On special generic maps of simply connected $2n$-manifolds into $\mathbb{R}^3$

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Abstract


The purpose of this paper is to study special generic maps into $\mathbb{R}^3$. We prove the congruence formula and equality which show relations between the source manifold and the singular point set of a map. As corollaries, we determine the homeomorphism type of the source manifold in the 4-dimensional case and give an unknotting result for the singular point set of a special generic map of $S^4$ into $\mathbb{R}^3$.

Keywords: Special generic map, stable map, fold, cusp, swallow tail.

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1. Introduction

Let $f$ be a smooth map from a closed $n$-dimensional manifold $M^n$ into a $p$-dimensional manifold $N^p$ ($n \geq p$). Homological properties of the singular point set of $f$ are one of the most interesting problems in singularity theory. However, most of the known results are in mod 2 (e.g. the real Thom polynomial in [14], Whitney-Thom-Levine's result on the number of cusp points in [7, 14, 15]). We want to know their homological properties in finer forms (i.e., modulo 4, 8, etc.) and to evaluate the number of connected components of the singular point set.

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In this paper we restrict ourselves to special generic maps of a simply connected smooth manifold $M$ into $\mathbb{R}^3$: A $C^\infty$-map $f : M \to \mathbb{R}^3$ is special generic if every singular point of $f$ is a definite fold, i.e., there exist local coordinate systems $(x_1, \ldots, x_n)$ centered at $p$ and $(y_1, y_2, y_3)$ centered at $f(p)$ under which $f$ is of the form:

\begin{align*}
y_1 &= x_i, \quad i = 1, 2, \\
y_3 &= x_3^2 + \cdots + x_n^2.
\end{align*}

In the paper we prove the following Theorems A and B.

**Theorem A.** Let $M^4$ be a closed, simply connected 4-dimensional manifold and $f : M^4 \to \mathbb{R}^3$ be a special generic map. Then we have

$$\sigma(M^4) = S(f) \cdot S(f) \pmod{16},$$

where $S(f)$ is the singular point set of $f$, $\sigma(M^4)$ denotes the signature of $M$, and $S(f) \cdot S(f)$ stands for the self-intersection number of $S(f)$ in $M^4$.

**Theorem B.** Let $M$ be a closed, simply connected 2n-dimensional manifold ($n \geq 2$). For a special generic map $f : M \to \mathbb{R}^3$, we have

(i) $S(f)$ is a union of 2-spheres,

(ii) $\chi(S(f)) = 2\#S(f) = \chi(M),$

where $\#S(f)$ denotes the number of connected components of $S(f)$ and $\chi(X)$ is the Euler characteristic of $X$.

As corollaries, we determine the homeomorphism type of the source manifold in the 4-dimensional case and show that the set of singular points of special generic maps of $S^4$ into $\mathbb{R}^3$ is unknotted:

**Corollary 6.2.** Let $M$ be a closed, simply connected 4-manifold. If $M$ admits a special generic map $f : M \to \mathbb{R}^3$ such that $S(f)$ is connected, then $M$ is homeomorphic to $S^4$.

**Corollary 6.4.** For a special generic map $f : S^4 \to \mathbb{R}^3$ such that its Stein factorization $W_f$ is a 3-ball (for the definition of the Stein factorization see Section 5), $S(f)$ is a 2-sphere and unknotted.

In a more generalized setting, we have the following congruence formula for the self-intersection number of $S(f)$.

**Theorem C.** Let $M^4$ be a closed, oriented 4-dimensional manifold with $H_1(M^4; \mathbb{Z}) = 0$ and $f : M^4 \to \mathbb{R}^3$ be a stable map. Then we have

$$\sigma(M^4) = -S(f) \cdot S(f) \pmod{4}.$$
2. Euler characteristics of the source manifold and the singular point set

In this section we recall Fukuda’s results on the relations between the source manifold and set of singular points when the map has only $A_k$-type ($1 \leq k \leq n$) singularities. At the end of this section we will study the nonorientability of the singular point set $S(f)$ of a map which has only fold singularities. Let $f: M^n \to \mathbb{R}^3$ be a smooth map which has only fold singularities. If $p \in S(f)$, then we can choose local coordinate systems $(x_1, \ldots, x_n)$ centered at $p$ and $(y_1, \ldots, y_p)$ centered at $f(p)$ so that $f$ has the following forms:

$$
y_i = x_i, \quad 1 \leq i \leq p - 1,
$$

$$
y_p = \pm x_p^2 \pm \cdots \pm x_n^2.
$$

From the normal form we see that $S(f)$ is a $(p - 1)$-dimensional manifold and the restricted map $f|S(f)$ is a smooth immersion.

If a smooth map $f: M^n \to \mathbb{R}^p$ ($n \geq p$) admits only definite fold points in the above, i.e., $y_p = x_p^2 + \cdots + x_n^2$, such a map is called special generic (this terminology is originally due to Burlet and de Rham [2]).

Now we recall Fukuda’s results in [4]. Let $A_k(f)$ be the set of $A_k$-type singularities ($1 \leq k \leq p$) for a smooth map $f: M^n \to \mathbb{R}^p$ (see Morin [10], in which $A_k$-type singularities are referred to as $\Sigma^{n-p+1,1,...,1,0}$ in the language of the Thom-Boardman symbols).

**Lemma 2.1** (Fukuda [4]). Let $M^n$ be a closed $n$-manifold and $f: M^n \to \mathbb{R}^p$ ($n \geq p$) be a smooth map which has only $A_k$-type singularities ($1 \leq k \leq p$). Then we have

$$
\chi(M^n) = \chi(\overline{A_k(f)}) \pmod{2},
$$

where $\overline{A_k(f)}$ is the topological closure of $A_k(f)$.

In particular, if $f$ has only fold singularities ($A_1$-type), then the Euler characteristic of $M^n$ has the same parity as that of the singular point set $S(f)$.

**Definition 2.2.** Suppose that $n \geq p$ and $n - p + 1$ is even. For a smooth map $f: M^n \to \mathbb{R}^p$ which admits only fold singularities, a point $p \in S(f)$ is called a fold point with index $\lambda \pmod{2}$ if $f$ has the following normal form using local coordinates at $p$ and $f(p)$:

$$
y_i = x_i, \quad 1 \leq i \leq p - 1,
$$

$$
y_p = -x_p^2 - \cdots - x_{p+\lambda-1}^2 + x_{p+\lambda}^2 + \cdots + x_n^2.
$$

We set

$$S^+(f) = \{ p \in S(f) : \text{index } \lambda \text{ is even} \},
$$

$$S^-(f) = \{ p \in S(f) : \text{index } \lambda \text{ is odd} \}.
$$

These two sets are clearly well defined since $n - p + 1$ is even.
Lemma 2.3 (Fukuda [4]). Let $M^n$ be a closed $n$-manifold. Suppose that $n \geq p$ and $n - p + 1$ is even. Let $f : M^n \to \mathbb{R}^p$ be a smooth map which has only fold singularities. Then we have
\[
\chi(M^n) = \chi(S^+(f)) - \chi(S^-(f)).
\]

Remark 2.4. When $f : M^n \to \mathbb{R}^3$ has only fold singularities, Lemma 2.1 says that if the Euler characteristic of $M^n$ is odd, then the singular point set $S(f)$ contains a nonorientable surface with odd genus.

Lemma 2.3 plays a fundamental role in the proof of Theorem B stated in the introduction.

We end this section by generalizing this remark.

Proposition 2.5. Let $M^n$ be a closed $n$-manifold and $f : M^n \to \mathbb{R}^p$ ($n \geq p \geq 3$) be a smooth map which admits only fold singularities. If $\chi(M^n)$ is odd, then $S(f)$ is unorientable.

Proof. As usual, we define the normal bundle, $\nu(\tilde{f})$, of the immersion $\tilde{f} := f|S(f)$ by the exactness of
\[
0 \to \tau(S(f)) \to \tilde{f}^*\tau(\mathbb{R}^p) \to \nu(\tilde{f}) \to 0,
\]
where $\tau(S(f))$ is the tangent bundle of $S(f)$ and $\tilde{f}^*\tau(\mathbb{R}^p)$ the induced bundle. We then have
\[
\tau(S(f)) \oplus \nu(\tilde{f}) = \tilde{f}^*\tau(\mathbb{R}^p).
\]
Note that $\tilde{f}^*\tau(\mathbb{R}^p)$ is trivial. This implies
\[
w(S(f)) \cdot w(\nu(\tilde{f})) = w(\tilde{f}^*\tau(\mathbb{R}^p)) = 1,
\]
where $w(X)$ is the total Stiefel–Whitney class of $X$.

Since $S(f)$ is a $(p - 1)$-dimensional manifold, the normal bundle $\nu(\tilde{f})$ is a line bundle over $S(f)$. Then we set $w(\nu(\tilde{f})) = 1 + \alpha$, where $\alpha \in H^1(S(f); \mathbb{Z}/2)$. Thus we have
\[
w(S(f)) = 1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1},
\]
where the powers are cup products. Hence we have $w_i(\nu(\tilde{f})) = \alpha = w_i(S(f))$. Using the Poincaré–Hopf Theorem modulo 2 and applying Lemma 2.1, we have
\[
\chi(M^n) = \chi(S(f)) \quad (\text{mod} \ 2) \quad (\text{Lemma 2.1})
\]
\[
= \langle w_{p-1}(S(f)), [S(f)]_2 \rangle \quad (\text{mod} \ 2)
\]
\[
= \langle \alpha^{p-1}, [S(f)]_2 \rangle \quad (\text{mod} \ 2)
\]
\[
= \langle (w_1(S(f)))^{p-1}, [S(f)]_2 \rangle \quad (\text{mod} \ 2).
\]
The assumption that $\chi(M^n)$ be odd implies that $w_1(S(f))$ is nontrivial, which means that $S(f)$ is nonorientable. This completes the proof. \qed
3. Proof of Theorem C

In this section we prove the following Theorem C.

**Theorem C.** Let $M^4$ be a closed, oriented 4-dimensional manifold with $H_1(M^4; \mathbb{Z}) = 0$ and $f: M^4 \rightarrow \mathbb{R}^3$ be a stable map. Then we have

$$\sigma(M^4) = -S(f) \cdot S(f) \quad (\text{mod } 4).$$

Let $M$ be a closed $n$-dimensional manifold and $f: M \rightarrow \mathbb{R}^3$ be a stable map. If $p \in S(f)$, then there exist local coordinates $(x, y, z_1, \ldots, z_{n-2})$ and $(y_1, y_2, y_3)$ centered at $p$ and $f(p)$ such that $f$ has one of the following normal forms:

1. $(x, y, z_1, \ldots, z_{n-2}) \rightarrow (x, y, \pm z_1^2 \pm \cdots \pm z_{n-2}^2), \text{ fold},$
2. $(x, y, z_1, \ldots, z_{n-2}) \rightarrow (x, y, z_1^2 + xy + z_2^2 + \cdots + z_{n-2}^2), \text{ cusp},$
3. $(x, y, z_1, \ldots, z_{n-2}) \rightarrow (x, y, z_1^2 + xy^2 + xy \pm z_2^2 \pm \cdots \pm z_{n-2}^2), \text{ swallow tail}.$

In what follows, we will investigate the relation between the self-intersection number of $S(f)$ in $M^4$ and the signature of $M^4$.

**Lemma 3.1.** Let $M^4$ be a closed 4-manifold and $f: M^4 \rightarrow \mathbb{R}^3$ be a stable map. Then we have

$$\chi(M^4) \equiv \chi(S(f)) \quad (\text{mod } 2).$$

**Proof.** By Lemma 2.1 we have

$$\chi(M^4) = \chi(A_1(f)) + \chi(A_2(f)) + \#A_3(f) \quad (\text{mod } 2), \quad (*)$$

where $\#A_3(f)$ denotes the number of $A_3$-type (swallow tail) singular points. Since $A_2(f)$ is a union of circles, we have

$$\chi(A_2(f)) = 0. \quad (**)$$

According to Ando [1], the Thom polynomial of $A_3(f)$ is $w_1^2 + w_1 w_3$. Hence we have

$$\#A_3(f) \equiv (w_1^2 + w_1 w_3, [M^4]_2) \quad (\text{mod } 2). \quad (***$$

Since $M^4$ is oriented, $w_1 = 0$. Therefore $\#A_3(f) \equiv 0 \pmod{2}$. Since $\overline{A_3(f)}$ is $S(f)$, the conclusion follows from $(*)$, $(**)$ and $(***). \quad \square$

**Definition 3.2.** Let $M$ be a closed manifold. A closed 2-dimensional submanifold $F$ of $M$ is called a *characteristic surface* of $M$ if the mod 2 cycle $[F]_2 \in H_2(M; \mathbb{Z}/2)$ is Poincaré dual to the 2nd Stiefel–Whitney class $w_2(M) \in H^2(M; \mathbb{Z}/2)$.

The following lemma was first given by Rohlin [13] and fully proved in a generalized form by Guillou and Marin [5].
Lemma 3.3 [5, 13]. Let $M$ be a closed, oriented 4-dimensional manifold with $H_1(M; \mathbb{Z}) = 0$ and $F$ be a characteristic surface of $M$. Then we have
\[ \sigma(M) = F \cdot F + 2\chi(F) \pmod{4}. \]

Lemma 3.4 (Thom [14]). Let $f: M^4 \rightarrow \mathbb{R}^3$ be a stable map. Then $S(f)$ is a characteristic surface of $M^4$.

Proof of Theorem C. Let $f: M^4 \rightarrow \mathbb{R}^3$ be a stable map. From Lemma 3.4, $S(f)$ is a characteristic surface of $M^4$. Then from Lemma 3.3 we have
\[ \sigma(M^4) = S(f) \cdot S(f) + 2\chi(S(f)) \pmod{4}. \] (1)

As we will see later, we have
\[ \sigma(M^4) = \chi(S(f)) \pmod{2}. \] (2)

Hence
\[ 2\sigma(M^4) = 2\chi(S(f)) \pmod{4}. \] (3)

Combining (1) and (3), we obtain the required result
\[ \sigma(M^4) = -S(f) \cdot S(f) \pmod{4}. \]

We have the above congruence (2) as follows. We decompose $H^2(M^4; \mathbb{Q})$ into the positive eigenspace $H^+$ and the negative eigenspace $H^-$ of the symmetric bilinear form defining the signature of $M^4$:
\[ H^2(M^4; \mathbb{Q}) = H^+ \oplus H^- . \]

Then we have
\[ \sigma(M^4) = \dim H^+ - \dim H^- = \dim H^+ + \dim H^- \pmod{2} \]
\[ = 2\text{nd betti number of } M^4 \]
\[ = \chi(M^4) \pmod{2} \]
\[ = \chi(S(f)) \pmod{2}, \]
where the last congruence follows from Lemma 3.1. This completes the proof of the theorem. \( \square \)

The above congruence (2) implies

Corollary 3.5. Let $M^4$ be an oriented 4-dimensional manifold and $f: M^4 \rightarrow \mathbb{R}^3$ be a stable map. If the signature of $M^4$ is odd, then $S(f)$ contains an unorientable surface with odd genus.
4. Proof of Theorem A

In this section we prove the following

**Theorem 4.1.** Let $M^4$ be a closed, oriented 4-manifold and $N^3$ be an oriented 3-manifold. If $f : M^4 \to N^3$ is a stable map whose singular point set is a union of 2-spheres, then we have

$$\sigma(M^4) = S(f) \cdot S(f) \pmod{16}.$$

As we will see later in Section 6, for a special generic map of a simply connected 4-manifold $M^4$ into $\mathbb{R}^3$, the singular point set is a disjoint union of 2-spheres (see Lemma 6.1). Therefore Theorem 4.1 implies Theorem A.

**Lemma 4.2** (Thom [14]). Let $M^4$ be a closed, oriented 4-manifold and $N^3$ be an oriented 3-manifold. For a stable map $f : M^4 \to N^3$, $S(f)$ is a characteristic surface of $M^4$.

**Proof.** Since any oriented 3-manifold is parallelizable, $w_j(N^3) = 0$ (1 $\leq j \leq 3$). Hence $f^*w_j(N^3)$ do not appear in the Thom polynomial $P(\Sigma^{2,0}) = P(w(M^4), f^*w_j(N^3))$. The desired conclusion follows easily. □

**Proof of Theorem 4.1.** The method of the proof is similar to [6]. First fix an orientation of $M^4$. We assume that $S(f)$ has $k$ connected components and set $S(f) = S_1 \cup \cdots \cup S_k$. Moreover, we set

$$n_i = S_i \cdot S_i \geq 0, \quad 1 \leq i \leq p,$$

$$m_j = S_j \cdot S_j < 0, \quad p + 1 \leq j \leq k.$$ 

We construct a spin manifold $M_k$ by surgering the singular point set out and by induction on $i$ and $j$.

As the first step we construct a manifold $\tilde{M}_i$ such that $w_j(\tilde{M}_i) = [S_1 \cup \cdots \cup S_k]^2 \in \mathbb{H}^2(\tilde{M}_i; \mathbb{Z}/2)$ and that $\sigma(\tilde{M}_i) = \sigma(M^4) - S_i \cdot S_i$. Let $CP^2$ and $\overline{CP}^2$ be the complex projective plane and the one with the opposite orientation, respectively. Then $CP^i \subseteq CP^2$ (1 $\leq i \leq n_i + 1$) and $[CP^i] = w_i(\overline{CP}^2)$. Set $M_i = M^4 \# CP^2 \# \cdots \# CP^2_{n_i + 1}$. We construct $\tilde{M}_i$ from $M_i$ as follows.

Consider the connected sum $S_i \# CP^1 \# \cdots \# CP^1_{n_i + 1}$ in $M_i$. Set $\tilde{S}_i = S_i \# CP^1 \# \cdots \# CP^1_{n_i + 1}$. Then $\tilde{S}_i$ is a smoothly embedded 2-sphere in $M_i$. Let $\xi \in H_2(M_i^4; \mathbb{Z})$ be the homology class represented by $\tilde{S}_i$ and $\eta_i \in H_2(CP^2_i; \mathbb{Z})$ (1 $\leq i \leq n_i + 1$) the homology class represented by $CP^2_i$, respectively. Then the homology class $\zeta = \xi + \Sigma \eta_i \in H_2(M_i^4; \mathbb{Z})$ can be represented by $\tilde{S}_i$, using the natural isomorphism

$$H_2(M^4; \mathbb{Z}) \oplus H_2(\overline{CP}^2; \mathbb{Z}) \oplus \cdots \oplus H_2(CP^2_{n_i + 1}; \mathbb{Z}) \cong H_2(M_i^4; \mathbb{Z}).$$

The self-intersection number of $\tilde{S}_i$ in $M_i$ is

$$\tilde{S}_i \cdot \tilde{S}_i = \xi \cdot \xi + \Sigma \eta_i \cdot \eta_i = n_i - (n_i + 1) = -1.$$
Hence the tubular neighborhood of $\tilde{S}_1$ in $M_1$ is the $D^2$-bundle over $\tilde{S}_1$ with Euler number $-1 \in \pi_1(SO(2))$, which is denoted by $N(\tilde{S}_1)$. Then $\partial N(\tilde{S}_1)$ is the $(-1)$-Hopf bundle and diffeomorphic to $S^3$. We now set $\tilde{M}_1 = (M_1 - \text{Int } N(\tilde{S}_1)) \cup D^4$. Note that

$$
\tilde{M}_1 \# \mathbb{CP}^2 = (M_1 - \text{Int } D^4) \cup_{\partial} (\mathbb{CP}^2 - \text{Int } D^4)
$$

$$
= (M_1 - \text{Int } N(\tilde{S}_1)) \cup_{\text{id}} N(\tilde{S}_1) = M_1
$$

$$
= M^4 \# \mathbb{CP}_1^2 \# \cdots \# \mathbb{CP}_{n+1}^2.
$$

From the above construction we see

**Lemma 4.3.** $S_1 \cup \mathbb{CP}_1^1 \cup \cdots \cup \mathbb{CP}_{n+1}^1 (\subset M_1)$ lies in $N(\tilde{S}_1) = \mathbb{CP}^2 - \text{Int } D^4$ of the decomposition (*) of $M_1 = \tilde{M}_1 \# \mathbb{CP}^2$.

This lemma will be used at the end of this section.

The additivity of the signature implies

$$
\sigma(\tilde{M}_1) - 1 = \sigma(M^4) - (n_1 + 1).
$$

Hence we have

$$
\sigma(\tilde{M}_1) = \sigma(M^4) - S_1 \cdot S_1.
$$

\[\text{(X}_1)\]

Moreover, as we will see later, we have

$$
\text{w}_2(\tilde{M}_1) = [S_2 \cup \cdots \cup S_k]_{\mathbb{Z}/2} \in H^2(M_1; \mathbb{Z}/2).
$$

\[\text{(W}_1)\]

This completes the first step of our induction.

Next for $i = 2, \ldots, p$ we can construct $\tilde{M}_i$ and $M_i$ from $\tilde{M}_{i-1}$ inductively in the same way such that

$$
\sigma(\tilde{M}_i) = \sigma(\tilde{M}_{i-1}) - n_i = \sigma(M^4) - \sum S_i \cdot S_i.
$$

\[\text{(X}_i)\]

$$
\text{w}_2(\tilde{M}_i) = [S_{i+1} \cup \cdots \cup S_k]_{\mathbb{Z}/2} \in H^2(\tilde{M}_i; \mathbb{Z}/2).
$$

\[\text{(W}_i)\]

Hence we have

$$
\sigma(\tilde{M}_p) = \sigma(M^4) - (n_1 + \cdots + n_p) = \sigma(M^4) - \sum S_i \cdot S_i.
$$

\[\text{(X}_p)\]

$$
\text{w}_2(\tilde{M}_p) = [S_{p+1} \cup \cdots \cup S_k]_{\mathbb{Z}/2} \in H^2(\tilde{M}_p; \mathbb{Z}/2).
$$

\[\text{(W}_p)\]

Next for $j = p+1, \ldots, k$ we will make a similar process as above. Let $M_{p+1} = M_p \# \mathbb{CP}_1^2 \# \cdots \# \mathbb{CP}_{m_1}^2$, where $m_1 = |m_{p+1}| + 1$ and consider the connected sum $\tilde{S}_{p+1} = S_{p+1} \# \mathbb{CP}_1^2 \# \cdots \# \mathbb{CP}_{m_1}^2$. Then $\tilde{S}_{p+1}$ is also a smoothly embedded 2-sphere with self-intersection number $+1$ in $M_{p+1}$. Then we set

$$
\tilde{M}_{p+1} = (M_{p+1} - \text{Int } N(\tilde{S}_{p+1})) \cup \partial D^4.
$$

We see

$$
\tilde{M}_{p+1} \# \mathbb{CP}^2 = M_{p+1} = \tilde{M}_p \# \mathbb{CP}_1^2 \# \cdots \# \mathbb{CP}_2^2.
$$
Moreover, in the same way as \((X_i)\) we see
\[
\sigma(\tilde{M}_{p+1}) = \sigma(\tilde{M}_p) + |m_{p+1}| = \sigma(\tilde{M}_p) - \sum S_{p+1} \cdot S_{p+1}.
\]
\[(X_{p+1})\]
\[
w_2(\tilde{M}_{p+1}) = [S_{p+2} \cup \cdots \cup S_k]^2 \in H^2(\tilde{M}_{p+1}; \mathbb{Z}/2).
\]
\[(W_{p+1})\]
Repeating the same constructions until all the 2-spheres that are obstructions to being a spin manifold are surgered out, we have
\[
\sigma(\tilde{M}_k) - \sigma(\tilde{M}_{k-1}) - m_k - \cdots - \sigma(M^4) - \sum n_i - \sum m_j
\]
\[
= \sigma(M^4) - S(f) \cdot S(f),
\]
\[(X_k)\]
\[
w_2(\tilde{M}_k) = 0.
\]
\[(W_k)\]
Hence \(M_k\) is a spin manifold. From Rohlin’s theorem [11], \(\sigma(\tilde{M}_k) = 0 \pmod{16}\). Thus from \((X_k)\) we have the required result.

**Proof of \((W_i)\).** First we prove \((W_i)\). According to Wu’s formula [9, p. 1361], on a closed, oriented smooth 4-manifold, \(w_2\) is characterized by the property that \(w_2 \cup v = v \cup v\) for any \(v \in H^2(M; \mathbb{Z}/2)\). So it is sufficient to show that \([S_2 \cup \cdots \cup S_k]^2 \cup v = v \cup v\) for all \(v \in H^2(M; \mathbb{Z}/2)\). Equivalently, by the Poincaré duality, it suffices to show that \([S_2 \cup \cdots \cup S_k]_2 \cdot y = y \cdot y \pmod{2}\) for all \(y \in H_2(M; \mathbb{Z}/2)\). From Lemma 4.2 we have
\[
[S(f)]_2 \cdot x = x \cdot x \pmod{2}
\]
for all \(x \in H_2(M_i; \mathbb{Z}/2)\).

We set \([F] = [S_2 \cup \cdots \cup S_k]_2\) and \(m = n_1 + 1\). We have the following isomorphism.
\[
H_2(M_i) \oplus H_2(\mathbb{C}P^2_i) \cong H_2(M^4) \oplus H_2(\mathbb{C}P^2_1) \oplus \cdots \oplus H_2(\mathbb{C}P^2_m).
\]
Then every element \(y \in H_2(M_i)\) has the form
\[
y = x + a_1 v_1 + \cdots + a_m v_m \pmod{2},
\]
where \(x \in H_2(M^4), v_i \in H_2(\mathbb{C}P^2_i)\) \((1 \leq i \leq m)\).

Since \((S_2 \cup \cdots \cup S_k) \cap (\mathbb{C}P^2_1 \cup \cdots \cup \mathbb{C}P^2_m) = \emptyset\), we see that \([F] \cdot v_i = 0\) for \(i = 1, \ldots, m\). Hence we have
\[
[F] \cdot y = [F] \cdot x - [S(f)]_2 \cdot x \cdot x.
\]
On the other hand,
\[
[S_1] \cdot x + a_1 + \cdots + a_m = [S_1] \cdot x + a_1 v_1 + v_1 + \cdots + a_m v_m \cdot v_m
\]
\[
= [S_1] \cdot x + v_1 \cdot a_1 v_1 + \cdots + v_m \cdot a_m v_m
\]
\[
= ([S_1] + v_1 + \cdots + v_m) \cdot y
\]
\[
= 0,
\]
\[(1)\]
where we note that \(v_i \cdot x = 0\) and \(v_i \cdot v_j = 0\) \((i \neq j)\), since \(S_1 \cap \mathbb{C}P^4_i = \emptyset\) and \(\mathbb{C}P^4_i \cap \mathbb{C}P^4_j = \emptyset\) for \(i \neq j\).
The last equality in (2) can be seen as follows: From Lemma 4.3 we see \([S_1] + v_1 + \cdots + v_m) \in H_2(CP^2; \mathbb{Z}/2) \subset H_2(\tilde{M}_1; \mathbb{Z}/2) \oplus H_2(CP^2; \mathbb{Z}/2) \cong H_2(M_1; \mathbb{Z}/2)\). On the other hand, \(y \in H_2(\tilde{M}_1; \mathbb{Z}/2) \subset H_2(\tilde{M}_1; \mathbb{Z}/2) \oplus H_2(CP^2; \mathbb{Z}/2) \cong H_2(M_1; \mathbb{Z}/2)\). Thus \(([S_1] + v_1 + \cdots + v_m) \cdot y = 0\).

Moreover, we get
\[
a_1^2 + \cdots + a_m^2 = a_1 + \cdots + a_m + a_1(a_1 - 1) + \cdots + a_m(a_m - 1)
= a_1 + \cdots + a_m \pmod{2}.
\]
Therefore, from (1), (2) and (3) we have
\[
[F] \cdot y = [S(f)]_2 \cdot x + a_1^2 + \cdots + a_m^2
= x \cdot x + a_1 v_1 \cdot a_1 v_1 + \cdots + a_m v_m \cdot a_m v_m
= y \cdot y \pmod{2}.
\]
Thus from the characterization of \(w_2\), we have \([F] = [S \cup \cdots \cup S_i] = w_2(M_i)\). This completes the proof of (Wi).

In the same way we can prove (Wi) for \(i = 2, \ldots, k\). This completes the proof of Theorem 4.1. \(\square\)

5. Special generic maps and their Stein factorization

Let \(f: M \to \mathbb{R}^3\) be a stable map. It induces on \(M\) an equivalence relation, that is: \(x \sim x'\) if and only if \(f(x) = f(x') = y\) and \(x, x'\) belong to the same connected component of \(f^{-1}(y)\). We denote the natural projection by \(q: M \to M/\sim = W_f\) and let \(q': W_f \to \mathbb{R}^3\) be the map defined by \(f = q' \circ q\). This factorization of \(f\) is known in algebraic geometry as the Stein factorization (cf. [8]).

In what follows, we restrict ourselves to the case of a special generic map into \(\mathbb{R}^3\).

**Lemma 5.1.** Let \(M\) be a closed \(n\)-manifold. For a special generic map \(f: M \to \mathbb{R}^3\), we have the following properties:

1. \(W_f\) is a compact 3-manifold with boundary.
2. \(\partial W_f\) is homeomorphic to \(S(f)\).
3. \(q': W_f \to \mathbb{R}^3\) is a smooth immersion.
4. \(S(f)\) is orientable.

**Proof.** Let \(p \in S(f)\). Then there exist local coordinates \((x_1, \ldots, x_n)\) and \((y_1, y_2, y_3)\) centered at \(p\) and \(f(p)\) respectively, such that \(f\) is given by the following normal form
\[
y_i = x_n \quad i = 1, 2,
= x_3^2 + \cdots + x_n^2.
\]
Then we choose an open $\varepsilon$-neighborhood $U = \{x_1^2 + \cdots + x_n^2 < \varepsilon^2\}$ of $p$ in $M$. Then $f$ maps $U$ to $V = \{y_1^2 + y_2^2 + y_3^2 < \varepsilon^2, y_3 \equiv 0\}$. In addition, the open 2-disk $\{x_1^2 + x_2^2 < \varepsilon^2, x_3 = \cdots = x_n = 0\}$, which is a coordinate neighborhood around $p$ in $S(f)$, is mapped homeomorphically to $\{y_1^2 + y_2^2 < \varepsilon^2, y_3 = 0\}$. From the definition of the Stein factorization, $q(U)$ is homeomorphic to $V$. Then $\{q(U), f|q(U)\}$ is a chart of $W_f$. This proves (1). Evidently, $q^*: W_f \to \mathbb{R}^3$ is a smooth immersion. Hence $\partial W_f$ is orientable. It is also easy to see that $q(S(f))$ is homeomorphic to $\partial W_f$. Thus $S(f)$ is orientable. \hfill \square

**Remark 5.2.** It is easy to see from the normal form that the quotient map $q: M \to W_f$ induces the surjective homomorphism $q_*: \pi_1(M) \to \pi_1(W_f)$.

### 6. Proof of Theorem B

In this section we prove the following

**Theorem B.** Let $M$ be a closed, simply connected $2n$-dimensional manifold ($n \geq 2$). For a special generic map $f: M \to \mathbb{R}^3$, we have

(i) $S(f)$ is a union of 2-spheres,
(ii) $\chi(S(f)) - 2\#S(f) - \chi(M)$,

where $\#S(f)$ denotes the number of connected components of $S(f)$.

This theorem is an immediate conclusion combining the following Lemma 6.1 and Lemma 2.3.

**Lemma 6.1.** Let $M$ be a closed, simply connected $2n$-dimensional manifold ($n \geq 2$). For a special generic map $f: M \to \mathbb{R}^3$, $S(f)$ consists of only 2-spheres.

**Proof of Theorem B.** From Lemma 2.3 we have the following equality for a special generic map $f: M \to \mathbb{R}^3$, since $S^+(f) = S(f)$ and $S^-(f) = \emptyset$:

$$\chi(M) = \chi(S(f)).$$

Then by the above lemma, we have

$$\chi(S(f)) = 2\#S(f).$$

This completes the proof. \hfill \square

**Proof of Lemma 6.1.** Since $q^*: \pi_1(M) \to \pi_1(W_f)$ is surjective and $M$ is simply connected, $W_f$ is also simply connected. Hence $H_1(W_f; \mathbb{Z}) = 0$ and $H^1(W_f; \mathbb{Z}) = 0$. Consider the homology exact sequence of the pair $(W_f, \partial W_f)$

$$\cdots \to H_2(W_f, \partial W_f; \mathbb{Z}) \to H_1(\partial W_f; \mathbb{Z}) \to H_0(W_f; \mathbb{Z}) \to \cdots$$

From the Poincaré–Lefschetz duality,

$$H_2(W_f, \partial W_f; \mathbb{Z}) \cong H^1(W_f; \mathbb{Z}) = 0.$$
Therefore we have
\[ H_i(\partial W_f; \mathbb{Z}) = 0. \]

By the classification of 2-manifolds, \( \partial W_f \) consists of only 2-spheres. Hence by Lemma 5.1, \( S(f) \) is a union of 2-spheres. This completes the proof. \( \square \)

As stated in the introduction, we obtain the following corollary.

**Corollary 6.2.** Let \( M^4 \) be a closed, oriented, simply connected 4-manifold. If \( M^4 \) admits a special generic map \( f: M^4 \to \mathbb{R}^3 \) such that \( S(f) \) is connected, then \( M^4 \) is homeomorphic to the 4-sphere.

**Proof.** By Theorem B we have \( \chi(M^4) = 2 \). Since \( M^4 \) is simply connected, \( M^4 \) is a homotopy 4-sphere. The conclusion follows from [3]. \( \square \)

**Corollary 6.3.** Let \( M \) be a closed, simply connected \( 2n \)-manifold (\( n \geq 2 \)). If the Euler characteristic of \( M \) is odd, then there exist no special generic maps over \( M \) into \( \mathbb{R}^3 \).

For example, \( \mathbb{C}P^2 \) admits no special generic maps into \( \mathbb{R}^3 \).

**Corollary 6.4.** For a special generic map \( f: S^4 \to \mathbb{R}^3 \) such that \( W_f \) is a 3-ball, then \( S(f) \) is a 2-sphere and unknotted.

**Proof.** We define the composite map
\[
(S^4, S(f)) \xrightarrow{a} (W_f, \partial W_f) \xrightarrow{\psi} (D^3, S^2) \xrightarrow{h} \mathbb{R},
\]
where \( \psi \) is a diffeomorphism and \( h \) is a height function. We set \( \rho = h \circ \psi \circ q \). Then \( \rho(S(f)) \) has only two critical points. Hence \( S(f) \) is unknotted. This completes the proof. \( \square \)

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**References**

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