induced topology on N is also the I-adic topology.*

Proof. Clearly, any fundamental open neighborhood of 0 in the induced topology has the form

$$(\mathrm{Ker}\, f) \cap N = \mathrm{Ker}\,(f \mid N), \qquad f : M \to I^n.$$

Conversely, any fundamental open neighborhood of 0 in the I-adic topology has the form Ker g, where $g : N \to I^n$. Since I^n is injective, we may extend g to $f : M \to I^n$, so. that Ker g = (Ker f) $\cap N$.

When R is a commutative ring and P is a prime ideal, one usually introduces the P-adic topology on M by taking as fundamental open neighborhoods of 0 all submodules of the form MP^n for some natural number n. The P-adic topology behaves nicely with respect to quotient modules, but not, in general, with respect to submodules. The following result has been proved elsewhere [6]:

1.5. Proposition. *If R is a commutative Noetherian local ring and P is a prime ideal, then the $I(R/P)$-adic topology coincides with the P-adic topology on every finitely generated module.*

As corollaries one may obtain common versions of the Artin-Rees Lemma and the Krull Intersection Theorem. We record the following trivial observation:

1.6. Proposition. *If N is a submodule of M and M/N is I-torsionfree, then N is closed in the I-adic topology. In particular, every I-torsionfree module is Hausdorff in the I-adic topology.*

Proof. Let $m \in M$, $m \notin N$. Since M/N is I-torsionfree, there exists $f : M \to I$ such that $f(m) \neq 0$ and $f(N) = 0$. Therefore $\{m\}$ + Ker f does not meet N. But it is an open neighborhood of m, hence N is closed.

§ 2. Exactness of the quotient functor

We shall regard the quotient functor Q not as a reflector but as an endo-functor of Mod R, hence as the functor belonging to an idempotent triple on Mod R. We shall examine some conditions on I which guarantee different degrees of smoothness of Q.

Given any flat left R-module ${}_RF$, we shall write

$$F^* = \mathrm{Hom}_{\mathbf{Z}}(F, \mathbf{Q}/\mathbf{Z}).$$

Then F^* is an injective right R-module. In the F^*-torsion theory a module M is torsion if and only if $M \otimes_R F = 0$. In particular, a cyclic module R/D is torsion if and only if $DF = F$ (see [8]).

2.1. Proposition. *Q is isomorphic to the identity functor on* Mod R *if and only if it is determined by an injective F^*_R, where $_RF$ is a free left R-module.*

Proof. In the F^*-torsion theory, M is torsion if and only if $M \otimes_R F = 0$, that is, $M = 0$. Thus every module is torsionfree, and so $Q(M) = M$.

Conversely, if $Q(M) = M$ for every module M, we see that $T(M) = \text{Ker}\,(M \to Q(M))$ is zero, hence M is torsionfree. Then the torsion theory coincides with that obtained from an injective of the form F^*, where $_RF$ is free.

2.1.1. Corollary. *Every injective cogenerator G is similar to F^* for free $_RF$.*

Proof. If m is a nonzero element of the module M, there exists $f : M \to G$ such that $f(m) \neq 0$ Thus every module M is G-torsionfree, hence the G-torsion theory is the same as the F^*-torsion theory for free $_RF$.

Of special interest are those torsion theories for which the approximation functor of Proposition 1.2 (5) is an equivalence. Various conditions for this to happen were stated by Gabriel, Maranda, Walker and Walker, and Goldman. In particular, this is the case if and only if $Q \cong (\quad) \otimes_R Q(R)$; $Q(R)$ is then a flat left R-module and $R \to Q(R)$ is an epimorphism of rings.

The following is a variant of known results; see e.g. [11, Corollary 13.12] and [5, Proposition 2.7] [2]. We give a proof for completeness.

2.2. Proposition. Q *is isomorphic to $(\quad) \otimes_R Q(R)$ if and only if it is determined by an injective F^*, where $R \to F$ is an epimorphism of rings and $_RF$ is flat.*

Proof. Q is the reflector from Mod R to the full subcategory of torsionfree divisible modules followed by the inclusion functor. The left adjoint of Mod $(R \to Q(R))$ is $(\quad) \otimes Q(R)$. Assume that this is isomorphic to the reflector, then it is left exact, hence $Q(R)$ is flat as a left R-module. Also Mod $(R \to Q(R))$ is then isomorphic to the inclusion functor, hence it is faithful and full, hence $R \to Q(R)$ is an epimorphism of rings. Moreover, a module M is torsion if and only if $\text{Ker}\,(M \to Q(M)) = T(M) = M$, that is, $M \otimes_R Q(R) \cong Q(M) = 0$. Thus the torsion theory is that determined by $Q(R)^*$.

Conversely, consider the torsion theory determined by F^*. To show that $Q(M) \cong$

[2] Incidentally in the proof of the latter [5, Proposition 2.7] lines 7 to 9 on p. 46 should be corrected as follows: "$[S_R, I_R] = \text{Hom}_S(S, I_S) = 0$; hence, if $f : S_R \to I_R$ and $fh(R) = 0$, then for any $s \in S$, $f(s) = f(h(r_0)s) = 0$."

$\cong M \otimes_R F$ we use Proposition 1.1. and check the following:

Q1. The kernel and cokernel of $M \to M \otimes_R F$ are torsion.

Q2. $M \otimes_R F$ and $I(M \otimes F)/(M \otimes F)$ are torsionfree.

Then, taking $M_R = R_R$, we see that $Q(R) \cong F$ as R-modules. That they are isomorphic as rings follows from the fact that, for given $q \in Q(R)$, there is a unique way of extending the mapping $r \mapsto qr$, $r \in R$, to an R-homomorphism $Q(R) \to Q(R)$. It remains to check Q1 and Q2.

Q1. If $m \in M$ is in the kernel of $g : M \to M \otimes F$, then $mR \otimes F = 0$, hence m is torsion.

Let $n \in M \otimes F$ with equivalence class $[n]$ modulo $g(M)$. We claim that $[n]\ R \otimes F = 0$. In view of the exact sequence

$$0 \to g(M) \otimes F \xrightarrow{\alpha} (g(M) + nR) \otimes F \to [n]\ R \otimes F \to 0,$$

it suffices to show that α is a surjection, that is, that $nR \otimes F \subseteq g(M) \otimes F$. Put $n = \Sigma_i m_i \otimes f_i$, then

$$n \otimes 1 = \Sigma_i m_i \otimes (f_i \otimes 1) = \Sigma_i m_i \otimes (1 \otimes f_i) = \Sigma_i (m_i \otimes 1) \otimes f_i,$$

and so $nR \otimes F = (n \otimes 1)\ F \subseteq g(M) \otimes F$. This argument depends on the fact that $1 \otimes f = f \otimes 1$ for all $f \in F$, a consequence of the observation that $f \mapsto 1 \otimes f$ and $f \mapsto f \otimes 1$ agree when $f \in h(R)$ and the assumption that $h : R \to F$ is an epimorphism of rings.

Q2. First, we show that every F-module N is torsionfree as an R-module. Indeed, let T be any torsion module, so that $T \otimes F = 0$. Then

$$\mathrm{Hom}_R(T, N) \cong \mathrm{Hom}_R(T, \mathrm{Hom}_F(F, N)) \cong \mathrm{Hom}_F(T \otimes F, N) = 0.$$

Next, we claim that if N is an F-module, then also $I(N_R)/N$ is torsionfree. Indeed, by a trick of Findlay, it is contained in the F-module J/N, where $J = I(N_F)$. For $J_R \cong \mathrm{Hom}_F(F, J)$ is injective, since J_F is injective and ${}_RF$ is flat.

The proof is now complete.

In Proposition 2.2, we considered the situation in which Q preserves all colimits. More generally, one may ask when Q is exact. Goldman [4] has considered a number of equivalent formulations of this property, among which:

(G) Every torsionfree factor module of a torsionfree divisible module is divisible.

We shall obtain another criterion for this to happen. Note that if A is a divisible submodule of a torsionfree module B, then B/A is torsionfree.

2.3. Proposition. *Given a torsionfree divisible module A, then B/A is divisible for every torsionfree divisible module B extending A if and only if $I(A)/A$ is divisible.*

Proof. The necessity of the condition follows if we take $B = I(A)$.

Conversely, assume the condition, and let B be a torsionfree divisible module containing A. Now B/A is a submodule of $I(B)/A$ with factor module $I(B)/B$. Since B is divisible and $I(B)$ is torsionfree, $I(B)/B$ is torsionfree. Therefore B/A will be divisible if $I(B)/A$ is divisible, and this remains to be shown.

Put $I(B) = I(A) \oplus K$, then K is injective. Hence $I(B)/A = I(A)/A \oplus K$, and this is divisible, because the first summand is divisible by assumption and the second by injectivity.

2.3.1. Corollary. *Q is exact if and only if $I(A)/A$ is divisible for all torsionfree divisible modules A.*

Proof. The necessity of the condition is clear. Conversely, assume B is torsionfree divisible and B/A torsionfree. Then A is torsionfree, and the condition asserts that $I(A)/A$ is divisible. Then B/A is divisible, by Proposition 2.3., hence Q is exact, by Goldman's condition (G).

This criterion for exactness of Q is not ideal, we should prefer something like Proposition 2.2. Still, it may be applied, as the corollary to our next proposition will show.

We recall that the singular submodule of a module consists of all those elements which are annihilated by essential right ideals of R.

2.4. Proposition. *Given an injective module I_R, the following statements are equivalent:*

(1) *I has zero singular submodule.*

(2) *$I(M)/M$ is I-torsion for every module M.*

(3) *Every I-divisible module is injective.*

Proof. We show that (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2). We claim that $\mathrm{Hom}_R(I(M)/M, I) = 0$. Let $f : I(M) \to I$ be such that $f(M) = 0$. Take any $j \in I(M)$ and put $f(j) = i \in I$. Then, for all $r \in R$, $jr \in M \Rightarrow ir = 0$, that is, $i(j^{-1}M) = 0$. Since M is an essential submodule of $I(M)$, $j^{-1}(M)$ is an essential right ideal of R. Hence, by (1), $i = 0$, and so $f = 0$, as was to be shown.

(2) ⇒ (3). Let M be divisible. Then $I(M)/M$ is both torsion and torsionfree, hence $M = I(M)$.

(3) ⇒ (1). Let E be an essential right ideal of R. Then $R/E \subseteq I(E)/E = D(E)/E$ is torsion, hence $\mathrm{Hom}_R(R/E, I) = 0$. Suppose $i \in I$ is such that $iE = 0$, then $[r] \mapsto ir$ is a mapping $R/E \to I$, and so $i = 0$.

2.4.1. Corollary (See [6, Example 2].) *When I has zero singular submodule, then Q is exact.*

Proof. If A is torsionfree and divisible, then $I(A)/A$ is zero, hence injective. Now apply Corollary 2.3.1.

If (S, η, μ) is a triple on Mod R, let $\psi_A : A \otimes_R S(R) \to S(A)$ be the canonical mapping such that, for all $a \in A$ and $s \in S(R)$,

$$\psi_A(a \otimes s) = S(a)(s).$$

We note that a may be regarded as a homomorphism $R \to A$, hence $S(a) : S(R) \to S(A)$.

2.5. Proposition. *Let (S, η, μ) be a triple on* Mod R *and assume that S is exact. Then the following statements are true*:

(1). *If C is finitely generated, then ψ_C is epi.*

(2). *If C is finitely presented, then ψ_C is iso.*

(3). *If R is right Noetherian, then ψ_C is mono, for any module C, and $S(R)$ is flat as a left R-module.*

Proof. Given an exact sequence of R-modules

$$0 \to A \overset{m}{\to} B \overset{e}{\to} C \to 0,$$

left exactness of S yields the following commutative diagram with two exact rows:

$$\begin{array}{ccccccccc}
 & & A \otimes S(R) & \overset{m \otimes 1}{\to} & B \otimes S(R) & \overset{e \otimes 1}{\to} & C \otimes S(R) & \to & 0 \\
 & & \downarrow{\scriptstyle \psi_A} & & \downarrow{\scriptstyle \psi_B} & & \downarrow{\scriptstyle \psi_C} & & \\
0 & \to & S(A) & \underset{S(m)}{\to} & S(B) & \underset{S(e)}{\to} & S(C) & &
\end{array}$$

(1). Assume that C is finitely generated, then we may suppose that B is finitely generated and free, hence ψ_B will be an isomorphism. By the Two Square Lemma, or by routine diagram chasing, it then follows that

$$(*) \qquad \text{Cok}\ \psi_A \cong \text{Ker}\ \psi_C.$$

If S is exact, $S(e)$ is epi, and we see from the second square that ψ_C is epi.

(2). Assume that C is finitely presented, then, in addition to the above, we may suppose that A is finitely generated. Hence, by (1), ψ_A is epi. Therefore, by (*), ψ_C is mono.

(3). Any R-module C is the direct limit of finitely generated submodules C_i with injections $k_i : C_i \to C$. Then also $C \otimes S(R)$ is the direct limit of the $C_i \otimes S(R)$ with injections $k_i \otimes 1$. Take any element $d \in C \otimes S(R)$ and suppose $\psi_C(d) = 0$. Since $C \otimes S(R)$ is the union of the images of the $C_i \otimes S(R)$, we may put

$$d = (k_i \otimes 1)(d_i),$$

for some i, where $d_i \in C_i \otimes S(R)$. Then

$$S(k_i)\ \psi_{C_i}(d_i) = \psi_C(k_i \otimes 1)(d_i) = \psi_C(d) = 0,$$

and $S(k_i)$ is mono, hence $\psi_{C_i}(d_i) = 0$.

Now assume that R is right Noetherian. Then the finitely generated module C_i is finitely presented. Hence, by (2), ψ_{C_i} is mono, and therefore $d_i = 0$. Hence $d = 0$, and so ψ_C is mono.

Assume that $m : A \to B$ is any monomorphism. As we have just proved that ψ_A is mono, it follows that $m \otimes 1$ is mono. Therefore $S(R)$ is flat as a left R-module.

2.5.1. Corollary. *If Q is exact, then $Q(C) \cong C \otimes_R Q(R)$ for any finitely presented module C.*

If Q is exact and R is right Noetherian, then Q preserves all colimits (see [4]). If we didn't know already that $Q(R)$ is then a flat left R-module, we could deduce it from the above.

§ 3. The triple associated with I

Let E be the ring of endomorphisms of the injective module I_R. Then I becomes a bimodule ${}_EI_R$. If M is a right R-module, we write

$$I \phi M = \mathrm{Hom}_R(M, I),$$

and this is understood to be a left E-module. On the other hand, if N is a left E-module, we write

$$N \obslash I = \mathrm{Hom}_E(N, I),$$

and this is understood to be a right R-module. Thus we obtain a pair of functors

$$I \phi (\ ) : \mathrm{Mod}R \to (E\,\mathrm{Mod})^{\mathrm{op}},$$

$$(\ ) \obslash I : (E\,\mathrm{Mod})^{\mathrm{op}} \to \mathrm{Mod}\,R,$$

the former being left adjoint to the latter.

We shall use the convention that homomorphisms of left modules are written on the right of their arguments and that they compose by associativity, thus $((\alpha)\varphi)\psi = (\alpha)(\varphi\psi)$.

The above pair of adjoint functors gives rise to a triple (S, η, μ) on Mod R as follows:

$$S(M) = (I \phi M) \obslash I;$$

$$(f)(S(g)(s')) = (fg)s';$$

$$(f)(\eta(M)(m)) = f(m);$$

$$(f)(\mu(M)(\sigma)) = (f^*)\sigma.$$

Here M and M' are R-modules, $f\epsilon I \phi M$, $g: M' \to M$, $s'\epsilon S(M')$, $m\epsilon M$, $\sigma\epsilon S^2(M)$. Moreover $f^*: S(M) \to I$ is defined by

$$f^*(s) = (f)s,$$

where $s\epsilon S(M)$.

We note that $S(R) = (I \phi R) \obslash I \cong I \obslash I$ is the bicommutator of I, the opposite of the usual ring $\mathrm{End}_E(I)$, and that $S(I) = E \obslash I \cong I$.

3.1. Proposition. *The following assertions hold with respect to the I-torsion theory*:

(1). Ker $\eta(M) = T(M)$.

(2). *$S(M)$ is torsionfree and divisible.*

(3). *There is a unique homomorphism $Q(M) \to S(M)$ over M, and this is a monomorphism.*

(4). *$Q(R) \to S(R)$ is a ring homomorphism.*

Proof. (1). $m \in \operatorname{Ker} \eta(M)$ if and only if, for all $f \in I \phi M$, $f(m) = (f)(\eta(M)(m)) = 0$. This is the same as saying that $\operatorname{Hom}_R(mR, I) = 0$, that is, $m \in T(M)$.

(2). Let $0 \neq s \in S(M)$, then $0 \neq (f)s = f^*(s)$, for some $f \in I \phi M$. Here $f^*: S(M) \to I$, and so $S(M)$ is torsionfree.

To see that $S(M)$ is divisible, take any right ideal D of R such that R/D is torsion, and let $\varphi: D \to S(M)$. Then, for any $d \in D$ and $f \in I \phi M$, we write

$$_f\varphi(d) = (f)(\varphi(d)).$$

It is clear that $_f\varphi: D \to I$. Since I_R is torsionfree and divisible, there exists a unique $_fi \in I$ such that, for all $d \in D$,

$$_f\varphi(d) = {_fi}d.$$

Now $f \mapsto {_fi}$ is a mapping $I \phi M \to I$, and it is easily seen to be an E-homomorphism. Hence there exists a unique $s \in S(M)$ such that

$$(f)s = {_fi},$$

for all $f \in I \phi M$, and therefore, for all $d \in D$,

$$(f)sd = {_fi}d = {_f\varphi}(d) = (f)(\varphi(d)),$$

hence $sd = \varphi(d)$.

(3). Since $T(M) = \operatorname{Ker}(M \to Q(M))$, we see from (1) that there exists a unique map $Q(M) \to S(M)$ such that $M \to Q(M) \to S(M)$ is $\eta(M)$. Moreover, this is a monomorphism, since $Q(M)$ is an essential extension of $M/T(M)$ and $M/T(M) \to S(M)$ is mono.

(4). We omit the verification that $Q(R) \to S(R)$ is a ring homomorphism, as this is routine and has already been shown elsewhere [5].

In view of Proposition 3.1., we may write

$$M/T(M) = \bar{M} \subseteq Q(M) \subseteq S(M).$$

Since $S(M)$ is torsionfree and $Q(M)$ is divisible, $S(M)/Q(M)$ is torsionfree. Since $Q(M)/\bar{M}$ is torsion, we have, by Proposition 1.1.,

$$(**) \qquad Q(M)/\bar{M} = T(S(M)/\bar{M}).$$

Thus, in the definition of $Q(M)$, $S(M)$ could have been used in place of the injective hull $I(M)$.

3.1.1. Corollary. *$Q(M) \to S(M)$ is the equalizer of the pair of maps $\eta S(M)$, $S\eta(M)$: $S(M) \to S^2(M)$.*

Proof. Take any $s \in S(M)$ and $g \in I \phi S(M)$, then

$$(g)(S\eta(M)(s)) = (g\eta(M))s = (g\eta(M))^*(s)$$

and

$$(g)(\eta S(M)(s)) = g(s).$$

Put $g' = g - (g\eta(M))^*$, then the equalizer of the given pair of maps is $K = \bigcap_{g \in I \phi M} \operatorname{Ker} g'$. We claim that $K = Q(M)$.

In view of (**) above, $s \in Q(M)$ if and only if, for all $g \in I \phi S(M)$,

$$g\eta(M) = 0 \Rightarrow g(s) = 0.$$

Since $f^* \eta(M) = f$ for all $f \in I \phi M$, we have

$$g'\eta(M) = g\eta(M) - (g\eta(M))^* \eta(M) = 0.$$

Now suppose $s \in Q(M)$, then, putting g' in place of g above, we see that $g'(s) = 0$. Thus $Q(M) \subseteq K$.

Conversely, suppose $s \in K$. Then, if $g\eta(M) = 0$, also $g(s) = g'(s) = 0$, hence $s \in Q(M)$. Thus $K \subseteq Q(M)$.

It follows from a result by Fakir [3] that the idempotent triple determined by Q is the best co-approximation of the triple (S, η, μ) by an idempotent triple. In particular, the latter is never idempotent unless $S = Q$.

It is natural to ask when $Q(M) = S(M)$. We shall return to this question after some preliminary calculations.

We have already pointed out that $S(I) \cong I$, and it follows that $S(I^n) \cong I^n$ for every finite cardinal n. Actually, the canonical mapping $\eta(I^n)$ is an isomorphism, and we shall describe its inverse explicitly.

Define $\delta_n : S(I^n) \to I^n$ by

$$\delta_n(s) = \langle (p_1)s, ..., (p_n)s \rangle = \sum_{i=1}^{n} k_i((p_i)s),$$

for any $s \in S(I^n)$, where $p_i : I^n \to I$ and $k_i : I \to I^n$ are the canonical projections and injections respectively.

In what follows, we shall have to consider the algebras of the triple (S, η, μ). We recall [2] that an *S-algebra* is a pair (A, α), where $A \in \text{Mod } R$ and $\alpha \in \text{Hom}_R(S(A), A)$ such that

$$\alpha\eta(A) = 1, \alpha\mu(A) = \alpha S(\alpha).$$

Furthermore, an S-homomorphism between S-algebras (A, α) and (B, β) is an R-homomorphism $g : A \to B$ such that $g\alpha = \beta S(g)$.

3.2. Lemma. *δ_n is the inverse of $\eta(I^n)$, (I^n, δ_n) is an S-algebra, and every R-homomorphism $g : I^n \to I^k$ is an S-homomorphism.*

Proof. It is easily seen that $\delta_n\eta(I^n) = 1$ and $\eta(I^n)\delta_n = 1$. We shall only check the second equality, by taking any $s \in S(I^n)$ and $f \in I \,\phi\, I^n$ and computing:

$$(f)(\eta(I^n)\delta_n(s)) = f(\delta_n(s)) = \sum_{i=1}^{n} fk_i((p_i)s) = \sum_{i=1}^{n} (fk_ip_i)s = (f)s,$$

since s is an E-homomorphism and $fk_i \in E$, and since $\sum_{i=1}^{n} k_ip_i = 1$.

To show that (I^n, δ_n) is an S-algebra, it remains to verify that $\delta_nS(\delta_n) = \delta_n\mu(I^n)$. Take any $\sigma \in S^2(I^n)$ and any $f \in I \,\phi\, I^n$, then

$$(f)(S(\delta_n)(\sigma)) = (f\delta_n)\sigma.$$

Now define $f^* \in I \,\phi\, S(I^n)$ by taking any $s \in S(I^n)$ and putting

$$f^*(s) = (f)s = \sum_{i=1}^{n} (fk_ip_i)s = \sum_{i=1}^{n} fk_i(p_i)s = f\delta_n(s)$$

so that

$$f^* = f\delta_n.$$

Thus

$$(f)(S(\delta_n)(\sigma)) = (f^*)\sigma = (f)(\mu(I^n)(\sigma)).$$

and so

$$\delta_n S(\delta_n)(\sigma) = \delta_n \mu(I^n)(\sigma),$$

as was to be shown. (The reader will note that the f^* defined here is a special case of that at the beginning of § 3, with $M = I^n$.)

Finally, by naturality of η,

$$S(g)\eta(I^n) = \eta(I^k)g.$$

Multiplying by δ_k on the left and δ_n on the right, we obtain

$$\delta_k S(g) = g\delta_n,$$

and so g is an S-homomorphism.

3.3. Lemma. *For each R-homomorphism $f:M \to I^n$ there exists a unique S-homomorphism $f^*:(S(M), \mu(M)) \to (I^n, \delta_n)$ such that $f^*\eta(M)) = f$. Moreover, $f^*S(M)/f^*Q(M)$ is torsion.*

Proof. Recall that $(S(M), \mu(M))$ is the free S-algebra generated by M, with adjunction $\eta(M)$. Therefore, given any R-homomorphism $f:M \to I^n$, there exists a unique S-homomorphism $f^*:(S(M), \mu(M)) \to (I^n, \delta_n)$ such that $f^*\eta(M) = f$. We compute, for any $s \in S(M)$,

$$f^*(s) = f^*\mu(M)\,S\eta(M)(s) = \delta_n S(f^*)S\eta(M)(s) = \delta_n S(f)(s)$$

$$= \langle (p_1)(S(f)(s)), \ldots, (p_n)(S(f)(s))\rangle = \langle (p_1 f)s, \ldots, (p_n f)s\rangle.$$

The f^* introduced at the beginning of the present section is a special case of this, with $n = 1$.

To see that $\mathrm{Hom}_R(f^*S(M)/f^*Q(M), I) = 0$, take any $g:f^*S(M) \to I$ and suppose $gf^*Q(M) = 0$. Extend g to $h:I^n \to I$, then hf^* is the unique S-homomorphism $(S(M), \mu(M)) \to (I, \delta_1)$ such that $hf^*\eta(M) = 0$, since h and f^* are both S-homomorphisms, by Lemma 3.2 and the above. But also $0\eta(M) = 0$, and 0 is clearly an

S-homomorphism, hence $hf^* = 0$. Therefore $g = 0$, and our proof is complete.

3.4. Proposition. *If $I \phi M$ is a finitely generated E-module, then $S(M) = Q(M)$.*

Proof. Let the generators of $I \phi M$ be $g_1, ..., g_n$, and put $g = \langle g_1, ..., g_n \rangle$. Then $g^*: S(M) \to I^n$ is defined, for each $s \in S(M)$, by

$$g^*(s) = \langle (g_1)s, ..., (g_n)s \rangle.$$

Suppose $g^*(s) = 0$, then s annihilates all the generators of $I \phi M$, hence all of $I \phi M$, and so $s = 0$. Thus Ker $g^* = 0$.

Now, by Lemma 3.3, $g^*S(M)/g^*Q(M)$ is torsion. Since g^* is a monomorphism, it follows that $S(M)/Q(M)$ is torsion. But, since $S(M)$ is torsionfree and $Q(M)$ is divisible, $S(M)/Q(M)$ is also torsionfree, hence zero.

The following special case was obtained independently by Morita and the present author.

3.4.1. Corollary. *If $_EI$ is finitely generated, in particular, if $_EI$ is principal, then $S(R) = Q(R)$.*

The hypothesis of Proposition 3.4 can be given by a number of equivalent formulations.

3.5. Proposition. *Given any injective I_R, the following statements are equivalent:*

(1). *$T(M)$ is an open subset of M in the I-adic topology.*

(2). *There exists $g: M \to I^n$ such that $T(M) =$* Ker g.

(3). *$I \phi M$ is a finitely generated E-module.*

(4). *For some natural number n, $I^n \phi M$ is a principal* $\text{End}_R(I^n)$ *module.*

(5). *There exists $g: M \to I^n$ such that* Ker $g^* = 0$.

Proof. We show that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (3) ⇒ (5) ⇒ (2) ⇒ (1).

(1) ⇒ (2). By (1), the intersection $T(M)$ of all fundamental open neighborhoods of 0 in M is open. Thus $\{0\}+ \text{Ker}\, g \subseteq T(M) \subseteq \text{Ker}\, g$, for some $g: M \to I^n$, hence (2).

(2) ⇒ (3). Suppose $f \in I \phi M$. Then Ker $g = T(M) \subseteq$ Ker f, hence there exists $h: I^n \to I$ such that $f = hg$. Let $g = \Sigma_{i=1}^n k_i g_i$, where $g_i: M \to I$, and $h = \Sigma_{i=1}^n e_i p_i$, where $e_i: I \to I$. Then

$$f = hg = \sum_{i=1}^n e_i p_i \sum_{j=1}^n k_j g_j = \sum_{i=1}^n e_i g_i,$$

and this establishes (3).

(3) ⇒ (4). Let $I \phi M$ have generators $g_1, \ldots, g_n$, say. Then $g = \Sigma_{h=1}^{n} k_h g_h$ is a generator of $I^n \phi M$. Indeed, take any $f \in I^n \phi M$, then

$$f = \sum_{i=1}^{n} k_i f_i, \; f_i = \sum_{j=1}^{n} e_{ij} g_j,$$

where $f_i \in I \phi M$ and $e_{ij} \in E$, hence

$$f = \sum_{i,j} k_i e_{ij} g_j = \sum_{i,j} k_i e_{ij} p_j \sum_{i,j} k_h g_h = e'g,$$

where $e' \in I^n \phi I^n$. Thus (4) holds.

(4) ⇒ (3). Let $I^n \phi M$ have generator g, say. Then the $p_j g$ generate $I \phi M$. Indeed, take any $f \in I \phi M$, then $k_1 f \in I^n \phi M$, hence $k_1 f = e'g$, for some $e' \in I^n \phi I^n$. Therefore

$$f = p_1 k_1 f = p_1 e' g = p_1 e' \sum_j k_j p_j g = \sum_j p_1 e' k_j p_j g,$$

and $p_1 e' k_j \in E$. Thus (3) holds.

(3) ⇒ (5). This is already contained in the proof of Proposition 3.4.

(5) ⇒ (2). Assume (5), then

$$\operatorname{Ker} g = \operatorname{Ker} (g^* \eta(M)) = \operatorname{Ker} \eta(M) = T(M),$$

and so (2) holds.

(2) ⇒ (1). This is clear.

In view of the equivalent conditions (1)–(5), it is easy to find examples for which $S(M) = Q(M)$. For instance, if M is Artinian, condition (2) is clearly satisfied. Or again, if $I = I(R)$, ${}_E I$ is principal, and so (4) holds for $M = R$ and $n = 1$.

§ 4. The density theorem

How does $Q(M)$ sit inside $S(M)$? As it is not true in general that $Q(M) = S(M)$, one may wish to prove a density theorem. However, it follows from Proposition 1.6 that $Q(M)$ is closed in the I-adic topology of $S(M)$. We shall investigate another topology on $S(M)$ which also induces the I-adic topology on $Q(M)$.

Noting that $S(M) = (I \phi M) \otimes I \subseteq I^{I \phi M}$, we mean by the *finite* topology on $S(M)$ that which is induced by the product topology of $I^{I \phi M}$ when I is taken to be discrete (see [9]). We shall describe this finite topology explicitly.

Basic open sets of $S(M)$ have the form

$$V = \{ s \in S(M) \mid (f_1)s = i_1, \ldots, (f_n)s = i_n \},$$

where the $i_k \in I$ and the $f_k \in I \phi M$. Now $f = \langle f_1, \ldots, f_n \rangle : M \to I^n$ gives rise to the canonical homomorphism $f^* : S(M) \to I^n$ considered earlier, e.g., in Lemma 3.3, and $i = \langle i_1, \ldots, i_n \rangle \in I^n$ may be written as $f^*(s_0)$, where $s_0 \in S(M)$, unless V is empty. Therefore $V = \{ s_0 \} + \operatorname{Ker} f^*$.

We thus see that the finite topology on $S(M)$ is linear, a fundamental system of open neighborhoods of 0 consisting of all $\operatorname{Ker} f^*$, where $f \in I^n \phi M$. If $g : (S(M), \mu(M)) \to (I^n, \delta_n)$ is any S-homomorphism, we may put $g\eta(M) = f$, and then $g = f^*$. Therefore, in view of Lemma 3.3, the fundamental open neighborhoods of 0 may also be described as kernels of S-homomorphisms of the free S-algebra generated by M into some (I^n, δ_n).

4.1. Proposition. *The finite topology on $S(M)$ induces the I-adic topology on $Q(M)$ and on $\bar{M} = M/T(M)$. Moreover, $S(M)$ is Hausdorff and complete in the finite topology.*

Proof. Any $f \in I^n \phi M$ gives rise to a unique $f' : \bar{M} \to I^n$ such that $f'([m]) = f(m)$, for all $m \in M$, since $T(M) \subseteq \operatorname{Ker} f$. Again, f' may be extended to a unique $f'' : Q(M) \to I^n$, since $Q(M)/\bar{M}$ is torsion and I is torsionfree divisible.

Let us look at the induced topology of $Q(M)$. A fundamental open neighborhood of 0 has the form $\operatorname{Ker} f^* \cap Q(M) = \operatorname{Ker} f''$. Here $f'' : Q(M) \to I^n$ is any R-homomorphism, or any $Q(R)$-homomorphism, in view of Proposition 1.2(4). Therefore the induced topology of $Q(M)$ is the same as the I-adic topology, and it does not matter whether we regard I^n and $Q(M)$ as R-modules or as $Q(R)$-modules.

What has been said about $Q(M)$ goes, mutatis mutandum, for $\bar{M}$.

Suppose $s \in S(M)$ lies in the intersection of all fundamental open neighborhoods of 0. Then, in particular, for each $f \in I \phi M$, $(f)s = f^*(s) = 0$, and so $s = 0$. Thus $S(M)$ is Hausdorff.

Finally, we shall prove that $S(M)$ is complete[3]. Given any directed set $(X, \leq)$, let $\{ s_x \mid x \in X \}$ be a Cauchy net on $S(M)$. Thus, for each $f \in I^n \phi M$, there exists $x_f \in X$ such that

(i) $\qquad (\forall y, z \geq x_f) s_y - s_z \in \operatorname{Ker} f^*$.

[3] This seems to be known (see [9]), but I have not seen a proof.

Replacing x_f by f, we easily deduce that

(ii) $\quad \operatorname{Ker} g^* \subseteq \operatorname{Ker} f^* \Rightarrow s_f - s_g \in \operatorname{Ker} f^*.$

Now define the set mapping $s : I \not\in M \rightarrow$ by

(iii) $\quad (f)s = (f)s_f.$

We claim that s is an E-homomorphism. For example,

$$(ef)s = (ef)s_{ef}\,(\text{by (iii)}) = (ef)^*(s_{ef}) = (ef)^*(s_f)(\text{by (ii)}) = (ef)s_f$$

$$= e((f)s_f) = e((f)s)(\text{by (iii)}).$$

The crucial step in proving that $(f+g)s = (f)s + g(s)$ is to note that $f, g : M \rightarrow I$ give rise to a single map $\langle f, g\rangle : M \rightarrow I^2$, such that $\operatorname{Ker} \langle f, g\rangle^* = \operatorname{Ker} f^* \cap \operatorname{Ker} g^*$. We omit the details.

Now take any $x \geq x_f$, then $s_f - s_x \in \operatorname{Ker} f^*$, by (i). Moreover, $s - s_f \in \operatorname{Ker} f^*$, by (iii). Hence, by addition, $s - s_x \in \operatorname{Ker} f^*$. This shows that s is the limit of the given net, and our proof is complete.

4.2. Theorem. *Assume that every torsionfree factor module of $Q(M)$ is divisible. Then the following assertions are true:*

(1). *$Q(M)$ is a dense submodule of $S(M)$ in the finite topology.*

(2). *$S(M)$ is the completion of $Q(M)$ if the latter is endowed with the I-adic topology.*

(3). $S(M) = \varprojlim \{\operatorname{Im} g \mid g : Q(M) \rightarrow I^n\}$

Proof. A fundamental open neighborhood of a point $s \in S(M)$ has the form $\{s\} + \operatorname{Ker} f^*$, where $f : M \rightarrow I^n$. We claim that every such set meets $Q(M)$, in other words, that $f^*(s) \in f^*Q(M)$. We shall prove that $f^*S(M) = f^*Q(M)$.

Indeed, we know from Lemma 3.3 that $f^*S(M)/f^*Q(M)$ is torsion. Now, $f^*Q(M) \subseteq I^n$ is torsionfree. Under the assumption of the theorem it is therefore divisible. Since $f^*S(M) \subseteq I^n$ is torsionfree, $f^*S(M)/f^*Q(M)$ is also torsionfree, hence zero.

Thus $Q(M)$ is dense in $S(M)$. In view of Proposition 4.1, $S(M)$ is the completion of $Q(M)$. It follows that $S(M)$ is the inverse limit of all

$$Q(M)/(\operatorname{Ker} f^* \cap Q(M)) \cong f^*Q(M),$$

where $f: M \to I^n$. If $g: Q(M) \to I^n$ is any R-homomorphism, let $f: M \to I^n$ be obtained by composing with $M \to Q(M)$, then $f^* \mid Q(M) = g$, since f has a unique "extension" to $Q(M)$, see the proof of Proposition 4.1. Thus $f^* Q(M) = \mathrm{Im}\, g$.

4.2.1. [6]. **Corollary.** *If every torsionfree factor module of $Q(R)$ is divisible, then the I-adic completion of $Q(R)$ is the bicommutator of I_R. If R is commutative, this is the center of the ring of endomorphisms of I_R.*

Michler has constructed an Artinian ring R for which $Q(R) = S(R)$, by Propositions 3.4 and 3.5, but for which the assumption of Corollary 4.2.1 is not satisfied.

4.2.2. Remark. If R is commutative Noetherian, P a prime ideal, and $I_R = I(R/P)$, Matlis [7] has proved more than the above: $S(R)$ is actually the ring of endomorphisms of I_R, hence this ring is commutative. Matlis also showed that $_E I = I_{S(R)}$ is then injective. It follows that S is exact, hence we may apply Proposition 2.5 and deduce that $S(R)$ is a flat left R-module and that $S(M) \cong M \otimes_R S(R)$ for every finitely generated module M. If R is a commutative Noetherian local ring and P is its maximal ideal, this specializes, in view of Proposition 1.5, to the well-known result [1] that the P-adic completion $\hat{R}$ of R is flat and that the P-adic completion of any finitely generated module M is $M \otimes \hat{R}$.

The functor $M \mapsto C(M)$ is obtained in three steps:

(i) endow M with the I-adic topology,

(ii) complete,

(iii) forget the topology.

4.2.3. Corollary. *If Q is exact, then $S = CQ$.*

Proof. This is an immediate consequence of Theorem 4.2 and Goldman's condition (G) for exactness of Q, see § 2.

Examples of the latter result are provided by Propositions 2.1, 2.2 and 2.4, also by $I = I(R/P)$ when R is commutative and P is prime, and by any injective I when R is right hereditary. In particular, we have the following:

4.2.4. Corollary. *If $I = \mathrm{Hom}_{\mathbf{Z}}(F, \mathbf{Q}/\mathbf{Z})$, where $_R F$ is free, then $S = C$.*

§ 5. The algebras of the triple S

We have made some use of the S-algebras and S-homomorphisms of the triple

(S, η, μ) associated with the injective I. These form the so-called Eilenberg-Moore category of the triple. It may be of interest to identify them, at least in case Q is exact.

To get our bearing, let us first look at the free algebra $(S(M), \mu(M))$. We have seen that $S(M)$ is torsionfree and divisible in the I-torsion theory and that it is Hausdorff in the I-adic topology. It is also complete, not in general in the I-adic topology, but in the finite topology, which is coarser than the I-adic one. Actually, every torsionfree divisible module which is Hausdorff in the I-adic topology and complete in a coarser topology is in fact the underlying module of an S-algebra. However, we do not know that all S-algebras must be of this form. Instead of requiring completeness in a coarser topology, we shall postulate a limit process which makes all Cauchy nets converge.

5.1. Definition. Let A be a Hausdorff topological R-module and λ a function which assigns to each Cauchy net $\{a_x \mid x \in X\}$ an element of A denoted by $\lambda\{a_x \mid x \in X\} = \lambda_{x \in X} a_x$. We call λ a *limit operation* on A provided it satisfies the following conditions:

L1. $\lambda_{x \in X}(a_x r) = (\lambda_{x \in X} a_x)r$.

L2. $\lambda_{x \in X}(a_x + b_x) = \lambda_{x \in X} a_x + \lambda_{x \in X} b_x$.

L3. If $\{a_x \mid x \in X\}$ converges to a in the given topology, then $\lambda_{x \in X} a_x = a$.

L4. If $\{a_{x,y} \mid (x, y) \in X \times Y\}$ and $\{\lambda_{y \in Y} a_{x,y} \mid x \in X\}$ are Cauchy nets, then $\lambda_{(x,y) \in X \times Y} a_{x,y} = \lambda_{x \in X} \lambda_{y \in Y} a_{x,y}$.

If A is complete, there is exactly one limit operation, the usual limit. But even if A becomes Hausdorff and complete in a module topology which is coarser than the given topology, the limit in the new topology will be a limit operation.

We point out two consequences of the axioms.

L3′. If $a_x = a$ for all $x \in X$, then $\lambda_{x \in X} a_x = a$.

L4′. If $\{a_{f(y)} \mid y \in Y\}$ is a cofinal subnet of the Cauchy net $\{a_x \mid x \in X\}$, then $\lambda_y a_{f(y)} = \lambda_x a_x$.

Indeed, L3′ follows immediately from L3. To show L4′, consider the net $\{a_x - a_{f(y)} \mid (x, y) \in X \times Y\}$. This is not only Cauchy, but actually converges to 0 in the given topology. For let U be any neighborhood of 0, then there exists $x_U \in X$ such that

$$(\forall x, x' \geq x_U)\ a_x - a_{x'} \in U.$$

Now, by cofinality, there exists $y_U \in Y$ such that $f(y_U) \geq x_U$, hence also

$$(\forall y \geq y_U) f(y) \geq x_U.$$

It follows that

$$(\forall x \geq x_U)(\forall y \geq y_U) a_x - a_{f(y)} \in U.$$

Thus $\lambda_{(x,y)}(a_x - a_{f(y)}) = 0$, by L3. On the other hand, by L4, it is

$$\lambda_x \lambda_y (a_x - a_{f(y)}) = \lambda_x a_x - \lambda_y a_{f(y)},$$

using L1, L2 and L3′, hence $\lambda_x a_x = \lambda_y a_{f(y)}$.

5.2. Definition. Let (A, λ) and (A', λ') be Hausdorff topological modules equipped with limit operations λ and λ', respectively. By a *limit preserving* homomorphism $\varphi : (A, \lambda) \to (A', \lambda')$ we mean a homomorphism $\varphi : A \to A'$ such that

$$\varphi(\lambda_{x \in X} a_x) = \lambda'_{x \in X} \varphi(a_x)$$

whenever $\{ a_x \mid x \in X \}$ and $\{ \varphi(a_x) \mid x \in X \}$ happen to be Cauchy nets. We do not require that φ be continuous in the given topologies.

We note that, when A and A' are complete, φ preserves the usual limits if and only if it is continuous.

5.3. Lemma. *Let (A, λ) be a Hausdorff topological module A equipped with a limit operation λ, and let $\hat{A}$ be the completion of A. Then there exists a unique limit preserving homomorphism $\alpha : (\hat{A}, \lim) \to (A, \lambda)$ extending the identity mapping on A.*

Proof. Let N be the set of all open neighborhoods of 0 in A, directed by stipulating that $V \geq U$ means $V \subseteq U$. For each $c \in \hat{A}$ and $U \in N$, $\{ c \} + U$ meets A, hence there exists $a_U \in A$ such that $c - a_U \in U$.

It follows that $\{ a_U \mid U \in N \}$ is a Cauchy net on A, and we may consider $\lambda_{U \in N} a_U$. Suppose also $c - a'_U \in U$ for all $U \in N$, then $a_U - a'_U \in A \cap (U \backslash U)$, and we easily see that $\{ a_U - a'_U \mid U \in N \}$ converges to 0 in A, hence also

$$\lambda_U a_U - \lambda_U a'_U = \lambda_U (a_U - a'_U) = 0,$$

by L1, L2 and L3. Thus we may define $\alpha(c) = \lambda_{U \in N} a_U$, and it is easily verified that α is an R-homomorphism $A \to A$. Moreover, by L3′, $\alpha(a) = \lambda_{U \in N} a = a$, for all $a \in A$, hence α extends the identity map $A \to A$.

We shall now prove that α is limit preserving. Consider any Cauchy net $\{c_x \mid x \in X\}$ on $\hat{A}$ with limit c. Then, for each $U \in N$, there exists $\xi(U) \in X$ such that

$$(\forall x \geq \xi(U))c - c_x \in U.$$

Moreover, by density, there exist $a_{x,U} \in A$ such that

$$c_x - a_{x,U} \in U.$$

Pick $U' \in N$ such that $U' + U' \subseteq U$. Then, for all $x \geq \xi(U')$ and all $V \geq U'$, we have

$$c - a_{x,V} = (c - c_x) + (c_x - a_{x,V}) \in U' + V \subseteq U' + U' \subseteq U.$$

It follows that $\{a_{x,V} \mid (x, V) \in X \times N\}$ is a Cauchy net on A and that

$$\alpha(c) = \lambda_{(x,V)} a_{x,V}.$$

To see that α is limit preserving, we may assume that $\{\alpha(c_x) \mid x \in X\}$ is also a Cauchy net. Then, by L4,

$$\alpha(c) = \lambda_x \lambda_V a_{x,V} = \lambda_x \alpha(c_x).$$

Finally, since α preserves limits, its definition is forced, hence α is unique.

5.4. Theorem. *Assume Q is exact. Then the following two categories are isomorphic:*

(i) *the category of S-algebras and S-homomorphisms;*

(ii) *the category of I-torsionfree and I-divisible modules, equipped with a limit operation for the I-adic topology, and limit preserving homomorphisms.*

Proof. We recall from Proposition 1.6 that when A is I-torsionfree then it is Hausdorff in the I-adic topology. We shall establish a one-to-one correspondence between the pairs (A, λ) in the second category and the S-algebras (A, α) with the same underlying module A, and we shall prove that an R-homomorphism is limit preserving if and only if it is an S-homomorphism. The proof will consist of five parts.

(1). Let (A, λ) be given, where A is torsionfree divisible and λ is a limit operation. Then $\eta(A): A \to S(A)$ is mono and embeds A as a dense submodule into its completion $S(A)$, by Theorem 4.2. Now, by Lemma 5.3, there exists $\alpha: (S(A), \lim) \to$

(A, λ) such that $\alpha\eta(A) = 1$. It remains to show that $\alpha S(\alpha) = \alpha\mu(A)$.

First, we note that

$$\alpha S(\alpha)\,\eta S(A) = \alpha\eta(A)\alpha = \alpha = \alpha\mu(A)\,\eta S(A),$$

by naturality of η, and because $\mu(A)\eta S(A) = 1$ in any triple.

Next, we observe that $\mu(A)$ and $S(\alpha)$ are continuous maps from $S^2(A)$ to $S(A)$, if both are equipped with the finite topology. Indeed, easy calculations show that for each $f : A \to I^n$,

$$\mu(A)^{-1}\,\mathrm{Ker}\, f^* = \mathrm{Ker}\,(f^*)^*,$$

and

$$S(\alpha)^{-1}\,\mathrm{Ker}\, f^* = \mathrm{Ker}\,(f\alpha)^*.$$

Thus $\alpha\mu(A)$ and $\alpha S(\alpha)$ are limit preserving homomorphisms $(S^2(A), \lim) \to (A, \lambda)$.

Take any $\sigma \in S^2(A)$, then for each $g : S(A) \to I^n$, there exists $s_g \in S(A)$ such that

$$\sigma - \eta S(A)(s_g) \in \mathrm{Ker}\, g^*.$$

Thus

$$\sigma = \lim_g \eta S(A)(s_g)$$

in the finite topology of $S^2(A)$. We plan to compare $\alpha\mu(A)(\sigma)$ with $\alpha S(\alpha)(\sigma)$, but first we must verify that $\{\, s_g \mid g : S(A) \to I^n \,\}$ and $\{\, \alpha(s_g) \mid g : S(A) \to I^n \,\}$ are Cauchy nets in the I-adic topologies of $S(A)$ and A respectively. We write $g' \geq$ for $\mathrm{Ker}\, g'^* \subseteq \mathrm{Ker}\, g$.

For each $g' \geq g$, we have

$$\eta S(A)(s_g) - \eta S(A)(s_{g'}) \in \mathrm{Ker}\, g^*,$$

whence it easily follows that

$$s_g - s_{g'} \in \mathrm{Ker}\, g^*\,\eta S(A) = \mathrm{Ker}\, g,$$

and so the first net is Cauchy. Take any $f : A \to I^n$, and suppose $g, g' \geq f\alpha$, then by the above

$$s_g - s_{g'} \in \operatorname{Ker} f\alpha,$$

hence

$$\alpha(s_g) - \alpha(s_{g'}) \in \alpha\operatorname{Ker} f\alpha \subseteq \operatorname{Ker} f,$$

and so the second net is Cauchy. Therefore $\lambda_g\alpha(s_g)$ exists. On the one hand this is

$$\lambda_g \alpha\eta(A)\alpha(s_g) = \lambda_g \alpha A(\alpha)\eta S(A)(s_g) = \alpha S(\alpha)\,\lambda_g\, \eta S(A)(s_g) = \alpha S(\alpha)(\sigma),$$

by naturality of η, and since $\alpha S(\alpha)$ is limit preserving. On the other hand it is

$$\lambda_g \alpha\mu(A)\,\eta S(A)(s_g) = \alpha\mu(A)\,\lambda_g \eta S(A)(s_g) = \alpha\mu(A)(\sigma),$$

since $\alpha\mu(A)$ is limit preserving. Thus $\alpha S(\alpha)(\sigma) = \alpha\mu(A)(\sigma)$, as was to be shown.

(2). Let (A, α) be a given S-algebra. Since $\alpha\eta(A) = 1$, $\eta(A)$ is mono, hence A is torsionfree. We may regard A as a submodule of $Q(A)$. By Corollary 3.1.1, for each $q \in Q(A)$, $S\eta(A)(q) = \eta S(A)(q)$, hence

$$q = S(\alpha)S\eta(A)(q) = S(\alpha)\,\eta S(A)(q) = \eta(A)\alpha(q),$$

and so $q \in \operatorname{Im} \eta(A)$. Thus A is divisible.

Let $\{ a_x \mid x \in X \}$ be any Cauchy net on A in the I-adic topology. We define

$$\lambda_{x\in X} a_x = \alpha \lim_{x \in X} \eta(A)(a_x).$$

We claim that λ is a limit operation on A. Indeed, conditions L1 to L3 are easily checked; we shall skip the verification. The proof of L4 is a little more interesting, as it involves the equation $\alpha S(\alpha) = \alpha\mu(A)$.

Assume that $\{ a_{x,y} \mid (x, y) \in X \times Y\}$ and $\{ \lambda_{y\in Y} a_{x,y} \mid x \in X\}$ are Cauchy nets on A. Then

$$\begin{aligned}
\lambda_x\lambda_y a_{x,y} &= \alpha \lim_x \eta(A)\,\alpha \lim_y \eta(A)(a_{x,y}) \\
&= \alpha \lim_x S(\alpha)\,\eta S(A) \lim_y \eta(A)(a_{x,y}) \\
&= \alpha S(\alpha) \lim_x \eta S(A) \lim_y \eta(A)(a_{x,y}) \\
&= \alpha\mu(A) \lim_x \eta S(A) \lim_y \eta(A)(a_{x,y})
\end{aligned}$$

$$= \alpha \lim_x \mu(A)\eta S(A) \lim_y \eta(A)(a_{x,y})$$

$$= \alpha \lim_x \lim_y \eta(A)(a_{x,y}) = \alpha \lim_{(x,y)} \eta(A)(a_{x,y}) = \lambda_{(x,y)} a_{x,y}.$$

(3). We have shown in (1) how to define α in terms of λ, and in (2) how to define λ in terms of α. It is easily seen that these two definitions give a one-to-one correspondence between pairs (A, λ) and pairs (A, α).

For the remaining two parts of the proof we require a lemma.

5.5. Lemma. *Let $\varphi : A \to B$ be an R-homomorphism (not necessarily continuous). Then:*

(i) φ sends I-adic Cauchy nets on A onto I-adic Cauchy nets on B.

(ii) $S(\varphi)$ is a continuous homomorphism of $S(A)$ to $S(B)$ in their finite topologies.

Proof. (i). Let $\{a_x \mid x \in X\}$ be a Cauchy net on A. Then, for each $f: A \to I^n$, there exists $\xi(f) \in X$ such that

$$(\forall x \geq \xi(f))\, a_x - a_{\xi(f)} \in \operatorname{Ker} f.$$

Now consider any $g: B \to I^n$. Then

$$(\forall x \geq \xi(g\varphi))a_x - a_{\xi(g\varphi)} \in \varphi^{-1} \operatorname{Ker} g,$$

that is,

$$\varphi(a_x) - \varphi(a_{\xi(g\varphi)}) \in \operatorname{Ker} g.$$

(ii). Take any fundamental open neighborhood $\operatorname{Ker} g^*$ of 0 on $S(B)$, where $g: B \to I^n$, then

$$S(\varphi)^{-1} \operatorname{Ker} g^* = \operatorname{Ker} \delta_n\, S(g)S(\varphi) = \operatorname{Ker} (g\varphi)^*,$$

which is clearly a fundamental open neighborhood of 0 on $S(A)$.

We now continue with the proof of Theorem 5.4.

(4). Suppose $\varphi : (A, \alpha) \to (A', \alpha')$ is an S-homomorphism. Let $\{a_x \mid x \in X\}$ be a Cauchy net on A. By Lemma 5.5, $\{\varphi(a_x) \mid x \in X\}$ is a Cauchy net on A' and $S(\varphi)$ is limit preserving. Therefore

$$\lambda'_{x\in X}\varphi(a_x) = \alpha' \lim_{x\in X} \eta(A')\varphi(a_x) = \alpha' \lim_{x\in X} S(\varphi)\eta(A)(a_x)$$

$$= \alpha' S(\varphi) \lim_{x\in X} \eta(A)(a_x) = \varphi\alpha \lim_{x\in X} \eta(A)(a_x) = \varphi\lambda_{x\in X} a_x,$$

and therefore φ is limit preserving.

(5). Suppose $\varphi : (A, \lambda) \to (A', \lambda')$ is a limit preserving homomorphism. Take any element $s \in S(A)$, then $s = \lim_{x\in X} \eta(A)(a_x)$, where $\{ a_x \mid x \in X \}$ is a Cauchy net on A. By Lemma 5.5, $\{ \varphi(a_x) \mid x \in X \}$ is a Cauchy net on A' and $S(\varphi)$ is limit preserving. Therefore

$$\varphi\alpha(s) = \varphi\lambda_{x\in X} a_x = \lambda'_{x\in X}\varphi(a_x) = \alpha' \lim_{x\in X} \eta(A')\varphi(a_x)$$

$$= \alpha' \lim_{x\in X} S(\varphi)\eta(A)(a_x) = \alpha' S(\varphi) \lim_{x\in X} \eta(A)(a_x) = \alpha' S(\varphi)(s),$$

and therefore φ is an S-homomorphism.

The method employed in the proof of Theorem 5.4 can be used to establish a more general result.

Let P be a class of modules closed under isomorphic images, finite products and submodules. By pro-P we shall understand the category of pro-P modules, that is, inverse limits of modules in P, and continuous homomorphisms (in the inverse limit topology). The inclusion pro-$P \to$ Mod R has a left adjoint (see [5, § 3]): with each R-module M we associate

$$C_P(M) = \varprojlim \{ M/K \mid M/K \in P \}.$$

We may regard C_P as an endo-functor of Mod R, then it gives rise to a triple (C_P, η_P, μ_P).

5.6. Proposition. *The category of algebras of the triple (C_P, η_P, μ_P) is isomorphic to the following category: Its objects are pairs (A, λ), where A is an R-module and λ is a limit operation on A with respect to the P-topology on A; its maps are limit preserving homomorphisms.*

References

[1] M.F. Atiyah and I.G. Macdonald, Introduction to commutative algebra (Addison-Wesley, Reading, Mass., 1969).

[2] S. Eilenberg and J.C. Moore, Adjoint functors and triples, Illinois J. Math. 9 (1965) 381–398.
[3] S. Fakir, Monade idempotente associée à une monade, C.R. Acad. Sci. Paris 270 (1970) 99–101.
[4] O. Goldman, Rings and modules of quotients, J. Algebra 13 (1969) 10–47.
[5] J. Lambek, Torsion theories, additive semantics, and rings of quotients, Lecture Notes in Math. 177 (Springer, Berlin, 1971).
[6] J. Lambek, Bicommutators of nice injectives, J. Algebra 21 (1972) 60–73.
[7] E. Matlis, Injective modules over Noetherian rings, Pacific J. Math. 8 (1958) 511–528.
[8] K. Morita, Flat modules, injective modules and quotient rings, Math. Z. 120 (1971) 25–40.
[9] B.J. Muller, Linear compactness and Morita duality, J. Algebra 16 (1970) 60–66.
[10] B. Rattray, Non-additive torsion theories, to appear.
[11] B. Stenström, Rings and modules of quotients, Lecture Notes in Math. 237 (Springer, Berlin, 1971).