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Topology and its Applications 155 (2008) 2027–2030
Topology and its Applications
www.elsevier.com/locate/topol

Automorphisms in spaces of continuous functions on Valdivia compacta

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Received 4 September 2006; accepted 24 July 2007

Abstract

We show that there are no automorphic Banach spaces of the form $C(K)$ with K continuous image of Valdivia compact except the spaces $c_0(\Gamma)$. Nevertheless, when K is an Eberlein compact of finite height such that $C(K)$ is not isomorphic to $c_0(\Gamma)$, all isomorphism between subspaces of $C(K)$ of size less than $\aleph_\omega$ extend to automorphisms of $C(K)$.

MSC: 46B26

Keywords: Automorphism; Automorphic space; Eberlein compact; Valdivia compact; Space of continuous functions

Introduction

A Banach space X is said to be automorphic if for every isomorphism $T : Y_1 \to Y_2$ between two (closed) subspaces of X with $\mathrm{dens}(X/Y_1) = \mathrm{dens}(X/Y_2)$ there exists an automorphism $\tilde{T} : X \to X$ which extends T, that is, $\tilde{T}|_{Y_1} = T$. It has been shown in [9] that a necessary condition for a Banach space X to be automorphic is to be extensible, which means that for every subspace $E \subset X$ and every operator $T : E \to X$, there exists an operator $\tilde{T} : X \to X$ that extends T. Clearly every Hilbert space $\ell_2(\Gamma)$ is automorphic and on the other hand, Lindenstrauss and Rosenthal [7] have proven that c_0 is automorphic and also that ℓ_∞ has a partial automorphic character, namely that isomorphisms $T : Y_1 \to Y_2$ can be extended provided that ℓ_∞/Y_i is nonreflexive for $i = 1, 2$, though ℓ_∞ is not automorphic. Moreno and Plichko [9] have recently shown that $c_0(\Gamma)$ is automorphic for every set Γ. It remains open the question posed in [7] whether the only automorphic separable Banach spaces are ℓ_2 and c_0 and also the more general question whether all automorphic Banach spaces are isomorphic either to $\ell_2(\Gamma)$ or to $c_0(\Gamma)$ for some set Γ. Our aim in this note is to address this latter problem for the case of Banach spaces $C(K)$ of continuous functions on compact spaces. Must

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[1] The author was supported by a Marie Curie Intra-European Felloship MCEIF-CT2006-038768 (E.U.) and research projects MTM2005-08379 and Séneca 00690/PI/04 (Spain).
[2] The author was partially supported by project MTM2004-02635.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.07.007

be an automorphic $C(K)$ space isomorphic to $c_0(\Gamma)$? We provide a positive answer to this problem in the case when K is a continuous image of a Valdivia compact, which is a large class of compact spaces originated from functional analysis and which includes for example all Eberlein and all dyadic compact spaces.

Namely, a compact space is said to be a Valdivia compact if it is homeomorphic to some $K \subset \mathbb{R}^\Gamma$ in such a way that the elements of K of countable support are dense in K (the support of $x \in \mathbb{R}^\Gamma$ is the set of nonzero coordinates). If such K can be found so that all elements of K have countable support, then the compact is said to be a Corson compact, and if moreover it can be taken so that $K \subset c_0(\Gamma) \subset \mathbb{R}^\Gamma$, then it is called an Eberlein compact.

Theorem 1. *Let K be a continuous image of a Valdivia compact. If $C(K)$ is extensible, then $C(K)$ is isomorphic to $c_0(\Gamma)$.*

Although it is a standard notion, we recall now what a scattered compact is. The derived space of a topological space X is the space X' obtained by deleting from X its isolated points. The derived sets $X^{(\alpha)}$ are defined recursively setting $X^{(0)} = X$, $X^{(\alpha+1)} = [X^{(\alpha)}]'$ and $X^{(\beta)} = \bigcap_{\gamma<\beta} X^{(\gamma)}$ for β a limit ordinal. The space X is scattered if $X^{(\alpha)} = \emptyset$ for some α, and in this case the minimal such α is called the height of X.

In Theorem 1, if K is not a scattered compact of finite height, then the extensible property fails at the separable level, meaning that in that case $C(K)$ contain both a complemented and noncomplemented copy of the same separable space. The most delicate case of Theorem 1 happens when K is a scattered compact of finite height and it relies on some results of [2]. In this situation K is indeed an Eberlein compact and the special behavior of $c_0(\Gamma)$ when $|\Gamma| < \aleph_\omega$ studied in [2,3] and [5] combined with the general results about $c_0(\Gamma)$ from [9] yield that we have the extensible and automorphic properties for subspaces of density less than $\aleph_\omega$.

Theorem 2. *Let K be an Eberlein compact of finite height.*

(1) *For every isomorphism $T : Y_1 \to Y_2$ between two subspaces of X with $\operatorname{dens}(Y_1) = \operatorname{dens}(Y_2) < \aleph_\omega$ and $\operatorname{dens}(C(K)/Y_1) = \operatorname{dens}(C(K)/Y_2)$ there exists an automorphism $\tilde{T} : C(K) \to C(K)$ that extends T.*
(2) *For every subspace $Y \subset C(K)$ with $\operatorname{dens}(Y) < \aleph_\omega$ and every operator $T : Y \to C(K)$, there exists an operator $\tilde{T} : C(K) \to C(K)$ that extends T.*

Only recently, Bell and Marciszewski [3] have constructed an Eberlein compact space of height 3 and weight $\aleph_\omega$ that is not isomorphic to $c_0(\Gamma)$, where $|\Gamma| = \aleph_\omega$. It was shown by Godefroy, Kalton and Lancien [5] that if K is an Eberlein compact of finite height and weight less than $\aleph_\omega$, then $C(K)$ is isomorphic to $c_0(\Gamma)$, cf. also [3] and [8].

The most typical example of scattered compact which is not Eberlein is a Mrówka space, that is, a separable uncountable scattered compact space K of height three and $|K^{(2)}| = 1$. In this case $C(K)$ is not extensible, cf. Proposition 4. However, it is unclear to us whether there may exist a scattered compact space such that $C(K)$ is extensible but not isomorphic to any $c_0(\Gamma)$.

This research was done when both authors were visiting the National Technical University of Athens. We are specially indebted to Spiros Argyros for his valuable help and suggestions.

Proof of Theorem 1

Let us first observe that if $C(K)$ is an extensible space then K does not contain any copy of the ordinal interval $[0, \omega^\omega]$. Indeed, if we had $[0, \omega^\omega] \subset K$, by the Borsuk–Dugundji extension theorem, $C[0, \omega^\omega]$ is a complemented subspace of $C(K)$. But it is known (see [10]) that $C[0, \omega^\omega]$ contains an uncomplemented copy of itself, so $C(K)$ contains both complemented and uncomplemented copies of $C[0, \omega^\omega]$ and so $C(K)$ is not extensible.

A result of Kalenda asserts that a continuous image of a Valdivia compact which does not contain the ordinal interval $[0, \omega_1]$ is a Corson compact. Any Corson compact is monolithic (that is, every separable closed subset has countable weight) and for monolithic spaces we have the following, which is probably a known fact:

Proposition 3. *Let K be a monolithic compact which does not contain any copy of $[0, \omega^\omega]$, then K is a scattered compact of finite height.*

Proof. First, if K was not scattered, then there is a continuous surjection from K onto the unit interval, $f: K \to [0,1]$. For every rational point $q \in [0,1]$ we choose $x_q \in K$ with $f(x_q) = q$. Let L be the closure of the set of points $\{x_q\colon q \in \mathbb{Q} \cap [0,1]\}$, which is a metrizable compact since K is monolithic. Moreover, f maps L onto $[0,1]$, so L contains a perfect compact set, which being metrizable, contains a further copy of the Cantor set $\{0,1\}^{\mathbb{N}}$ and in particular a copy of $[0, \omega^\omega]$, contrary to our assumption. Thus, we proved that K must be scattered.

Suppose now that K was a scattered compact of infinite height. For every $n \in \mathbb{N}$ let $A_n = K^{(n)} \setminus K^{(n+1)}$ be the nth level of K. Since the height of K is infinite, A_n is nonempty for every $n \in \mathbb{N}$. Indeed, A_n is an infinite set which is dense in $K^{(n)}$. We observe that for $n > 0$, every element $x \in A_n$ is the limit of a sequence of elements of A_{n-1} (take U a clopen set which isolates x inside $K^{(n)}$, then $U \cap A_{n-1}$ is infinite and any sequence contained in $U \cap A_{n-1}$ converges to x). For every n we take countable sets $B_{n,n} \subset A_n$, $B_{n,n-1} \subset A_{n-1}, \ldots, B_{n,0} \subset A_0$ in the following way:

- $B_{n,n}$ is an arbitrary countably infinite subset of A_n.
- $B_{n,n-1}$ is a countable subset of A_{n-1} such that every element of $B_{n,n}$ is the limit of a sequence of elements of $B_{n,n-1}$.
- $B_{n,k}$ is a countable subset of A_k such that every element of $B_{n,k+1}$ is the limit of a sequence of elements of $B_{n,k}$.

Let $B_n = B_{n,n} \cup B_{n,n-1} \cup \cdots \cup B_{n,0}$. Notice that B_n is a scattered topological space of height $n+1$ with $B_n^{(k)} = B_{n,n} \cup B_{n,n-1} \cup \cdots \cup B_{n,k}$. Let $L = \overline{\bigcup_{n=0}^{\infty} B_n}$. The compact L is a scattered compact of infinite height and it is moreover metrizable because K is monolithic. Any metrizable scattered compact is homeomorphic to an ordinal interval, and since L has infinite height, $[0, \omega^\omega] \subset L$. □

In order to prove Theorem 1 we shall assume by contradiction that there exists some compact space K which is a continuous image of Valdivia compact, with $C(K)$ extensible and not isomorphic to $c_0(\Gamma)$. The previous discussion shows that any such K must be scattered compact of finite height. Hence, we can choose one such compact K_0 of minimal height. We shall work with this K_0 towards getting a contradiction.

Let Δ be the set of isolated points of K_0, so that $K_0' = K_0 \setminus \Delta$, $K_0 = K_0' \cup \Delta$. We consider the restriction operator $S: C(K_0) \to C(K_0')$ for which $\ker(S) = c_0(\Delta)$ and we have a short exact sequence

$$0 \to c_0(\Delta) \to C(K_0) \to C(K_0') \to 0. \tag{$\star$}$$

By [2, Theorem 1.2], there exists $\tilde{\Delta} \subset \Delta$ with $|\tilde{\Delta}| = |\Delta|$ such that $c_0(\tilde{\Delta})$ is complemented in $C(K_0)$. Since $C(K_0)$ is extensible, it follows that, being $c_0(\tilde{\Delta})$ a complemented subspace of X, also $c_0(\Delta)$ is a complemented subspace of $C(K_0)$. Therefore, the short exact sequence ($\star$) splits and we have

$$C(K_0) = c_0(\Delta) \oplus C(K_0').$$

In particular, $C(K_0')$ is a complemented subspace of $C(K_0)$ and therefore $C(K_0')$ is also extensible. Moreover, K_0' has height one unit less than the height of K_0, so by the minimality property used to choose K_0 we conclude that $C(K_0')$ is isomorphic to $c_0(\Gamma)$ for some Γ. But then

$$C(K_0) \cong c_0(\Delta) \oplus C(K_0') \cong c_0(\Delta) \oplus c_0(\Gamma) \cong c_0(\Delta \cup \Gamma),$$

a contradiction since $C(K_0)$ was not isomorphic to any $c_0(\Lambda)$. This finishes the proof of Theorem 1.

Let us note that we did not use the full strength of the assumption of $C(K)$ being extensible in the hypothesis of Theorem 1. We only needed that $C(K)$ does not contain both complemented and uncomplemented copies of the same space X, for the spaces $X = C[0, \omega^\omega]$ and $X = c_0(\Gamma)$.

We include now the proof that Mrówka compacta do not provide extensible Banach spaces, which uses similar ideas as in the preceding arguments:

Proposition 4. *Let K be a Mrówka space. Then $C(K)$ contain both complemented and uncomplemented copies of c_0.*

Proof. K contains convergent sequences, that is, a copy of $[0, \omega]$, so by the Borsuk–Dugundji extension theorem, it contains a complemented copy of $C[0, \omega] \cong c_0$. On the other hand, let Δ be the countable set of the isolated points. Like above, $c_0(\Delta)$ is the kernel of the restriction operator $C(K) \to C(K')$. It is well known that $c_0(\Delta)$ is not complemented in $C(K)$ in this case. One argument to see this is the following: Suppose $c_0(\Delta)$ was complemented in $C(K)$. Then $C(K) \cong c_0(\Delta) \oplus C(K')$. The space K' is homeomorphic to the one point compactification of a discrete set Γ, so $C(K') \cong c_0(\Gamma)$ and $C(K) \cong c_0(\Delta) \oplus c_0(\Gamma)$. This implies $C(K)$ is a weakly compactly generated space, and therefore K is an Eberlein compact. Every separable Eberlein compact has countable weight and a Mrówka space is separable but has uncountable weight (we refer to [4] for reference to standard properties of weakly compactly generated spaces and Eberlein compact spaces). □

Proof of Theorem 2

Let us first note that a continuous image K of Valdivia compact which is scattered compact of finite height is an Eberlein compact. We use again Kalenda's result [6] that K must be either Corson or contain a copy of $[0, \omega_1]$, and the latter possibility cannot happen since K has finite height. It is a result of Alster [1] that every scattered Corson compact is an Eberlein compact.

We state now a result from [5] mentioned in the introduction:

Theorem 5 *(Godefroy, Kalton, Lancien). If Q is an Eberlein compact of finite height and $w(Q) = \aleph_m < \aleph_\omega$, then $C(Q)$ is isomorphic to $c_0(\aleph_m)$.*

In the view of this and of the fact that $c_0(\Gamma)$ is an automorphic and hence also extensible space, we are concerned in Theorem 2 with the case when $w(K) \geqslant \aleph_\omega$. So let K be an Eberlein compact of finite height and weight not lower than $\aleph_\omega$ and let $T : Y_1 \to Y_2$ be an isomorphism between subspaces of $C(K)$ such that $\operatorname{dens}(Y_1) = \operatorname{dens}(Y_2) = \aleph_n < \aleph_\omega$. Our aim is to find an automorphism of $C(K)$ that extends T.

Let Z be a subspace of $C(K)$ of density character $\aleph_{n+1}$ such that $Y_1 + Y_2 \subset Z$. We define an equivalence relation $\sim$ on K in the following way:

$$p \sim q \quad \Leftrightarrow \quad y(p) = y(q) \quad \text{for every } y \in Z.$$

The quotient $L = K/\sim$ with the quotient topology is a compact space and the quotient map $K \to L$ a continuous surjection which allows us to view $C(L)$ as a subspace of $C(K)$ such that $Z \subset C(L)$. Moreover, since the space Z separates the points of L and has density character $\aleph_{n+1}$, $w(L) = \aleph_{n+1}$. Now, by Theorem 5, $C(L)$ is isomorphic to $c_0(\aleph_{n+1})$ and we know by [9] that this space is automorphic. Hence, since $\operatorname{dens}(C(L)/Y_1) = \aleph_{n+1} = \operatorname{dens}(C(L)/Y_2)$, there exists an automorphism $\hat{T} : C(L) \to C(L)$ that extends T. Finally, by [2, Theorem 1.1] every copy of $c_0(\aleph_{n+1})$ in a weakly compactly generated space is complemented, so $C(L)$ is complemented in $C(K)$ and this allows us to obtain an automorphism $\tilde{T} : C(K) \to C(K)$ that extends $\hat{T}$. This finishes the proof of part (1) of Theorem 2.

Part (2) of Theorem 2 is a consequence of part (1) by [9, Theorem 3.1] (this theorem only states that an automorphic space must be extensible but the proof shows that if the automorphic property holds for a given subspace then so does the extensible property). Alternatively, part (2) can also be proven by an argument which is completely analogous to that of part (1).

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