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## Neutron beta-decay in effective field theory

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### Abstract

Radiative corrections to the lifetime and angular correlation coefficients of neutron beta-decay are evaluated in effective field theory. We also evaluate the lowest order nucleon recoil corrections, including weak-magnetism. Our results agree with those of the long-range and model-independent part of previous calculations. In an effective theory the model-dependent radiative corrections are replaced by well-defined low-energy constants. The effective field theory allows a systematic evaluation of higher order corrections to our results to the extent that the relevant low-energy constants are known.

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### 1. Introduction

The radiative corrections for beta-decay have been intensively investigated by a number of authors, and the prime issue for such studies has been to deduce the value of the Cabbibo–Kobayashi–Maskawa (CKM) matrix element  $V_{ud}$  from nuclear beta-decay data. An accurate value for  $V_{ud}$  is important for testing the unitarity of the CKM matrix. The most precise values of  $V_{ud}$  have been obtained from the accurate data of super-allowed  $0^+ \rightarrow 0^+$  nuclear beta-decays [1]. Neutron beta-decay measurements provide an alternative method of determining  $V_{ud}$ , a method which does not depend on the accuracy of nuclear models. Neutron beta-decay experiments also provide the most precise determination of the axial-vector coupling constant,  $g_A$ , which plays an important role in hadronic weak-interaction reactions including many astrophysical processes. Theoretically, pion beta-decay can also be used for determining  $V_{ud}$ . Unfortunately, however, the currently available experimental data on pion beta-decay are not accurate enough to allow us to take full advantage of this merit, see, e.g., Cirigliano et al. [2].

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To extract an accurate value of  $V_{ud}$  from neutron decay data, the theoretical expression for the neutron decay rate including radiative corrections must be known with sufficient accuracy. The usual convention is to decompose radiative corrections of order  $\alpha$  into two parts, the “outer” and the “inner” corrections [3–5]. The “outer” correction is a universal function of the electron energy, independent of the details of the strong interactions. The “inner” correction stems from short-range terms and hadronic structure effects. This hadronic structure dependence (and additional nuclear structure dependence in the case of nuclear beta-decay) causes uncertainties in extracting fundamental quantities like  $V_{ud}$  from experimental data.<sup>1</sup>

In this communication we present the first calculation of radiative corrections to neutron beta-decay based on a low-energy effective field theory (EFT). EFT provides symmetry constraints required by the underlying theory and a systematic expansion scheme for the evaluation of the hadron current. As suggested by Weinberg [7], low-energy hadronic physics can be described by an effective field theory of QCD known as “chiral perturbation theory” ( $\chi$ PT). The effective chiral Lagrangian,  $\mathcal{L}_\chi$ , reflects the symmetries and the pattern of symmetry breaking of the underlying QCD. For massless quarks the QCD Lagrangian is chirally symmetric, but chiral symmetry is spontaneously broken generating the pions as massless Goldstone bosons. Since the  $u$  and  $d$  quark masses are very small compared with the QCD scale  $\Lambda_{\text{QCD}}$ , and since the finite pion mass generated by the quark masses is small compared to a typical strong interaction scale, it is reasonable to treat the explicit chiral symmetry breaking terms as small perturbations.  $\mathcal{L}_\chi$  is expanded in powers of  $Q/\Lambda_\chi \ll 1$  where  $Q$  denotes the typical four-momentum of the process in question or the pion mass,  $m_\pi$ , which represents the small explicit chiral symmetry breaking scale. The chiral scale,  $\Lambda_\chi \simeq 4\pi f_\pi \simeq 1$  GeV ( $f_\pi = 92.4$  MeV is the pion decay constant), is associated with the “high-energy” processes that have been integrated out in arriving at  $\mathcal{L}_\chi$  and with pion loops. The parameters appearing in  $\mathcal{L}_\chi$ , called the *low-energy constants* (LECs), effectively subsume the high-energy physics that has been integrated out. In principle, these LECs could be determined from the underlying theory, but in practice the LECs are determined phenomenologically from experimental data. Once the LECs are determined from appropriate empirical data, then  $\mathcal{L}_\chi$  represents a complete Lagrangian up to a specified chiral order. Furthermore, starting from  $\mathcal{L}_\chi$ , one can develop, for the amplitude of a given process, a well-defined perturbation scheme by organizing the relevant Feynman diagrams according to powers in  $Q/\Lambda_\chi$ . If all the Feynman diagrams up to a given power,  $\nu$ , in  $Q/\Lambda_\chi$  are taken into account, then the results depend only on the LECs up to this order, with the contributions of higher order terms suppressed by an extra power of  $Q/\Lambda_\chi$ .

Over the past decade  $\chi$ PT has been successfully applied to many processes; for reviews, see, e.g., Refs. [8,9]. Chiral Lagrangians including the photon field have been developed and applied to, e.g., pion–nucleon scattering, see Refs. [10,11]. Our present calculation of the radiative corrections to neutron beta-decay is an EFT based on the spirit of the chiral Lagrangian approach. Thus we write down an effective Lagrangian, appropriate to neutron beta-decay, obeying chiral symmetry and involving a minimum set of LECs and use the Lagrangian to estimate the relevant amplitudes to leading, next-to-leading, and next-to-next-to-leading orders (LO, NLO, N<sup>2</sup>LO) in the  $Q/\Lambda_\chi$  expansion. In fact, since the typical energy transfer of the reaction is much smaller than the pion mass, the “ $Q/\Lambda_\chi$  expansion” here has a special feature to be explained in the next section.

The results of our EFT calculation confirm the expression for the model-independent universal function derived by Sirlin [3]. Furthermore, our calculation provides expressions for corrections of order  $\alpha$  to the angular correlation coefficients in neutron beta-decay. We will show that the short-distance phenomena including the model-dependent hadronic radiative corrections can be condensed into two LECs, one relevant to the Fermi constant  $G_F$  and the other to the axial coupling constant  $g_A$ . The values of these LECs need to be determined by experiments. In order to have crude order-of-magnitude estimates of our LECs, we also compare our results with the “inner” radiative corrections obtained in the standard calculations. Furthermore, we shall argue that, provided the LECs involved in

<sup>1</sup> A new calculation of these radiative corrections, obtained with the standard model of electroweak interactions, has been reported in a recent preprint [6]. The results however seem to differ markedly from the classic calculations of Sirlin et al. [3–5].

our calculation are of a “natural” size, the neutron-decay rate and angular correlation coefficients calculated here are expected to have a precision better than  $10^{-3}$ .

## 2. Effective theory for neutron beta-decay

Since neutron beta-decay is a low energy process, it is natural to use here heavy-baryon chiral perturbation theory (HB $\chi$ PT), see, e.g., Refs. [8,9]. In fact the appropriate amplitude, however without radiative corrections, can be obtained from HB $\chi$ PT calculations of muon capture on a proton,  $\mu + p \rightarrow n + \nu$ , which have been carried out including N<sup>2</sup>LO correction terms [12–15]. Neutron beta-decay, however, has a feature not shared by muon capture, namely several different expansion scales. In particular, the maximum energy release,  $\Delta M = m_n - m_p - m_e = 0.782$  MeV, is very small compared to the pion mass  $m_\pi$  and the nucleon mass  $m_N = (m_p + m_n)/2$ . Correspondingly, if we denote by  $\bar{Q}$  the typical four-momentum transfer of the process,  $\bar{Q} \sim \Delta M$  is also very small. We therefore introduce here a particular “ $Q/\Lambda_\chi$ ” expansion in which  $Q$ , unlike most HB $\chi$ PT calculations, only represents  $\bar{Q}$ . The chiral symmetry breaking scale,  $m_\pi/\Lambda_\chi \simeq 0.14$ , will be accounted for separately. The nucleon recoil terms are governed by the scale  $\bar{Q}/m_N \simeq 0.8 \times 10^{-3}$ , and they are NLO corrections to the LO expression. The scale  $\bar{Q}/m_N \simeq \bar{Q}/\Lambda_\chi$  is numerically of the same magnitude as  $\alpha/(2\pi) \sim 10^{-3}$ , governing the radiative corrections, which are our primary interest ( $\alpha$  is the fine structure constant). Therefore, for our present purposes, we consider the  $\alpha/(2\pi)$  and  $\bar{Q}/m_N$  corrections to be of the same order.

The relevant effective Lagrangian,  $\mathcal{L}_\beta$ , for the neutron decay process reads

$$\mathcal{L}_\beta = \mathcal{L}_{e\nu\gamma} + \mathcal{L}_{NN\gamma} + \mathcal{L}_{e\nu NN}, \quad (1)$$

where  $\mathcal{L}_{e\nu\gamma}$  is the lepton–photon Lagrangian,  $\mathcal{L}_{NN\gamma}$  describes the heavy nucleon interacting with a photon, and  $\mathcal{L}_{e\nu NN}$  gives the effective  $V - A$  interaction between the lepton and the heavy nucleon current. Since the pion mass is much heavier than the typical momentum scale of the reaction,  $\bar{Q} \ll m_\pi$ , we suppress the pion fields of the chiral Lagrangian,  $\mathcal{L}_\chi$ , and in  $\mathcal{L}_\beta$  we have retained only the interactions between the heavy nucleon field, lepton current, and photons. Later in the text, we will discuss the role of the pions in the present calculation. Thus one has, through LO and NLO,

$$\mathcal{L}_{e\nu\gamma} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi_A}(\partial \cdot A)^2 + \left(1 + \frac{\alpha}{4\pi}e_1\right)\bar{\psi}_e(i\gamma \cdot D)\psi_e - m_e\bar{\psi}_e\psi_e + \bar{\psi}_\nu i\gamma \cdot \partial\psi_\nu, \quad (2)$$

$$\mathcal{L}_{NN\gamma} = \bar{N} \left[ 1 + \frac{\alpha}{8\pi}e_2(1 + \tau_3) \right] i v \cdot DN, \quad (3)$$

$$\begin{aligned} \mathcal{L}_{e\nu NN} = & -\frac{(\hat{G}_F V_{ud})}{\sqrt{2}}\bar{\psi}_e\gamma_\mu(1 - \gamma_5)\psi_\nu \\ & \times \left\{ \bar{N}\tau^+ \left[ \left(1 + \frac{\alpha}{4\pi}e_V\right)v^\mu - 2\hat{g}_A \left(1 + \frac{\alpha}{4\pi}e_A\right)S^\mu \right] N \right. \\ & \left. + \frac{1}{2m_N}\bar{N}\tau^+ [i(v^\mu v^\nu - g^{\mu\nu})(\vec{\partial} - \vec{\partial})_\nu - 2i\hat{\mu}_V[S^\mu, S \cdot (\vec{\partial} + \vec{\partial})] - 2i\hat{g}_A v^\mu S \cdot (\vec{\partial} - \vec{\partial})]N \right\}, \end{aligned} \quad (4)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  and  $D_\mu$  is the covariant derivative of QED. The  $\xi_A$  is the gauge parameter and we choose the Feynman gauge  $\xi_A = 1$ . The  $v^\mu$  is the velocity vector of the heavy-baryon formalism, which we take as  $v^\mu = (1, \vec{0})$ , and  $S^\mu$  is the nucleon spin operator  $2S^\mu = (0, \vec{\sigma})$ . The isovector magnetic moment in the NLO Lagrangian is  $\hat{\mu}_V \rightarrow \mu_V = 4.706$ . The quantities  $e_1$ ,  $e_2$ ,  $e_V$  and  $e_A$  are defined as the LECs of the theory. The LECs  $e_1$  and  $e_2$  are the  $\alpha$ -order corrections related to the wave-function normalization factors of the electron and proton, respectively. The LECs  $e_V$  and  $e_A$  are the  $\alpha$ -order corrections to the Fermi and Gamow–Teller amplitudes, where we

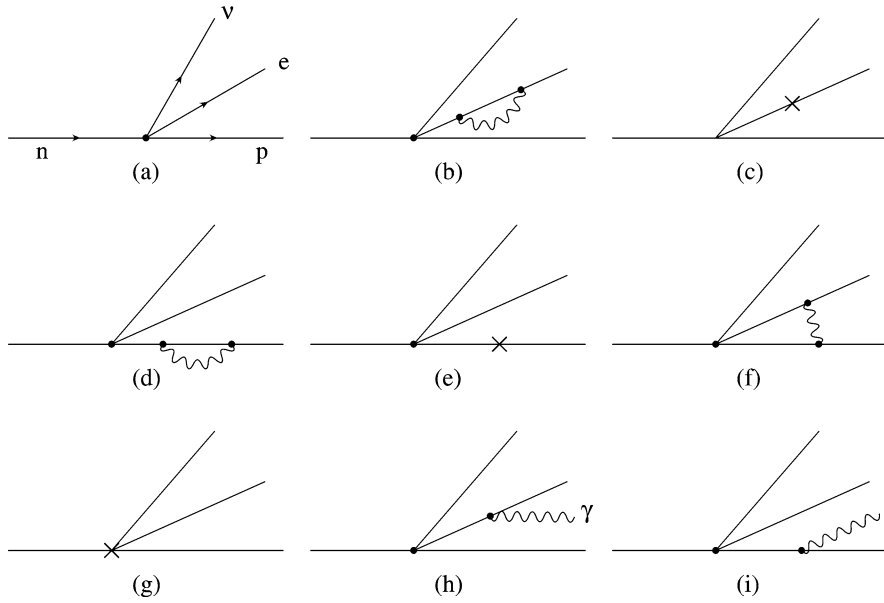


Fig. 1. Feynman diagrams for neutron beta-decay up to order  $\alpha$ . In diagram (a), the four-fermion vertex can represent either the leading order (LO) or next-to-leading order (NLO) vertex, the latter being a  $1/m_N$  correction to the former. The crosses on the electron and nucleon lines in diagrams (c) and (e) are vertices involving the LECs,  $e_1$  and  $e_2$ , respectively. The vertex of diagram (g) is given by the LECs  $e_V$  and  $e_A$ .

have factored out the common coefficient  $\hat{G}_F V_{ud}/\sqrt{2}$ . Those LECs are used to absorb infinities coming from the virtual photon-loops and take into account short-range radiative effects. We remark that some of those LECs contain contributions from, e.g.,  $g_i$ 's for the one nucleon sector without leptons [10] and  $X_i$ 's for the meson sector with leptons [16] in  $\chi$ PT.<sup>2</sup> As is conventional, the parameters of the initial Lagrangian, e.g., the Fermi constant  $\hat{G}_F$  and the axial coupling constant  $\hat{g}_A$ , are taken as the coupling constants in the absence of radiative corrections and in the chiral limit,  $m_\pi = 0$ . Thus in particular, we assume that the Fermi constant,  $\hat{G}_F \rightarrow G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2}$ , as determined from muon-decay. As we discuss in the next paragraph, higher order hadronic corrections, i.e., pion-loops, renormalize these “bare” couplings to their physical values in the absence of electromagnetic effects, e.g.,  $\hat{g}_A \rightarrow g_A$ . Furthermore, radiative effects give rise to additional corrections to the coupling constants,  $G_F$  and  $g_A$  which depend on the process being considered. These radiative corrections will be displayed explicitly in the present work.

We calculate the Feynman diagrams shown in Fig. 1, where the vertices are determined by the Lagrangian,  $\mathcal{L}_\beta$ , given above. Several remarks are in order on the diagrams in Fig. 1. Consider first diagram (a), which does *not* involve radiative corrections. Diagram (a) is a tree-diagram for the LO and NLO amplitudes. As regards the LO contribution, one may wonder why we do not consider here the pion-pole diagram (not shown). The pion-pole diagram, which is responsible for the induced pseudoscalar coupling, formally belongs to LO and hence would be included in normal circumstances. However, the extremely small momentum transfer involved in neutron beta-decay ( $\bar{Q} \ll m_\pi$ ) drastically suppresses the pion-pole diagram contribution. Due to the presence of the pion propagator and a momentum of order  $\bar{Q}$  at each vertex, the pion-pole diagram scales like  $(\bar{Q}/m_\pi)^2 \simeq 3 \times 10^{-5}$  relative to the dominant LO terms. The accuracy of our present treatment does not warrant the inclusion of this

<sup>2</sup> Unfortunately the connection between the LECs  $e_V$  and  $e_A$  and the  $g_i$  and  $X_i$  of Refs. [10,16] is not straightforward. The  $g_i$  and  $X_i$  originate in Lagrangians which involve only subsets of the degrees of freedom considered here and thus generate radiative corrections to only particular vertices in the diagrams for neutron beta-decay. Their contribution can be absorbed in  $e_V$  and  $e_A$ , but  $e_V, e_A$  would also contain contributions from the LECs of a yet-to-be-calculated Lagrangian involving the nucleon, lepton current, and photons simultaneously.

tiny pion-pole contribution, and we will not consider it in the main body of our calculation. In the concluding section, however, we will briefly discuss the LO pion-pole term and its radiative corrections. Diagram (a) in Fig. 1 also includes the NLO vertex coming from the nucleon recoil terms  $\propto \bar{Q}/m_N$  featuring in Eq. (4). Since we are treating the  $\bar{Q}/m_N$  and  $\alpha/(2\pi)$  corrections as contributions of the same order, we will discuss these recoil terms later in the text; however, in evaluating radiative corrections, we need not consider the recoil terms since these corrections would be of higher order  $\sim \alpha/(2\pi) \times \mu_V \bar{Q}/(2m_N) \sim 10^{-6}$ . At order N<sup>2</sup>LO there occur two kinds of contributions. Higher order recoil corrections scale as  $(\bar{Q}/m_N)^2 \simeq 10^{-6}$  and therefore can be neglected. The remaining N<sup>2</sup>LO terms (diagrams not shown) come from pion-loops and the corresponding hadronic LECs which would appear in HB $\chi$ PT Lagrangian at this order, see, e.g., Refs. [8,9]. The pion-loop diagrams which generate terms proportional to  $\bar{Q}^2$ , i.e., terms representing the hadronic vertex form factor effects, can be neglected, since their contributions are suppressed by a factor of  $(\bar{Q}/\Lambda_\chi)^2 \simeq 10^{-6}$  relative to the dominant LO terms. The remaining contributions of the pion-loops, which contain terms proportional to  $(m_\pi/\Lambda_\chi)^2$ , renormalize the bare quantities such as the “bare” axial-vector coupling constant  $\mathring{g}_A$ . These  $(m_\pi/\Lambda_\chi)^2$  terms and the corresponding hadronic LECs are absorbed into the renormalized  $g_A$  so that to N<sup>2</sup>LO order,  $g_A = \mathring{g}_A[1 + \mathcal{O}((m_\pi/\Lambda_\chi)^2)]$ , see, e.g., Eq. (4.50) in Ref. [8] or Eq. (50) in Ref. [12]. Radiative corrections to the pion loop diagrams are suppressed by a scale  $(m_\pi/\Lambda_\chi)^2 \simeq 2 \times 10^{-2}$  relative to the leading *radiative* corrections, and therefore their contributions can be ignored in the present calculation.

The above discussion indicates that, to the accuracy in question, we need only consider radiative corrections of the following type. Of the contributions topologically represented by diagram (a), consider those involving the LO vertex and evaluate all possible radiative corrections applied to these LO diagrams. Diagrams (b), (d), (f) in Fig. 1 are one-photon loop corrections for the electron propagator, the nucleon propagator, and the four-point vertex function, respectively. Meanwhile, diagrams (c), (e) and (g) represent the contributions of the counter terms, the  $e_1$ ,  $e_2$ ,  $e_V$  and  $e_A$  terms, in the Lagrangian. These LECs remove the ultraviolet divergence arising from the loop diagrams (b), (d) and (f). As is well known, the infrared divergences contained in diagrams (b), (d), (f) should be canceled by the infrared divergences in the bremsstrahlung diagrams (h) and (i),<sup>3</sup> and we have confirmed this cancellation explicitly.

### 3. The correlation coefficients and the decay rate from EFT

A general expression for the differential neutron decay rate  $d\Gamma$  is well known [18] for a case wherein only the neutron is polarized, and in which the nucleon recoil and radiative corrections are ignored:

$$\frac{d\Gamma}{dE_e d\Omega_{\hat{p}_e} d\Omega_{\hat{p}_v}} \simeq \frac{(G_F V_{ud})^2}{(2\pi)^5} (1 + 3g_A^2) |\vec{p}_e| E_e E_v^2 \times \left[ 1 + a(\vec{\beta} \cdot \hat{p}_v) + b\left(\frac{m_e}{E_e}\right) + \hat{n} \cdot \left( A\vec{\beta} + B\hat{p}_v + D\frac{\vec{p}_e \times \vec{p}_v}{E_e E_v} \right) \right]. \quad (5)$$

Here  $E_e$  and  $\vec{p}_e$  ( $E_v$  and  $\vec{p}_v$ ) are the electron (neutrino) energy and momentum,  $\hat{n}$  is the neutron spin polarization vector,  $\vec{\beta} = \vec{p}_e/E_e$ , and  $a, b, A, B, D$  are the correlation coefficients. If we calculate diagram (a) in Fig. 1 in the LO approximation, and if we neglect the nucleon recoil terms in the phase space factor, then our calculation reproduces Eq. (5), and furthermore we recover the standard lowest order expressions for the correlation coefficients as given in [18]

$$a = \frac{1 - g_A^2}{1 + 3g_A^2}, \quad A = \frac{-2g_A^2 + 2g_A}{1 + 3g_A^2}, \quad B = \frac{2g_A^2 + 2g_A}{1 + 3g_A^2}, \quad (6)$$

<sup>3</sup> Recently these bremsstrahlung diagrams have been studied by Bernard et al. for radiative neutron beta-decay,  $n \rightarrow p + \nu + e + \gamma$ , in EFT [17].

where  $g_A$  is the physical axial coupling constant. The coefficient  $b$  in Eq. (5), which reflects the presence of scalar and tensor weak couplings, vanishes in our LO calculation, since our Lagrangian only contains the standard vector and axial vector weak interaction. The parameter  $D$  in Eq. (5) is related to time-odd correlations and hence it also should vanish in the LO calculation since our Lagrangian is  $T$  invariant. However, “induced”  $D$  terms can appear at higher orders. For instance, interference between the weak magnetism and the radiative corrections would generate a  $D$  term of order  $10^{-5}$  [19].

As we proceed to include the higher order radiative diagrams generated by the Lagrangian of Eq. (1), we encounter infinities coming from the photon-loop diagrams in Fig. 1. In order to eliminate these infinities, we need to introduce counter terms with the corresponding LECs in our Lagrangian. We renormalize these LECs in the usual effective field theoretical method based on the dimensional regularization of loop integrals [8,9]. The finite LECs renormalized at the scale  $\mu$  are given by<sup>4</sup>

$$e_{V,A}^R(\mu) = e_{V,A} - \frac{1}{2}(e_1 + e_2) + \frac{3}{2} \left[ \frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + 1 \right] + 3 \ln \left( \frac{\mu}{m_N} \right). \quad (7)$$

This renormalization is adequate to remove all the infinities associated with virtual photons which we encounter in this calculation. The differential neutron decay rate including the radiative corrections and  $1/m_N$  corrections is found to be

$$\frac{d\Gamma}{dE_e d\Omega_{\vec{p}_e} d\Omega_{\vec{p}_\nu}} = \frac{(G_F V_{ud})^2}{(2\pi)^5} \frac{F(Z, E_e) |\vec{p}_e| E_\nu}{m_n [E_p + E_\nu + E_e (\vec{\beta} \cdot \hat{p}_\nu)]} |M|^2, \quad (8)$$

where we have retained the relativistic expression for the phase factor, and

$$\begin{aligned} |M|^2 = & m_n m_p E_e E_\nu \left( 1 + \frac{\alpha}{2\pi} e_V^R \right) \left( 1 + \frac{\alpha}{2\pi} \delta_\alpha^{(1)} \right) C_0(E_e) (1 + 3\tilde{g}_A^2) \\ & \times \left\{ 1 + \left( 1 + \frac{\alpha}{2\pi} \delta_\alpha^{(2)} \right) C_1(E_e) \vec{\beta} \cdot \hat{p}_\nu \right. \\ & \left. + \left( 1 + \frac{\alpha}{2\pi} \delta_\alpha^{(2)} \right) [C_2(E_e) + C_3(E_e) \vec{\beta} \cdot \hat{p}_\nu] \hat{n} \cdot \vec{\beta} + [C_4(E_e) + C_5(E_e) \vec{\beta} \cdot \hat{p}_\nu] \hat{n} \cdot \hat{p}_\nu \right\}. \quad (9) \end{aligned}$$

The explanation of the quantities appearing in this expression will be given below. We remark that, in order to arrive at this factored form, we have freely exploited the fact that terms of order  $(\alpha/2\pi)^2$ ,  $(\alpha/2\pi)(Q/m_N)$  and  $(Q/m_N)^2$  can be ignored to the order of accuracy of our concern.

In Eq. (8) the Coulomb part of the radiative correction has been extracted as an overall factor and incorporated into the usual Fermi function  $F(Z, E_e) \simeq 1 + (\alpha/2\pi) \delta_\alpha^{(\text{Coul})} = 1 + \alpha\pi/\beta$ , for  $Z = 1$ . In Eq. (9) the finite LEC,  $e_V^R$ , featuring in the factor  $(1 + \frac{\alpha}{2\pi} e_V^R)$  subsumes those short-range radiative corrections to the Fermi constant  $G_F^2$  which have been integrated out in arriving at our effective Lagrangian. This point will be further discussed in the final section. The axial coupling constant,  $g_A$ , which has been renormalized by pion loops, is multiplied by short-range radiative corrections involving the finite LEC  $e_A^R$  as well as  $e_V^R$ . For convenience, and to simplify the results, we incorporate this radiative correction to  $g_A$  into  $\tilde{g}_A$  defined by

$$\tilde{g}_A = g_A \left[ 1 + \frac{\alpha}{4\pi} (e_A^R - e_V^R) \right], \quad (10)$$

and this  $\tilde{g}_A$  has been used in Eq. (9). Recall that  $g_A$  corresponds to the physical value, with all short-range radiative corrections removed.

In Eq. (9),  $\delta_\alpha^{(1)}$  represents the model-independent radiative correction to  $G_F$ , which depends only on the kinematics of the electron, while  $\delta_\alpha^{(2)}$  gives the model-independent radiative corrections to the coefficients of the

<sup>4</sup> The convention for the dimensional parameter  $\varepsilon$  used here is:  $d = 4 - 2\varepsilon$ .

angular correlation terms,  $\vec{\beta} \cdot \hat{p}_\nu$  and  $\hat{n} \cdot \vec{\beta}$ . The explicit expressions for  $\delta_\alpha^{(1)}$  and  $\delta_\alpha^{(2)}$  are:

$$\begin{aligned} \delta_\alpha^{(1)} = & 3 \ln\left(\frac{m_N}{m_e}\right) + \frac{1}{2} + \frac{1 + \beta^2}{\beta} \ln\left(\frac{1 + \beta}{1 - \beta}\right) - \frac{1}{\beta} \ln^2\left(\frac{1 + \beta}{1 - \beta}\right) + \frac{4}{\beta} L\left(\frac{2\beta}{1 + \beta}\right) \\ & + 4 \left[ \frac{1}{2\beta} \ln\left(\frac{1 + \beta}{1 - \beta}\right) - 1 \right] \left[ \ln\left(\frac{2(E_e^{\max} - E_e)}{m_e}\right) + \frac{1}{3} \left( \frac{E_e^{\max} - E_e}{E_e} \right) - \frac{3}{2} \right] \\ & + \left( \frac{E_e^{\max} - E_e}{E_e} \right)^2 \frac{1}{12\beta} \ln\left(\frac{1 + \beta}{1 - \beta}\right), \end{aligned} \tag{11}$$

$$\begin{aligned} \delta_\alpha^{(2)} = & \frac{1 - \beta^2}{\beta} \ln\left(\frac{1 + \beta}{1 - \beta}\right) + \left( \frac{E_e^{\max} - E_e}{E_e} \right) \frac{4(1 - \beta^2)}{3\beta^2} \left[ \frac{1}{2\beta} \ln\left(\frac{1 + \beta}{1 - \beta}\right) - 1 \right] \\ & + \left( \frac{E_e^{\max} - E_e}{E_e} \right)^2 \frac{1}{6\beta^2} \left[ \frac{1 - \beta^2}{2\beta} \ln\left(\frac{1 + \beta}{1 - \beta}\right) - 1 \right]. \end{aligned} \tag{12}$$

Here  $E_e^{\max} = (m_n^2 - m_p^2 + m_e^2)/2m_n$  is the maximum electron energy, and  $L(z)$  is the Spence function defined by

$$L(z) = \int_0^z \frac{dt}{t} \ln(1 - t). \tag{13}$$

The factor  $C_0(E_e)$  contains the recoil corrections to the overall rate. It is given by

$$C_0(E_e) = 1 + \frac{1}{m_N(1 + 3\tilde{g}_A^2)} \left\{ (\tilde{g}_A^2 - 2\mu_V \tilde{g}_A + 1) E_e^{\max} - \frac{m_e^2}{E_e} (1 + \tilde{g}_A^2) + 2\mu_V \tilde{g}_A (\beta^2 + 1) E_e \right\}, \tag{14}$$

where we have used  $E_\nu = E_e^{\max} - E_e + \mathcal{O}(1/m_N)$ . The other coefficients  $C_i(E_e)$  ( $i = 1, 2, \dots, 5$ ) are given by

$$C_1(E_e) = \tilde{a} \left\{ 1 + \frac{1}{m_N} \left[ \frac{(\tilde{g}_A^2 + 2\mu_V \tilde{g}_A + 1) m_e^2}{1 + 3\tilde{g}_A^2} \frac{1}{E_e} + \frac{(\tilde{g}_A^2 + 1)[8\mu_V \tilde{g}_A E_e - 4E_e^{\max} \tilde{g}_A (\tilde{g}_A + \mu_V)]}{(\tilde{g}_A^2 - 1)(1 + 3\tilde{g}_A^2)} \right] \right\}, \tag{15}$$

$$C_2(E_e) = \tilde{A} \left\{ 1 + \frac{1}{m_N} \left[ \frac{(\tilde{g}_A^2 - 1)(\tilde{g}_A + \mu_V)}{2\tilde{g}_A(1 + 3\tilde{g}_A^2)} (E_e^{\max} - E_e) + \frac{E_e(\mu_V - 1)}{\tilde{g}_A - 1} - \beta^2 E_e \frac{\tilde{g}_A^2 + 2\tilde{g}_A \mu_V + 1}{1 + 3\tilde{g}_A^2} \right] \right\}, \tag{16}$$

$$C_3(E_e) = \tilde{A} \frac{E_e(\tilde{g}_A - \mu_V)}{2m_N \tilde{g}_A}, \tag{17}$$

$$C_4(E_e) = \tilde{B} \left\{ 1 + \frac{1}{m_N} \left[ \frac{E_e \beta^2 (\tilde{g}_A^2 - 1)(\tilde{g}_A - \mu_V)}{2\tilde{g}_A(1 + 3\tilde{g}_A^2)} + \frac{(\tilde{g}_A + \mu_V)(\tilde{g}_A - 1)^2}{(\tilde{g}_A + 1)(1 + 3\tilde{g}_A^2)} (E_e - E_e^{\max}) \right] \right\}, \tag{18}$$

$$C_5(E_e) = \tilde{B} \frac{(\tilde{g}_A + \mu_V)}{2m_N \tilde{g}_A} (E_e^{\max} - E_e), \tag{19}$$

where  $\tilde{a}$ ,  $\tilde{A}$ ,  $\tilde{B}$  are given by Eq. (6) with the substitution  $g_A \rightarrow \tilde{g}_A$ . It is to be noted that Eq. (9) exhibits angular dependences that are missing in Eq. (5). These extra angular dependences arise from the NLO contributions that have been included in the  $1/m_N$  corrections (which leads to Eq. (9)) but ignored in the LO evaluation (which leads to Eq. (5)). It has been a common practice to approximate the overall kinematic factor in Eq. (8) by applying an expansion in  $1/m_N$ . If convenient, one could use the following approximation:

$$\frac{m_p E_\nu^2}{E_p + E_\nu + E_e \vec{\beta} \cdot \hat{p}_\nu} \simeq (E_e^{\max} - E_e)^2 \left[ 1 + \frac{1}{m_N} (3E_e - E_e^{\max} - 3E_e \vec{\beta} \cdot \hat{p}_\nu) \right], \tag{20}$$



where we have used

$$E_\nu \simeq (E_e^{\max} - E_e) \left[ 1 + \frac{E_e}{m_N} (1 - \vec{\beta} \cdot \hat{p}_\nu) \right]. \quad (21)$$

The angular dependence appearing in Eq. (20) needs to be considered simultaneously with the angular dependences contained in Eq. (9).

The model-independent radiative correction  $\delta_\alpha^{(1)}$  in Eq. (11) agrees with that obtained by Sirlin [3], while  $\delta_\alpha^{(2)}$  in Eq. (12) also agrees with the result reported by Garcia and Maya [20]. We note that recoil corrections have also been calculated in the literature using the conventional methods. For instance, Wilkinson [21] evaluated corrections to the decay rate and the correlation coefficient  $A$ , and Bilen'kii et al. [22] computed corrections to the decay rate and the correlation coefficient  $a$ . Furthermore, Holstein [23] considered recoil corrections to all the observables for general nuclear beta-decays. Our results for the recoil corrections agree with those found in these previous studies.

#### 4. Discussion and conclusions

As mentioned in the introduction, a prime issue in the studies of neutron beta-decay is to deduce the precise value of  $V_{ud}$  from the experimental data. Another issue is the extraction of the value of  $g_A$  from the data. We shall discuss here the significance of our present calculation in connection with these two issues.

To obtain the actual numerical values of  $V_{ud}$  and  $g_A$  we need to know the values of the LECs,  $e_V^R$  and  $e_A^R$ , pertaining to the lepton-current nucleon-current vertex. These LECs parameterize short-distance physics not explicitly included in the effective Lagrangian,  $\mathcal{L}_\beta$ , and they need to be determined empirically using appropriate observables. This is an important line of studies for the future. Here, instead, we discuss simple order-of-magnitude estimates of the LEC  $e_V^R$ , which is the most important LEC in neutron beta-decay. Based on the general estimation of a photon-loop diagram, one may expect the natural scale for this parameter to be of the order of  $(\alpha/2\pi)e_V^R \sim 2 \times 10^{-2}$ , with  $e_V^R \sim \ln(m_e/\Lambda_\chi)$ . To obtain another rough estimate of  $e_V^R$  we may compare our result for the neutron decay rate obtained from Eq. (8) with Eq. (6) of Marciano and Sirlin [5]. Thus we introduce the premise

$$e_V^R \simeq -\frac{5}{4} - 4 \ln\left(\frac{m_W}{m_Z}\right) + 3 \ln\left(\frac{m_W}{m_N}\right) + \ln\left(\frac{m_W}{m_A}\right) + 2C + A_g, \quad (22)$$

where  $m_W$ ,  $m_Z$  are the masses of the  $W$ ,  $Z$  bosons and  $m_A$  is the axial mass scale. As is customary, we define the Fermi constant  $G_F$  of muon decay by absorbing the factor  $1 + (3\alpha/4\pi) \ln(m_W/m_Z)$  into  $G_F$  [24]. The contribution  $\ln(m_W/m_Z)$  in Eq. (22) is actually the difference between the contribution of the  $Z$ -box diagrams<sup>5</sup> in neutron beta-decay and the contribution of the  $Z$ -box diagrams in muon decay. In Eq. (22), the major contributions to the right-hand side originate from the short-range virtual photon corrections to the Fermi transition from the weak vector and axial-vector vertices. The former gives the contribution,  $3 \ln(m_W/m_N)$ , and the latter  $\ln(m_W/m_A)$ . The  $C$  in the expression is the long-range model-dependent correction coming from the axial-current and anomalous magnetic moments of the nucleon, and is proportional to  $(\mu_S g_A)$  where  $\mu_S$  is the isoscalar magnetic moment of the nucleon. A value of  $2C = 1.77$  was found in Ref. [5]. In an HB $\chi$ PT calculation, however, we have verified that a correction estimated from the diagrams of  $C$  is of higher order  $\propto \alpha/(2\pi)(\bar{Q}/m_N)^2$  and can be neglected (see Section 2). Finally, the  $A_g$  term, which includes a short-range strong-interaction correction, is very small:  $A_g \simeq -0.34$  [5]. In this connection, it might be of interest to decompose, following Cirigliano et al. [2], our  $e_V^R$  into two parts:  $e_V^R = e_V^{\text{SD}} + \tilde{e}_V^R$ . The  $e_V^{\text{SD}}$  term describes the universal short-distance physics of electroweak theory discussed by

<sup>5</sup> The  $Z$ -box diagrams here refer to diagrams like the one in Fig. 1(f), with the photon replaced by the  $Z$  boson; see Fig. 3 in Ref. [24].



Sirlin [24], while the  $\tilde{e}_V^R$  term describes short-distance hadronic physics. It is possible that the  $A_g$  term is associated with the  $\tilde{e}_V^R$  term. The above considerations lead to a rough estimate,  $e_V^R \simeq 20$ , i.e.,  $[\alpha/(2\pi)]e_V^R \sim 4 \times 10^{-2}$ , which is of a natural size as discussed above. The above comparison also leads us to expect that the dominant contribution to  $e_V^R$  comes from the short-range electroweak corrections.

The LEC  $e_A^R$  enters only as a radiative correction to  $g_A$  in Eq. (10), and therefore it may seem that there is no significant motivation to remove the radiative correction  $\frac{\alpha}{4\pi}(e_A^R - e_V^R)$  from  $\tilde{g}_A$  defined in Eq. (10) and deduce the values of  $g_A$ . Indeed, if we limit ourselves to neutron beta-decay, all the observables can be expressed using  $\tilde{g}_A$  without referring to  $g_A$ . However, since radiative corrections are specific to individual processes, there should be cases wherein the removal of  $\frac{\alpha}{4\pi}(e_A^R - e_V^R)$  from  $\tilde{g}_A$  has physical consequences and hence  $e_A^R$  does play a significant role. A possible example is the Goldberger–Treiman relation,  $g_A m_N = f_\pi g_{\pi N}$ , where  $g_{\pi N}$  is the pion–nucleon coupling constant. To elaborate on this point, it is useful to illustrate processes which necessitate the introduction of the LEC,  $e_A^R$ . To this end, we consider diagrams containing the exchange of a pion (pion-pole) plus a virtual photon. These diagrams involve three distinct one-particle-irreducible vertex functions. The first type is a nucleon–nucleon–lepton–lepton four-point vertex in which a virtual photon couples to both the nucleon and the leptonic currents. This class of diagrams requires a counter term involving  $e_A$  associated with  $g_A$ . The second type is a lepton–lepton–pion three-point vertex wherein a virtual photon only couples to the pion, the pion–lepton vertex or the lepton, and this vertex is related to the pion decay constant  $f_\pi$ . Some of the LECs arising from this type of diagrams can be found in the chiral Lagrangian considered by Knecht et al. [16]. These LECs are also related to the “inner” radiative corrections calculated for pion beta-decay, see, e.g., [25]. The third type is a nucleon–nucleon–pion vertex in which a virtual photon only couples to the pion, the pion–nucleon vertex or the nucleon, and this vertex is related to  $g_{\pi N}$ . The corresponding LECs are the  $g_i$ 's appearing in Müller and Meißner's work [10]. To our knowledge, however, no systematic HB $\chi$ PT study of the Goldberger–Treiman relation including the radiative corrections associated with each of the vertices has been done so far. In fact the radiative correction  $e_A^R$  really has not been fully studied yet in the standard approach. Instead it has usually been assumed that  $e_A^R \simeq e_V^R$ , which makes the radiative correction to  $g_A$  small. Such radiative corrections could contribute to the evaluation of the Goldberger–Treiman discrepancy, but there is clearly not yet enough information to determine whether they turn out to be significant in comparison with the chiral symmetry breaking term.

As discussed in Section 2, we have not included in our work radiative corrections involving the NLO vertex or pion loop diagrams. The former should be suppressed at least by a factor of  $\mu_V \bar{Q}/(2m_N) \simeq 2 \times 10^{-3}$ , and the latter by a factor of  $(m_\pi/\Lambda_\chi)^2 \simeq 2 \times 10^{-2}$  relative to the leading radiative corrections. Also omitted from our work are the isospin breaking effects, which are naturally incorporated in the N<sup>2</sup>LO heavy-baryon chiral Lagrangian [8] not explicitly written in this Letter. Recently, Kaiser [26] studied isospin violation corrections to  $G_F V_{ud}$  using HB $\chi$ PT and found that the isospin breaking corrections are of the order of  $10^{-5}$ . To the accuracy of our present concern, we can safely neglect the isospin violation corrections.

We now summarize. Using the effective field theory for neutron beta-decay, we have calculated the decay rate of the neutron and the angular correlation coefficients including recoil corrections and radiative corrections of order  $\alpha$ . We have included all non-radiative terms through N<sup>2</sup>LO except those which are negligible because of the extremely small value of  $\bar{Q}$  for neutron beta-decay. Our results reproduce the model-independent radiative corrections and recoil corrections in the literature. The short-range radiative corrections of the earlier calculations are replaced in our theory by the two finite radiative LECs,  $e_V^R$  and  $e_A^R$  where  $e_V^R$  affects  $G_F$  and the difference,  $e_V^R - e_A^R$ , affects  $g_A$ . Via comparison with the results of the existing model calculations, we have argued that the value of  $e_V^R$  is of a natural scale. An advantage of our EFT approach is the possibility of evaluating higher order corrections in a systematic way, and the possibility to parameterize the strong interaction dependent contributions in terms of well-defined LECs, which can in principle be obtained from independent experiments. The next order corrections in the EFT for neutron beta-decay are estimated to be of the order  $10^{-5}$  or smaller. Therefore, to the extent that the LECs involved in the present calculation are of a “natural” size (as discussed above), we expect our expressions for the rate and the angular correlation coefficients to be accurate to better than  $10^{-3}$ .

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