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Noncommutative field theories and gravity

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Abstract

We show that after the Seiberg–Witten map is performed the action for noncommutative field theories can be regarded as a coupling to a field dependent gravitational background. This gravitational background depends only on the gauge field. Charged and uncharged fields couple to different backgrounds and we find that uncharged fields couple more strongly than the charged ones. We also show that the background is that of a gravitational plane wave. A massless particle in this background has a velocity which differs from the velocity of light and we find that the deviation is larger in the uncharged case. This shows that noncommutative field theories can be seen as ordinary theories in a gravitational background produced by the gauge field with a charge dependent gravitational coupling.

Noncommutative (NC) theories have been studied in several contexts since a long time ago. More recently it was found that they arise as a limit of string theory with D-branes in a NS–NS background B field [1]. In this limit gravity decouples but still leaves some traces in the emerging NC field theory through the Moyal product, defined as

$$A(x) \star B(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x \partial_\nu^y} A(x)B(y)\big|_{y\to x}, \qquad (1)$$

where $\theta^{\mu\nu}$ is the NC parameter. As a consequence, NC theories are highly nonlocal and we would expect that they would be troublesome. However, upon quantization, the ultraviolet structure is not modified [2] but new infrared divergences appear and get mixed with the ultraviolet ones [3]. This mixing of divergences can be handled at one loop level but when higher loops are taken into account the infrared divergences are non-integrable turning the theory nonrenormalizable. The only known exceptions for $d > 2$ are supersymmetric non-gauge theories [4]. Even so this mixing of divergences have important consequences for many aspects of NC field theories [5]. From a classical point of view many solutions from the commutative field theory can be carried over to the NC corresponding one. Instantons, monopoles and vortex solutions were found for the NC Maxwell theory showing its resemblance with a non-Abelian theory. The main feature of these solutions is that they are non-singular and stable, properties usually not shared by their commutative counterparts [6].

An important property of NC theories, which distinguishes them from the conventional ones, is that translations in the NC directions are equivalent to

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gauge transformations [7]. This can be seen even for the case of a scalar field[1] which has the gauge transformation $\delta\hat{\phi} = -i[\hat{\phi}, \hat{\lambda}]_\star$, where $[A, B]_\star = A \star B - B \star A$ is the Moyal commutator. Under a global translation the scalar field transforms as $\delta_T\phi = \xi^\mu\partial_\mu\hat{\phi}$. Derivatives of the field can be rewritten using the Moyal commutator as $\partial_\mu\hat{\phi} = -i\theta^{-1}_{\mu\nu}[x^\nu, \hat{\phi}]_\star$ so that $\delta\hat{\phi} = \delta_T\hat{\phi}$ with gauge parameter $\hat{\lambda} = -\theta^{-1}_{\mu\nu}\xi^\mu x^\nu$. The only other field theory which has this same property is general relativity where local translations are gauge transformations associated to general coordinate transformations. This remarkable property shows that, as in general relativity, there are no local gauge invariant observables in NC theories.

An alternative approach to study NC theories makes use of commutative fields (with its usual properties) instead of the NC ones. They are related through the Seiberg–Witten (SW) map [1] which is presented as a series expansion in θ. In this way a local field theory is obtained at the expense of introducing a large number of non-renormalizable interactions [8]. Quantization is problematic due to the number of divergences that appear. It seems that at one loop level the SW map is just a field redefinition but at higher loop orders this is not true [9]. At the classical level, on the other side, it is possible to understand very clearly the breakdown of Lorentz invariance induced by the noncommutativity. The dispersion relation for plane waves in a magnetic background gets modified so that photons do not move with the velocity of light [10].

We can wonder how other properties of NC field theories show up in the commutative framework. In particular, the connection between translations and gauge transformations seems to be lost. A global translation on commutative fields cannot be rewritten as a gauge transformation. We will show in this Letter that another aspect concerning gravity emerges when commutative fields are employed. Noncommutative field theories can be interpreted as ordinary theories immersed in a gravitational background generated by the gauge field. Firstly we notice that the commutative theory can be regarded as an ordinary theory coupled to a field dependent gravitational background. We will show that the θ dependent terms in the commutative action can be interpreted as a gravitational background which depends on the gauge field. We then determine the metric which couples to real and complex scalar fields. We find that the uncharged field coupling is twice that of the charged one. So we can interpret the gauge coupling in NC theory as a particular gravitational coupling which depends on the charge of the field. We then show that the background describes a gravitational plane wave. We also determine the geodesics followed by a massless particle in this background. We find that its velocity differs from the velocity of light by an amount proportional to θ with the deviation for the uncharged case being twice that of the charged one. For the uncharged case the deviation is the same as that found for the gauge theory in flat space–time [10,11]. As a final check we derive these same velocities in a field theoretic context.

The action for the NC Abelian gauge theory in flat space–time is

$$S_A = -\frac{1}{4}\int d^4x\, \hat{F}^{\mu\nu} \star \hat{F}_{\mu\nu}, \tag{2}$$

where $\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_\star$. For a real scalar field in the adjoint representation of $U(1)$ the flat space–time action is

$$S_\varphi = \frac{1}{2}\int d^4x\, \hat{D}^\mu\hat{\varphi} \star \hat{D}_\mu\hat{\varphi}, \tag{3}$$

where $\hat{D}_\mu\hat{\varphi} = \partial_\mu\hat{\varphi} - i[\hat{A}_\mu, \hat{\varphi}]_\star$. On the other side, for a complex scalar field in the fundamental representation of $U(1)$ the action is

$$S_\phi = \int d^4x\, \hat{D}^\mu\hat{\phi} \star \big(\hat{D}_\mu\hat{\phi}\big)^\dagger, \tag{4}$$

with $\hat{D}_\mu\hat{\phi} = \partial_\mu\hat{\phi} - i\hat{A}_\mu \star \hat{\phi}$. The gauge transformations which leave the above actions invariant are given by

$$\begin{aligned} &\delta\hat{A}_\mu = \hat{D}_\mu\hat{\lambda}, \qquad \delta\hat{\varphi} = -i[\hat{\varphi}, \hat{\lambda}]_\star, \\ &\delta\hat{\phi} = i\hat{\lambda} \star \hat{\phi}, \qquad \delta\hat{\phi}^\dagger = -i\hat{\phi}^\dagger \star \hat{\lambda}. \end{aligned} \tag{5}$$

To go to the commutative framework we apply the SW map to the fields. We assume that there exists a conventional Abelian gauge field A_μ with the usual Abelian gauge transformation $\delta A_\mu = \partial_\mu\Lambda$ such that $\hat{A}_\mu(A) + \delta_{\hat{\Lambda}}\hat{A}_\mu(A) = \hat{A}_\mu(A + \delta_\Lambda A)$. For the NC real scalar field $\hat{\varphi}$ we assume the existence of a conventional uncharged scalar φ, with gauge transformation $\delta\varphi = 0$, such that $\hat{\varphi}(\varphi, A) + \delta_{\hat{\Lambda}}\hat{\varphi}(\varphi, A) = \hat{\varphi}(A +$

[1] For the gauge field a translation is equivalent to a gauge transformation plus a constant shift of the potential [7].

$\delta_\Lambda A, \varphi + \delta_\Lambda \varphi)$. For the NC complex scalar field $\hat{\phi}$ we associate a charged scalar field ϕ along the same lines. To first order in θ we find

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\alpha\beta} A_\alpha(\partial_\beta A_\mu + F_{\beta\mu}),$$
$$\hat{\varphi} = \varphi - \theta^{\alpha\beta} A_\alpha \partial_\beta \varphi,$$
$$\hat{\phi} = \phi - \frac{1}{2}\theta^{\alpha\beta} A_\alpha \partial_\beta \phi. \quad (6)$$

We can now expand the NC actions (2), (3) and (4) using (1) and apply the map (6) to get the corresponding commutative actions.

For the real scalar field we find, always to first order in θ,

$$S_\varphi = \frac{1}{2}\int d^4x \left[\partial^\mu\varphi\partial_\mu\varphi + 2\theta^{\mu\alpha} F_\alpha{}^\nu \left(-\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{4}\eta_{\mu\nu}\partial^\rho\varphi\partial_\rho\varphi\right)\right]. \quad (7)$$

It is worth to remark that the tensor inside the parenthesis is traceless. If we now consider this same field coupled to a gravitational background

$$S_{g,\varphi} = \frac{1}{2}\int d^4x \sqrt{-g}\, g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi, \quad (8)$$

and expand the metric $g_{\mu\nu}$ around the flat metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \eta_{\mu\nu} h, \quad (9)$$

where $h_{\mu\nu}$ is traceless, we get

$$S_{g,\varphi} = \frac{1}{2}\int d^4x \left(\partial^\mu\varphi\partial_\mu\varphi - h^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + h\partial^\rho\varphi\partial_\rho\varphi\right), \quad (10)$$

where indices are raised and lowered with the flat metric. Since both actions, (7) and (10), have the same structure we can identify a linearized background gravitational field

$$h^{\mu\nu} = \theta^{\mu\alpha} F_\alpha{}^\nu + \theta^{\nu\alpha} F_\alpha{}^\mu + \frac{1}{2}\eta^{\mu\nu}\theta^{\alpha\beta} F_{\alpha\beta},$$
$$h = 0. \quad (11)$$

Then, the effect of noncommutativity on the commutative scalar field is similar to a field dependent gravitational field.

The same procedure can be repeated for the complex scalar field. After the SW map the action (4) reduces to

$$S_\phi = \int d^4x \Big[D^\mu\phi(D_\mu\phi)^\dagger - \frac{1}{2}\left(\theta^{\mu\alpha} F_\alpha{}^\nu + \theta^{\nu\alpha} F_\alpha{}^\mu + \frac{1}{2}\eta^{\mu\nu}\theta^{\alpha\beta} F_{\alpha\beta}\right) \times D_\mu\phi(D_\nu\phi)^\dagger\Big]. \quad (12)$$

Note again that the tensor inside the parenthesis is traceless. Now the action for the charged scalar field in a linearized gravitational field is

$$S_{g,\phi} = \int d^4x \left[D^\mu\phi(D_\mu\phi)^\dagger - h^{\mu\nu} D_\mu\phi(D_\nu\phi)^\dagger + 2h D^\mu\phi(D_\mu\phi)^\dagger\right], \quad (13)$$

which has the same structure as the action (12). Hence, the background gravitational field in this case is

$$h^{\mu\nu} = \frac{1}{2}\left(\theta^{\mu\alpha} F_\alpha{}^\nu + \theta^{\nu\alpha} F_\alpha{}^\mu\right) + \frac{1}{4}\eta^{\mu\nu}\theta^{\alpha\beta} F_{\alpha\beta},$$
$$h = 0. \quad (14)$$

Then charged fields feel a gravitational background which is half of that felt by the uncharged ones. Therefore, the gravity coupling is now dependent on the charge of the field, being stronger for uncharged fields. Notice that the gauge field has now a dual role, it couples minimally to the charged field and also as a gravitational background.

We can now consider the gauge field. As it is well known, the SW map gives rise to the following action:

$$S_A = -\frac{1}{4}\int d^4x \left[F^{\mu\nu} F_{\mu\nu} + 2\theta^{\mu\rho} F_\rho{}^\nu \left(F_\mu{}^\sigma F_{\sigma\nu} + \frac{1}{4}\eta_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}\right)\right]. \quad (15)$$

Again, the tensor inside the parenthesis is traceless. At this point we could be tempted to consider this action as some gravitational action build up from the metric (11) or (14). Since the field strength always appears multiplied by θ inside the metric, all invariants constructed with it will be of order θ. Hence, they cannot give rise to (15), unless they appear in combinations involving the inverse of θ. If we insist in having an action which is polynomial in θ the best we can do is to regard the gauge field as having a double role again and couple it to gravitation as in the previous case. The linearized coupling of the Maxwell action is

$$S_{g,A} = -\frac{1}{4}\int d^4x\,\big(F^{\mu\nu}F_{\mu\nu} + 2h^{\mu\nu}F_\mu{}^\rho F_{\rho\nu}\big). \tag{16}$$

Since it does not couple to the trace part of the metric h remains arbitrary and $h^{\mu\nu}$ is given by (14). Since the NC gauge field resembles a non-Abelian gauge field we expect that its commutative counterpart couple to the same gravitational field as the charged one. It should also be remarked that in this case the gravitational field cannot be interpreted just as a fixed background since it depends on the dynamical gauge field.

Having determined the field dependent background metric we can now study its properties. We will consider the metric which couples to the charged fields (14). To consider the metric (11) we have just to replace θ by 2θ. The linearized metric is then

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2}\big(\theta_{\mu\alpha}F^\alpha{}_\nu + \theta_{\nu\alpha}F^\alpha{}_\mu\big) + \frac{1}{4}\eta_{\mu\nu}\theta^{\alpha\beta}F_{\alpha\beta}. \tag{17}$$

The geometric quantities can be evaluated without difficulty and we find

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\big[-\theta_{\alpha[\mu}\partial_{\nu]}\partial^\alpha F_{\sigma\rho} + \theta_{\rho\alpha}\partial^\alpha\partial_{[\mu}F_{\nu]\sigma} + \theta_{\sigma\alpha}\partial_\rho\partial_{[\mu}F^\alpha{}_{\nu]} + \theta^{\alpha\beta}(\eta_{\sigma[\mu}\partial_{\nu]}\partial_\rho F_{\alpha\beta} - \eta_{\rho[\mu}\partial_{\nu]}\partial_\sigma F_{\alpha\beta})\big], \tag{18}$$

$$R_{\mu\nu} = \frac{1}{4}\bigg(\theta_\mu{}^\alpha\partial_\alpha\partial^\beta F_{\beta\nu} + \theta_\nu{}^\alpha\partial_\alpha\partial^\beta F_{\beta\mu} + \frac{1}{2}\eta_{\mu\nu}\theta^{\alpha\beta}\Box F_{\alpha\beta}\bigg), \tag{19}$$

$$R = \frac{1}{4}\theta^{\alpha\beta}\Box F_{\alpha\beta}. \tag{20}$$

Notice that all of them, and also the Christoffel symbol, are first order in θ. Since on-shell and in the absence of matter $\partial^\mu F_{\mu\nu}$ is first order in θ, then $\Box F_{\mu\nu}$ is also first order.[2] This means that the Ricci tensor and the scalar curvature both vanish but not the Riemann tensor so that the metric (17) is not that of a flat space–time. Then, to order zero in θ, $F_{\mu\nu}$ satisfies the wave equation and is a function of $k_\mu x^\mu$ with $k^2 = 0$. Hence, the metric has all symmetries of a gravitational plane wave.

More rigorously, we find that the noncommutative parameter is covariantly conserved $D_\mu\theta^{\alpha\beta} = 0$. We then have a geometry equipped with a covariantly constant two-form. Since $\theta^{\alpha\beta}\theta_{\alpha\beta} = 0$ to first order then the two-form is also null. The existence of this covariantly null two-form guarantees that the metric (17) describes a pp-wave [12]. More than that, if we consider the solution for the gauge field to first order in θ and in the absence of matter as $F_{\mu\nu} = k_{[\mu}F_{\nu]}$, with k^μ a null vector and F_μ transversal, $k^\mu F_\mu = 0$, then $\partial_\alpha R_{\mu\nu\rho\sigma} = k_\alpha R_{\mu\nu\rho\sigma}$ and the complex Riemann tensor

$$P_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{i}{2}\epsilon_{\rho\sigma}{}^{\alpha\beta}R_{\mu\nu\alpha\beta} \tag{21}$$

satisfies $\partial_\alpha P_{\mu\nu\rho\sigma} = k_\alpha P_{\mu\nu\rho\sigma}$. This shows that the pp-wave is in fact a plane wave [12]. Then the metric (17) is that of a gravitational plane wave.

We can now turn our attention to the behavior of a massless particle in this background. Its geodesics is described by

$$ds^2 = \bigg(1 + \frac{1}{4}\theta^{\alpha\beta}F_{\alpha\beta}\bigg)dx^\mu\,dx_\mu + \theta_{\mu\alpha}F^\alpha{}_\nu\,dx^\mu\,dx^\nu = 0. \tag{22}$$

If we consider the case where there is no noncommutativity between space and time, that is $\theta^{0i} = 0$, and calling $\theta^{ij} = \epsilon^{ijk}\theta^k$, $F^{i0} = E^i$, and $F^{ij} = \epsilon^{ijk}B^k$, we find to first order in θ that

$$(1 - \vec{v}^2)(1 - 2\vec{\theta}\cdot\vec{B}) - \vec{\theta}\cdot(\vec{v}\times\vec{E}) + \vec{v}^2\vec{\theta}\cdot\vec{B} - (\vec{B}\cdot\vec{v})(\vec{\theta}\cdot\vec{v}) = 0, \tag{23}$$

where $\vec{v}$ is the particle velocity. Then to zeroth order, the velocity $\vec{v}_0$ satisfies $\vec{v}_0^2 = 1$ as it should. We can now decompose all vectors into their transversal and longitudinal components with respect to $\vec{v}_0$, $\vec{E} = \vec{E}_\mathrm{T} + \vec{v}_0 E_\mathrm{L}$, $\vec{B} = \vec{B}_\mathrm{T} + \vec{v}_0 B_\mathrm{L}$ and $\vec{\theta} = \vec{\theta}_\mathrm{T} + \vec{v}_0\theta_\mathrm{L}$. We then find that the velocity is

$$\vec{v}^2 = 1 + \vec{\theta}_\mathrm{T}\cdot(\vec{B}_\mathrm{T} - \vec{v}_0\times\vec{E}_\mathrm{T}). \tag{24}$$

Hence, a charged massless particle has its velocity changed with respect to the velocity of light by

[2] Notice that field equations for the gauge field are derived from (15) which is defined in flat space–time. Hence in the field equations and solutions the Minkowski metric is used.

an amount which depends on θ. For an uncharged massless particle

$$\vec{v}^2 = 1 + 2\vec{\theta}_{\rm T} \cdot (\vec{B}_{\rm T} - \vec{v}_0 \times \vec{E}_{\rm T}), \tag{25}$$

and the correction due to the noncommutativity is twice that of a charged particle.

We can now check the consistency of these results by going back to the original actions (7) and (12), and computing the group velocity for planes waves. Upon quantization they give the velocity of the particle associated to the respective field. For the uncharged scalar field we get the equation of motion

$$\left(1 - \frac{1}{2}\theta^{\mu\nu}F_{\mu\nu}\right)\Box\varphi - 2\theta^{\mu\alpha}F_\alpha{}^\nu\partial_\mu\partial_\nu\varphi = 0. \tag{26}$$

If the field strength is constant we can find a plane wave solution with the following dispersion relation

$$\left(1 - \frac{1}{2}\theta^{\mu\nu}F_{\mu\nu}\right)k^2 - 2\theta^{\mu\alpha}F_\alpha{}^\nu k_\mu k_\nu = 0, \tag{27}$$

and using the same conventions for vectors as before, it results in

$$\frac{\vec{k}^2}{\omega^2} = 1 - 2\vec{\theta}_{\rm T} \cdot \left(\vec{B}_{\rm T} - \frac{\vec{k}}{\omega} \times \vec{E}_{\rm T}\right), \tag{28}$$

where $k^\mu = (\omega, \vec{k})$. We then find that the phase and group velocities coincide and are given by (25) as expected. For the charged scalar field we have to turn off the gauge coupling in order to get a plane wave solution. In this case the equation of motion is

$$\left(1 - \frac{1}{4}\theta^{\mu\nu}F_{\mu\nu}\right)\Box\phi - \theta^{\mu\alpha}F_\alpha{}^\nu\partial_\mu\partial_\nu\phi = 0. \tag{29}$$

In a constant field strength background the dispersion relation for a plane wave reads as in (27) with θ replaced by $\theta/2$. Then we must perform the same replacement in the phase and group velocities and we get (24). Therefore, in both pictures, noncommutative and gravitational, we get the same results.

For the gauge field the situation is more subtle because of its double role. There is no clear way to split the action (15). What can be done is to break up the gauge field into a background plus a plane wave as in [10]. We then get the following dispersion relation

$$k^2 - 2\theta^{\mu\alpha}F_\alpha{}^\nu k_\mu k_\nu = 0, \tag{30}$$

where $F_\alpha{}^\nu$ is now the constant background. This leads to (28), that is, the dispersion relation for the uncharged scalar field. It also reproduces the result in [10,11] when the background is purely magnetic. This shows the dual role of the gauge field, since it couples to gravitation as a charged field but its dispersion relation is that of an uncharged field.

We have seen in this Letter that it is possible to regard noncommutative theories as conventional theories embedded in a gravitational background produced by the gauge field. This brings a new connection between noncommutativity and gravitation. We could imagine that this is a peculiarity of the first order term in the θ expansion of the SW map. If we consider the SW map for the scalar field to second order in θ we find that it is linear in the scalar field. A short calculation shows that the action (3) remains quadratic after the SW map and that it can be cast again in the form (8) since all terms of the form $\partial^2\varphi\partial^2\varphi$ cancel. Then it seems to be possible to keep the same interpretation to all orders in θ.

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