Infra-Mackey spaces, weak barrelledness and barrelledness

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Abstract

A weaker Mackey topology, infra-Mackey topology, is introduced. For an infra-Mackey space, dual local quasi-completeness, $c_0$-quasi-barrelledness, Ruess’ property (quasi-L) and $C$-quasi-barrelledness are equivalent to each other. Inspired by the definition of Mazur spaces, locally convex spaces are classified according to various conditions ensuring linear functionals continuous. In the classification, every class of special locally convex spaces is characterized by some completeness of the duals. From this, some new characterizations of quasi-barrelledness and barrelledness are given.

Keywords: Infra-Mackey spaces; Weak barrelledness; Barrelledness

1. Introduction

In this paper, every space will be assumed a Hausdorff locally convex space over the scalar field of real or complex numbers. Let $(E, t)$ be a space; then $(E, t)'$, or briefly $E'$, denotes the topological dual of $(E, t)$ and $E^\#$ denotes the algebraic dual of $E$. Mazur defined a space $E$ to be $C$-[quasi-]barrelled [1, Definition 8.2.6] if $U := \bigcap_{n=1}^{\infty} U_n$ is a neighborhood of 0 whenever $(U_n)$ is a sequence of absolutely convex closed neighborhoods of 0 such that any given singleton [bounded] set is contained in $U_n$ for almost all $n$. According to Webb, a space $(E, t)$ is called to be $c_0$-[quasi-]barrelled if each $\sigma(E', E)$-null [$\beta(E', E)$-null] sequence is $t$-equicontinuous (see, [1, Definition 8.2.22] or [2]). An increasing sequence $\sigma = \{A_n; n \in \mathbb{N}\}$ of absolutely convex subsets of a space $(E, t)$ is said to be absorbing (bornivorous) if for every $x$ in $E$ (every bounded subset $B$ of $E$) there

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is a positive integer $m$ such that $A_m$ absorbs $x$ (absorbs $B$); see [1, Definition 8.1.15]. We denote by $t_\sigma$ the finest locally convex topology on $E$ that induces the same topology as $t$ on each $A_n$. The topology $t_\sigma$ is defined by the family of those seminorms whose restrictions to the sets $A_n$ are continuous for the topology induced on $A_n$ by $t$; see [3]. Absorbing and bornivorous sequences of absolutely convex sets were considered by Valdivia, De Wilde, Houet, Garling, Roelcke, Ruess, et al. (cf. [1, 8.9]). Ruess [4] defined a space $(E, t)$ to have property ([quasi-L]) if $t = t_\sigma$ holds for each absorbing [bornivorous] sequence $\sigma$, and to have the weaker property ([quasi-LC]) if $(E, t_\sigma)' = (E, t)'$ holds. In the above definitions of Ruess’ four properties ([quasi-L] and ([quasi-LC]), it makes no difference whether the absorbing [bornivorous] sequences are required to be closed or not; see [1, Proposition 8.1.17(i)]. Obviously for a Mackey space $(E, t)$, the property ([quasi-LC]) is equivalent to the property ([quasi-L]). Recall that a space $E$ is locally complete if and only if every bounded closed absolutely convex subset of $E$ is a Banach disk; see [1, Proposition 5.1.6]. A space $E$ is said to be dual locally complete [5] if $(E', \sigma(E', E))$ is locally complete and is said to be dual locally quasi-complete [6] if $(E', \beta(E', E))$ is locally complete. For dual local [quasi-]completeness and Ruess’ property [quasi-LC], we have the following result (see [1, Proposition 8.1.29], [3,7,8]).

Theorem 1.1 (Ruess). For any space $E$, the following statements are equivalent:

(i) $E$ is dual locally [quasi-]complete.
(ii) $E$ has property ([quasi-LC]).
(iii) If $f \in E^#$ such that $f|_A$ is continuous, where $A$ is a [bornivorous] barrel in $E$, then $f \in E'$.

We know that [1, Observation 8.2.7]

$$C-[\text{quasi-}]-\text{barrelled} \Rightarrow \text{property ([quasi-L])} \Rightarrow \text{property ([quasi-LC])}$$

$$\iff \text{dual locally [quasi-] complete.}$$

Also, we have [1, Observation 8.2.23]

$$C-[\text{quasi-}]-\text{barrelled} \Rightarrow c_0-[\text{quasi-}]-\text{barrelled} \Rightarrow \text{property ([quasi-LC]).}$$

Saxon and Sánchez Ruiz [9, Theorem 3.2] proved that each Mackey dual locally complete space $E$ is C-barrelled. Their proof applies equally well to the quasi-inclusive case. Thus we have (see [9, Corollary 3.3] and [8, Theorem 4]):

Theorem 1.2 (Saxon and Sánchez Ruiz). For any Mackey space $E$, the following statements are equivalent:

(i) $E$ is dual locally [quasi-]complete.
(ii) $E$ is $c_0$-[quasi-]barrelled.
(iii) $E$ has property ([quasi-L]).
(iv) $E$ is $C$-[quasi-]barrelled.

Thus a space $E$ with dual local quasi-completeness, the weakest of the four “quasi” properties, suddenly enjoys C-quasi-barrelledness, the strongest of the four, when the space
is endowed with its Mackey topology. In fact, the Mackey topology is not unique compatible topology with the above equivalent relationship. In [8] the notion of quasi-Mackey spaces is introduced as follows.

**Definition 1.1.** A space $E$ is called a quasi-Mackey space if it has its quasi-Mackey topology; i.e., the topology induced by $(E'', \tau(E'', E'))$.

Köthe [10, §23, 4(6)] noted that the quasi-Mackey topology is compatible with the dual pair $(E, E')$ and may be strictly weaker than the Mackey topology $\tau(E, E')$.

**Theorem 1.3** [8, Theorem 5]. For a quasi-Mackey space $E$, the four statements with “quasi” in Theorem 1.2 are equivalent to each other.

From this, we have

**Theorem 1.4** [8, Corollary 2]. A space whose topology lies between its Mackey and quasi-Mackey topologies is $c_0$-quasi-barrelled if and only if it is dual locally quasi-complete.

For convenience we introduce the following notion.

**Definition 1.2.** A space $E$ is called a generalized quasi-Mackey space if its topology lies between its Mackey and quasi-Mackey topologies, or equivalently, if every $\sigma(E', E'')$-compact absolutely convex subset of $E'$ is equicontinuous.

Thus, Theorem 1.4 can be rewritten as follows:

“A generalized quasi-Mackey space is $c_0$-quasi-barrelled if and only if it is dual locally quasi-complete.”

In this paper, we first point out that there exists another compatible locally convex topology (which is strictly weaker than the quasi-Mackey topology) such that four statements with “quasi” in Theorem 1.2 are equivalent to each other. From this we obtain an improvement of Theorem 1.4. Inspired by the discussion on Mazur spaces (see [3,11]), we classify locally convex spaces according to various conditions ensuring linear functionals are continuous. In the classification, every class of special locally convex spaces is characterized by some completeness of their duals. This leads us to obtain some new characterizations of barrelledness and quasi-barrelledness.

### 2. Another kind of weak Mackey spaces

**Definition 2.1.** Let $(E, t)$ be a space and $E'$ be its topological dual. Denote $C$ the collection of all the absolutely convex $\beta(E', E)$-compact subsets of $E'$ and $C^\circ$ the polar taken in $E$ for every $C \in C$. Obviously $\{C^\circ: C \in C\}$ forms a base of 0-neighborhoods for some locally convex topology on $E$. We call the locally convex topology infra-Mackey topology and
denote it by $\kappa(E, E')$, the notation is quoted from [12, p. 235]. The space $(E, t)$ is called an infra-Mackey space if $t = \kappa(E, E')$, i.e., $\{ C^0: C \in C \}$ is a base of 0-neighborhoods in $(E, t)$. The space $(E, t)$ is called a generalized infra-Mackey space if every $C^0$ is a 0-neighborhood in $(E, t)$, where $C \in C$; or equivalently, every absolutely convex $\beta(E', E)$-compact subset of $E'$ is $t$-equicontinuous.

Clearly $\kappa(E, E')$ is a locally convex topology on $E$ which is compatible with the dual pair $\langle E', E \rangle$. Since the topology $\beta(E', E)$ is finer than one $\sigma(E', E'')$, a generalized quasi-Mackey space is always a generalized infra-Mackey space. However, the converse is not true, see the following:

Example 2.1. An infra-Mackey space which is not a generalized quasi-Mackey space.

Let $(E, \| \|)$ be an infinite-dimensional reflexive Banach space and $(E', \| \|') = (E', \beta(E', E))$ be its strong dual. Let $\kappa(E, E')$ denote the topology of the uniform convergence on absolutely convex compact subsets of $(E', \| \|')$, equivalently, the topology of the uniform convergence on compact subsets of $(E', \| \|')$; see [12, p. 235]. Then $(E, t) := (E, \kappa(E, E'))$ is an infra-Mackey space. We shall see that $(E, t)$ is not a generalized quasi-Mackey space. Remark that the dual ball $B' := \{ f \in E': \| f \|' \leq 1 \}$ is $\sigma(E', E)$-compact, i.e., $\sigma(E', E'')$-compact, but $B'$ is not relatively compact in $(E', \beta(E', E)) = (E', \| \|')$ since $E'$ is infinitely dimensional. Thus $B'$ is not $t$-equicontinuous.

Now we give a new parallel to Theorem 5 in [8].

Theorem 2.1. For an infra-Mackey space $(E, t)$, the following statements are equivalent:

(a) $(E, t)$ is dual locally quasi-complete.
(b) $(E, t)$ is $c_0$-quasi-barrelled.
(c) $(E, t)$ has property (quasi-L).
(d) $(E, t)$ is $C$-quasi-barrelled.

Proof. We know that $C$-quasi-barrelledness implies both $c_0$-quasi-barrelledness and Ruess’ property (quasi-L), either of which implies dual locally quasi-complete, respectively (see [1, Proposition 8.1.29, Observation 8.2.7 and 8.2.23]). So it suffices to prove that (a) $\Rightarrow$ (d).

Suppose that $(U_n)_{n \in N}$ is a sequence of absolutely convex closed 0-neighborhoods in $(E, t)$ such that any given bounded set is contained in $U_n$ for almost all $n$. Put $U := \bigcap_{n=1}^{\infty} U_n$, we shall show that $U$ is a 0-neighborhood in $(E, t)$. Taking any fixed 0-neighborhood $V$ in $(E', \beta(E', E))$, without loss of generality we may assume that $V = B^o$, where $B$ is a bounded set in $(E, t)$ and $B^o$ is the polar of $B$ in $E'$. Since $B$ is contained in all most $U_n$, there exists $p \in N$ such that $B \subset U_n$ for all $n \geq p$. Thus for all $n \geq p$,

$$U_n \subset B^o = V.$$ 

Since $(E, t)$ is an infra-Mackey space, for each $n$ there exists an absolutely convex $\beta(E', E)$-compact subset $K_n$ of $E'$ such that $U_n \supset K_n^o$, where $K_n^o$ denotes the polar of $K_n$.
in $E$. Obviously the $\beta(E', E)$-compact set $K_n$ is $\sigma(E', E)$-compact and hence $\sigma(E', E)$-closed. By bipolar theorem, $K_n^{oo} = K_n$, where the first polar is taken in $E$ and the second polar is taken in $E'$, and hence

$$U_n^o \subset K_n^{oo} = K_n.$$  

Since $K_n$ is $\beta(E', E)$-compact and $U_n^o$ is a $\beta(E', E)$-closed subset of $K_n$, we conclude that $C_n := U_n^o$ is $\beta(E', E)$-compact. By the assumption (a), $(E', \beta(E', E))$ is locally complete. Applying an interesting result of Saxon and Sánchez Ruiz [9, Lemma 3.1], we see that $\overline{\mathcal{T}(\bigcup_{n=1}^{\infty} C_n)}$ is $\beta(E', E)$-compact, where $\overline{\mathcal{T}(\bigcup_{n=1}^{\infty} C_n)}$ denotes the closed absolutely convex hull of $\bigcup_{n=1}^{\infty} C_n$ in $(E', \beta(E', E))$. Now the bipolar theorem implies that

$$\left(\overline{\mathcal{T}\left(\bigcup_{n=1}^{\infty} C_n\right)}\right)^o = \left(\bigcup_{n=1}^{\infty} C_n\right)^o = \bigcap_{n=1}^{\infty} C_n^{oo} = \bigcap_{n=1}^{\infty} U_n^{oo} = \bigcap_{n=1}^{\infty} U_n = U$$

is a 0-neighborhood in $(E, t)$. 

Since the topology of a generalized infra-Mackey space lies between its Mackey and infra-Mackey topologies, by Theorem 2.1 we obtain an improvement of Theorem 1.4 as follows.

**Theorem 2.2.** A generalized infra-Mackey space is dual locally quasi-complete if and only if it is $c_0$-quasi-barrelled.

**Theorem 2.3.** An infra-Mackey space $E$ has property (L) if and only if it is dual locally complete.

**Proof.** Obviously property (L) implies property (LC) and the latter is equivalent to dual locally complete (see [3, Theorem 2.3]). Conversely if $E$ is dual locally complete then $E$ is dual locally quasi-complete and $E$ has Banach–Mackey property (see [8, Theorem 3]). By Theorem 2.1, dual local quasi-completeness is equivalent to property (quasi-L). On the other hand, by [3, Theorem 2.4] $E$ having Banach–Mackey property is equivalent to that $E$ has property (B), i.e., every absorbing sequence is bornivorous. Thus $E$ having property (quasi-L) means that $E$ has property (L). 

**Example 2.2.** A space with property (L) which is not a generalized quasi-Mackey space.

Take a space $(E, t)$ as in Example 2.1, then $(E, t)$ is an infra-Mackey space. Clearly $(E', \sigma(E', E))$ is locally complete since here $(E', \sigma(E', E))$ and $(E', \beta(E', E))$ have the same dual and $(E', \beta(E', E))$ is locally complete. By Theorem 2.3, $(E, t)$ has property (L) but it is not a generalized quasi-Mackey space.

3. **Characterizations of quasi-barrelledness and barrelledness**

First we fix some notation. Let $(E, t)$ be a space and $E^\#$ an algebraic dual of $E$. Put
\[ E^b := \{ f \in E^\#: f \text{ maps bounded sets into bounded scalar sets} \}; \]
\[ E^s := \{ f \in E^\#: f \text{ is sequentially continuous} \}; \]
\[ E^q := \{ f \in E^\#: \text{the restrictions of } f \text{ to any bounded set is continuous} \}; \]
\[ E^u := \{ f \in E^\#: f|_U \text{ is continuous, where } U \text{ is a barrel in } (E, t) \}; \]
\[ E^v := \{ f \in E^\#: f|_V \text{ is continuous, where } V \text{ is a bornivorous barrel in } (E, t) \}. \]

Obviously
\[ E^\# \supset E^b \supset E^s \supset E^q \supset E^u \supset E^v \supset E'. \]

If \( E^b = E' \), we call \((E, t)\) a semi-bornological space [11, Problem 8-6-115].
If \( E^q = E' \), we call \((E, t)\) a Mazur space [11, Definition 8-6-3].
If \( E^u = E' \), we call \((E, t)\) a quasi-Mazur space.
Also we have the following equivalent relationships (for example, see [3, Theorem 2.3] and [8, Theorem 2]):
\[ E^u = E' \iff (E, t) \text{ has property (LC)}; \]
\[ E^v = E' \iff (E, t) \text{ has property (quasi-LC)}. \]

In fact, every class of locally convex spaces described above can be characterized by some completeness of the duals.

**Lemma 3.1.** Let \((E, t)\) be a space, then

(i) \((E, t)\) is semi-bornological if and only if \((E', T_{c_0})\) is complete, where \(T_{c_0}\) denotes the topology of uniform convergence on all the local null-sequences of \((E, t)\).

(ii) \((E, t)\) is a Mazur space if and only if \((E', T_{c_0})\) is complete, where \(T_{c_0}\) denotes the topology of uniform convergence on all the null-sequences of \((E, t)\).

(iii) \((E, t)\) is a quasi-Mazur space if and only if \((E', \beta(E', E))\) is complete.

(iv) \((E, t)\) has property (LC) if and only if \((E', \sigma(E', E))\) is locally complete, i.e., \((E, t)\) is dual locally complete.

(v) \((E, t)\) has property (quasi-LC) if and only if \((E', \beta(E', E))\) is locally complete, i.e., \((E, t)\) is dual locally quasi-complete.

(vi) \((E, t)\) has property (LC) if and only if \((E, t)\) has property (quasi-LC) and Banach–Mackey property.

**Proof.** In fact, [10, §28, 5(4)] gives the proof of (i). The statements (ii) and (iii) can be derived from Grothendieck’s completeness theorem (for example, see [10, pp. 269–272] or [11, Corollary 12-2-19]). Concerning the statements (iv) and (v), see [3, Theorem 2.3], [7] and [8, Theorem 2]. Keeping (iv) and (v) in mind, we see that the statement (vi) is nothing but only repeating Banach–Mackey theorem (for example, see [8, Theorem 3]). \[\square\]

The following implications are obvious:

- semi-bornological \(\Rightarrow\) Mazur \(\Rightarrow\) quasi-Mazur \(\Rightarrow\) property (quasi-LC);
- property (LC) \(\Rightarrow\) property (quasi-LC).
But none of the converses of the above implications is true.

**Example 3.1.** A Mackey–Mazur space which is not semi-bornological.

By [11, Problem 9.5-108], we know that \((l^{\infty}, \tau(l^{\infty}, l^1))\) is a Mackey Mazur space. But it is not semi-bornological. Clearly \((l^{\infty}, \tau(l^{\infty}, l^1))\) and \((l^{\infty}, \beta(l^{\infty}, l^1)) = (l^{\infty}, \| \|_\infty)\) have the same bounded sets. Since \((l^{\infty}, \| \|_\infty)' \supset l^1\) and \((l^{\infty}, \| \|_\infty)' \neq l^1\), there exists \(f \in (l^{\infty}, \| \|_\infty)\) such that \(f/ \notin l^1\). Thus
\[
f \in (l^{\infty}, \| \|_\infty)' = (l^{\infty}, \| \|_\infty)^b = (l^{\infty}, \tau(l^{\infty}, l^1))^b
\]
and
\[
f \notin l^1 = (l^{\infty}, \tau(l^{\infty}, l^1))'.
\]
Hence \((l^{\infty}, \tau(l^{\infty}, l^1))^b \neq (l^{\infty}, \tau(l^{\infty}, l^1))'\).

**Example 3.2.** A quasi-Mazur space with Banach–Mackey property which is not a Mazur space.

Since \((c_0, \beta(c_0, l^1)) = (c_0, \| \|_\infty)\) is complete, we know that \((E, t) := (l^1, \tau(l^1, c_0))\) is a quasi-Mazur space. Obviously \((c_0, \tau(c_0, l^1)) = (c_0, \beta(c_0, l^1))\) is complete, hence \((E, t)\) has Banach–Mackey property. Now we assert that \((E, t)\) is not a Mazur space. If not, \((E, t)\) is a Mackey Mazur space with Banach–Mackey property, which leads that \((E', \sigma(E', E)) = (c_0, \sigma(c_0, l^1))\) is sequentially complete (see [11, Problem 10-4-205], [13, Theorem 3.3]). But it is impossible (see [11, Problem 12-5-102]).

**Example 3.3.** A space with property (LC) which is not quasi-Mazur.

In fact, there exists a barrelled space which is not quasi-Mazur. Komura pointed out that there exist reflexive spaces which are not complete; see [14, p. 148] and [15]. Moreover, Knowies and Cook [16] gave a separable, non-complete, reflexive space. Let \((X, T)\) be a non-complete, reflexive space and let \((E, t) := (X', \beta(X', X))\). Obviously \((E, t)\) is barrelled, hence \((E', \sigma(E', E))\) is quasi-complete and \((E, t)\) has property (LC). But \((E', \beta(E', E)) = (X, \beta(X, X')) = (X, T)\) is not complete, i.e., \((E, t)\) is not a quasi-Mazur space.

**Example 3.4.** A space with property (quasi-LC) which does not have property (LC).

It is easy to construct such examples. As is well known, there exists a quasi-barrelled spaces \(E\) which is not barrelled (for example, see [1, Observation 4.1.2(c)], [12, p. 217]). Clearly \((E', \beta(E', E))\) is quasi-complete. Certainly \((E', \beta(E', E))\) is locally complete, i.e., \(E\) has property (quasi-LC). Since a quasi-barrelled space is barrelled if and only if it is dual locally complete (see [1, Corollary 5.1.35]), \((E', \sigma(E', E))\) is not locally complete, i.e., \(E\) does not have property (LC).

We know that a space \(E\) is barrelled if and only if \(E\) is a Mackey space and \((E', \sigma(E', E))\) is quasi-complete (for example, see [1, Corollary 4.1.15], [10, p. 305]). It seems that there exists a parallel result on quasi-barrelledness. It is easy to prove that a quasi-barrelled space \(E\) is a Mackey space with the quasi-complete strong dual \((E', \beta(E', E))\). However, it is unexpected that the converse is not true.
Example 3.5. A Mackey space with the quasi-complete strong dual which is not quasi-barrelled.

Let $(X, \|\|)$ be a non-reflexive Banach space and $X'$ be its topological dual. Put $(E, t) := (X', \tau(X', X))$, then $(E, t)$ is a Mackey space and $(E', \beta(E', E)) = (X, \beta(X, X'))$ is complete. Since $(X, \|\|)$ is not reflexive, there exists a bounded set $B$ in $(X, \|\|)$ which is not relatively compact in $(X, \sigma(X, X'))$. That is to say, there exists a bounded set $B$ in $(E', \beta(E', E))$ which is not relatively compact in $(E', \sigma(E', E))$. By Alaoglu–Bourbaki theorem, $B \subseteq E'$ is not $t$-equicontinuous and $(E, t)$ is not quasi-barrelled.

In the following we shall give some new characteristics of quasi-barrelledness and barrelledness. To this end, we need the following Šmulian’s criterion for weak compactness; see [17, pp. 142–143].

Lemma 3.2. Let $(E, t)$ be a space, $E'$ be its topological dual and $E^\#$ be its algebraic dual. An absolutely convex $\sigma(E', E)$-closed subset $B$ of $E'$ is $\sigma(E', E)$-compact if and only if $B^\circ \subseteq E$ is absorbing and every $f \in E^\#$ which is bounded on $B^\circ$, is continuous on $E$.

Theorem 3.1. A Mackey space $(E, t)$ is quasi-barrelled if and only if the following two conditions are satisfied:

(i) $(E, t)$ has property (quasi-LC), or equivalently, $(E, t)$ is dual quasi-complete; 
(ii) every $f \in E^\#$ which is bounded on a bornivorous barrel $W$, is continuous on $W$.

Proof. Suppose that $(E, t)$ is quasi-barrelled, then $(E', \beta(E', E))$ is quasi-complete and certainly is locally complete. By Lemma 3.1(v), $(E, t)$ has property (quasi-LC), i.e., the condition (i) is satisfied. Let $f \in E^\#$ be bounded on a bornivorous barrel $W$, then there exists $\lambda > 0$ such that $f \in \lambda W^\#$, where $W^\#$ denotes the polar of $W$ taken in $E^\#$, i.e., $W^\# = \{ f \in E^\#: |f(x)| \leq 1, \forall x \in W \}$. Without loss of generality, we may assume that $W = B^\circ$, where $B$ is an absolutely convex $\beta(E', E)$-bounded and $\sigma(E', E)$-closed subset of $E'$ and $B^\circ$ is the polar of $B$ taken in $E$. Since $(E, t)$ is quasi-barrelled, $B \subseteq E'$ is $t$-equicontinuous. By Alaoglu–Bourbaki theorem, $B$ is $\sigma(E', E)$-compact and hence it is compact in $(E^\#, \sigma(E^\#, E))$. By bipolar theorem, $W^\# = B^{\circ \#}$ is the closure of $B$ in $(E^\#, \sigma(E^\#, E))$, which is exactly $B$. Therefore

$$f \in \lambda W^\# = \lambda B^{\circ \#} = \lambda B \subseteq E'.$$

That is to say, the condition (ii) is satisfied.

Conversely suppose that a Mackey space $(E, t)$ satisfies conditions (i) and (ii). Let $B$ be any $\beta(E', E)$-bounded subset of $E'$. Since the $\sigma(E', E)$-closed absolutely convex hull of a $\beta(E', E)$-bounded set is still $\beta(E', E)$-bounded, we may assume that the $\beta(E', E)$-bounded set $B$ is absolutely convex $\sigma(E', E)$-closed. Clearly $W := B^\circ \subseteq E$ is a bornivorous barrel in $(E, t)$. Let $f \in E^\#$ be bounded on $W = B^\circ$, then by condition (ii), $f|_W$ is continuous. Since $(E, t)$ has property (quasi-LC), we know that $f \in E'$. By Lemma 3.2, we conclude that $B$ is $\sigma(E', E)$-compact. Hence $B$ is $t$-equicontinuous since $(E, t)$ is a Mackey space. Thus we have shown that $(E, t)$ is quasi-barrelled. \(\square\)
Remark 3.1. For a Mackey space, property (quasi-LC) is equivalent to property (quasi-L). Hence the condition (i) in Theorem 3.1 can be replaced by the following condition:

(i)' $(E, t)$ has property (quasi-L).

Similarly we can prove the following characterization of barrelledness.

**Theorem 3.2.** A Mackey space $(E, t)$ is barrelled if and only if the following two conditions are satisfied:

(i) $(E, t)$ has property (LC), or equivalently, $(E, t)$ is dual locally complete;
(ii) every $f \in E^#$ which is bounded on a barrel $W$, is continuous on $W$.

Remark 3.2. For a Mackey space, property (LC) is equivalent to property (L). Hence the condition (i) in Theorem 3.2 can be replaced by the following condition:

(i)' $(E, t)$ has property (L).

Since property (LC) implies Banach–Mackey property (see Lemma 3.1(vi)) and the latter says that every barrel is bornivorous (see [3, Theorem 2.4]), Theorem 3.2 can also be written as follows.

**Theorem 3.3.** A Mackey space $(E, t)$ is barrelled if and only if the following two conditions are satisfied:

(i) $(E, t)$ has property (LC);
(ii) every $f \in E^#$ which is bounded on a bornivorous barrel $W$, is continuous on $W$.

Next we give a variation of Theorem 3.1.

**Theorem 3.4.** A Mackey space $(E, t)$ is quasi-barrelled if and only if the following two conditions are satisfied:

(i) $(E, t)$ has property (quasi-LC);
(ii) for any $\beta(E', E)$-bounded subset $B$ of $E'$ and any $\epsilon > 0$, there exists an absolutely convex $\sigma(E', E)$-compact subset $C$ of $E'$ such that

$$B \subset \epsilon \overline{\sigma}(C \cup B),$$

where $\overline{\sigma}(C \cup B)$ denotes the $\sigma(E', E)$-closed absolutely convex hull of $C \cup B$.

**Proof.** Obviously the conditions (i) and (ii) are necessary for a Mackey space to be quasi-barrelled. We shall see that the condition (i) with (ii) is sufficient. Let $W$ be a bornivorous barrel in $(E, t)$ and let $f \in E^#$ be bounded on $W$. Since $(E, t)$ has property (quasi-LC), by Theorem 3.1 we only need prove that $f|_W$ is continuous at 0 (see [18, pp. 102–103]). Let $\lambda > 0$ such that $|f(x)| \leq \lambda$ for all $x \in W$. Put $B := W^\circ$, the polar of $W$ taken in $E'$, then
$B$ is $\beta(E',E)$-bounded. By condition (ii), for any $\epsilon > 0$ there exists an absolutely convex $\sigma(E',E)$-compact subset $C$ of $E'$ such that

$$B \subset \frac{\epsilon}{\lambda} \Gamma^\sigma (B \cup C).$$

Thus

$$W = W^{\circ \circ} = B^{\circ} \supset \frac{\lambda}{\epsilon} (T^{\sigma} (B \cup C))^{\circ} = \frac{\lambda}{\epsilon} (B^{\circ} \cap C^{\circ}) = \frac{\lambda}{\epsilon} (W \cap C^{\circ}).$$

From this, we have

$$\frac{\lambda}{\epsilon} x \in W, \quad \forall x \in W \cap C^{\circ}.$$

Hence

$$\left| f \left( \frac{\lambda}{\epsilon} x \right) \right| \leq \lambda, \quad \text{i.e.,} \quad \left| f(x) \right| \leq \epsilon, \quad \forall x \in W \cap C^{\circ}.$$

This means that $W \cap C^{\circ} \subseteq (|f| \leq \epsilon)$. Remarking that $(E, t)$ is a Mackey space and $C^{\circ}$ is a 0-neighborhood in $(E, t)$, we conclude that $f|_W$ is continuous at 0. \[ \square \]

Similarly we can prove the following variations of Theorems 3.2 and 3.3.

**Theorem 3.5.** A Mackey space $(E, t)$ is barrelled if and only if the following conditions (i) and (ii) are satisfied, or equivalently, (i) and (iii) are satisfied:

(i) $(E, t)$ has property (LC);

(ii) for any $\sigma(E', E)$-bounded subset $B$ of $E'$ and any $\epsilon > 0$, there exists an absolutely convex $\sigma(E', E)$-compact subset $C$ of $E'$ such that

$$B \subset \epsilon \Gamma^\sigma (C \cup B);$$

(iii) for any $\beta(E', E)$-bounded subset of $E'$ and any $\epsilon > 0$, there exists an absolutely convex $\sigma(E', E)$-compact subset $C$ of $E'$ such that

$$B \subset \epsilon \Gamma^\sigma (C \cup B).$$

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**References**

