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Faber polynomials, Cayley–Hamilton equation and Newton symmetric functions

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Abstract

We give an explicit formula for the Faber polynomials and for generalized Faber polynomials introduced by H. Airault and J. Ren in [H. Airault, J. Ren, An algebra of differential operators and generating functions on the set of univalent functions, Bull. Sci. Math. 126 (5) (2002) 343–367]. We introduce a new family of polynomials related to the Faber polynomials of the second kind. This allows us to give a generalized Cayley–Hamilton equation.

Résumé

Nous donnons une formule explicite des polynômes de Faber et des polynômes de Faber généralisés introduits par H. Airault et J. Ren dans [H. Airault, J. Ren, An algebra of differential operators and generating functions on the set of univalent functions, Bull. Sci. Math. 126 (5) (2002) 343–367]. Nous introduisons une nouvelle famille des polynômes liés aux polynômes de Faber. Ceci conduit à une généralisation de l'équation de Cayley–Hamilton.

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1. Introduction

The Faber polynomials introduced by Faber [3] play an important role in different areas of mathematics and there is a rich literature [2,7,8,5] describing their properties and their applications. In this paper, our goal is to show that elementary linear algebra techniques can provide new tools for the analysis of Faber polynomials. Let us briefly recall the basic definitions. Let K be a compact set in $\mathbb{C}$, not a single point, whose complement $\widehat{\mathbb{C}}\setminus K$ (with respect to the extended plane) is simply connected. By the Riemann theorem on conformal mapping there exists a unique function $z=\psi(w)$, meromorphic for $|w|>1$, which maps the domain $|w|>1$ onto $\widehat{\mathbb{C}}\setminus K$ and satisfies the conditions

$$\psi(\infty)=\infty, \qquad \psi'(\infty)>0.$$

This condition implies that the function $z=\psi(w)$, being analytic in the domain $|w|>1$ without the point $w=\infty$, has a simple pole at the point $w=\infty$.

The nth Faber polynomials of the first kind $F_n(z)$ and of the second kind $G_n(z)$ associated to ψ can be given from the following generating function [3,7]

$$\frac{\psi'(w)}{\psi(w)-z}=\sum_{m=0}^{\infty}F_m(z)w^{-m-1}, \tag{1.1}$$

$$\frac{1}{\psi(w)-z}=\sum_{m=0}^{\infty}G_m(z)w^{-m-1}. \tag{1.2}$$

In this paper, the Laurent expansion of the mapping ψ is given by:

$$\psi(w)=w+\sum_{k=0}^{\infty}\frac{b_{k+1}}{w^k}, \quad w\to\infty. \tag{1.3}$$

The variables $(b_1,b_2,\ldots,b_n,\ldots)$ are in the subset $\mathcal{M}$ of $\mathbb{C}^{\mathbb{N}}$ such that ψ is univalent outside of the unit disk. From (1.1) and (1.2), we observe that the Faber polynomials $F_n(z)$ and $G_n(z)$ depend on the parameters $b_1,b_2,\ldots,b_n$. We write

$$F_n(z)=:F_n(b_1-z,b_2,\ldots,b_n),$$
$$G_n(z)=:G_n(b_1-z,b_2,\ldots,b_n)$$

and in particular we have:

$$\frac{\psi'(w)}{\psi(w)}=\sum_{n=0}^{\infty}F_n(b_1,b_2,\ldots,b_n)w^{-n-1}, \tag{1.4}$$

$$\frac{1}{\psi(w)}=\sum_{n=0}^{\infty}G_n(b_1,b_2,\ldots,b_n)w^{-n-1}, \tag{1.5}$$

with

$$F_0=G_0=1 \quad \text{and} \quad F_1=G_1=-b_1.$$

On the submanifold $\mathcal{M}$, we introduce the family of the partial differential operators $(W_n)_{n\geqslant 1}$, the variables are $b_1, b_2, \ldots, b_n, \ldots,$ and $\frac{\partial}{\partial b_n}$ denotes the partial derivative with respect to the nth variable b_n,

$$W_n(b_1, b_2, \ldots, b_n, \ldots) = -\frac{\partial}{\partial b_n} - \sum_{i=1}^{\infty} b_i \frac{\partial}{\partial b_{n+i}}.$$

We prove that:

Theorem 1.1. *The Faber polynomials $F_n(b_1, b_2, \ldots, b_n)$ and $G_n(b_1, b_2, \ldots, b_n)$ verify the following differential equations for any $n \geqslant 1$ and $m \geqslant 0$:*

$$W_n(b_1, b_2, \ldots, b_n, \ldots)F_m = n\delta_{n,m}, \tag{1.6}$$

$$W_n(b_1, b_2, \ldots, b_n, \ldots)G_m = G_{m-n}. \tag{1.7}$$

If $\psi(z)$ has the form

$$\psi(z) = z + \sum_{k=0}^{\infty} \frac{b_{p(k+1)}}{z^{pk+p-1}}, \quad z \to \infty, \tag{1.8}$$

we obtain

Corollary 1.2. *The following holds*:

$$W_n(b_p, b_{2p}, \ldots, b_{np}, \ldots)F_m = m\delta_{n,\frac{m}{p}}.$$

The solutions of this system of differential equations are:

Theorem 1.3. *For any $k \geqslant 2$, the polynomials $F_k(b_1, b_2, \ldots, b_k)$ and $G_k(b_1, b_2, \ldots, b_k)$ are given by*:

$$F_k(b_1, b_2, \ldots, b_k) = \sum_{i_2=0}^{[\frac{k}{2}]} \cdots \sum_{i_k=0}^{[\frac{k}{k}]} A_{(i_2, i_3, \ldots, i_k)} b_2^{i_2} \cdots b_k^{i_k} b_1^{k-2i_2-3i_3-\cdots-ki_k}, \tag{1.9}$$

$$G_k(b_1, b_2, \ldots, b_k) = \sum_{i_2=0}^{[\frac{k+1}{2}]} \cdots \sum_{i_k=0}^{[\frac{k+1}{k}]} B_{(i_2, i_3, \ldots, i_k)} b_2^{i_2} \cdots b_k^{i_k} b_1^{k-2i_2-3i_3-\cdots-ki_k}, \tag{1.10}$$

where

$$A_{(i_2, i_3, \ldots, i_k)} := (-1)^{k+3i_2+4i_3+\cdots+(k+1)i_k} \frac{(k - i_2 - 2i_3 - \cdots - (k-1)i_k - 1)!k}{(k - 2i_2 - 3i_3 - \cdots - ki_k)!i_2! \cdots i_k!},$$

$$B_{(i_2, i_3, \ldots, i_k)} := (-1)^{k+3i_2+4i_3+\cdots+(k+1)i_k} \frac{(k - i_2 - 2i_3 - \cdots - (k-1)i_k)!}{(k - 2i_2 - 3i_3 - \cdots - ki_k)!i_2! \cdots i_k!}.$$

Remark 1.1. The Faber polynomials $F_k(b_1, b_2, \dots, b_k)$ can be written as:

$$F_k(b_1, b_2, \dots, b_k) = \sum_{i_1+2i_2+\cdots+ki_k=k} \mathcal{A}_{(i_1,i_2,i_3,\dots,i_k)} b_1^{i_1} b_2^{i_2} \cdots b_k^{i_k},$$

where

$$\mathcal{A}_{(i_1,i_2,i_3,\dots,i_k)} := (-1)^{k+2i_1+3i_2+\cdots+(k+1)i_k} \frac{(i_1+i_2+i_3+\cdots+i_k-1)!k}{i_1!i_2!\cdots i_k!}.$$

The first Faber polynomials F_n are given by:

- $F_2 = b_1^2 - 2b_2$
- $F_3 = -b_1^3 + 3b_1b_2 - 3b_3$
- $F_4 = b_1^4 - 4b_1^2b_2 + 2b_2^2 + 4b_1b_3 - 4b_4$
- $F_5 = -b_1^5 + 5b_1^3b_2 - 5b_1^2b_3 + 5b_2b_3 - 5b_1(b_2^2 - b_4) - 5b_5$
- $F_6 = b_1^6 - 6b_1^4b_2 - 2b_2^3 + 6b_1^3b_3 + 3b_3^2 + b_1^2(9b_2^2 - 6b_4) + 6b_2b_4$ $+ 6b_1(-2b_2b_3 + b_5) - 6b_6$
- $F_7 = -b_1^7 + 7b_1^5b_2 - 7b_1^4b_3 + 7b_1^3(-2b_2^2 + b_4) + 7b_1^2(3b_2b_3 - b_5)$ $+ 7b_1(b_2^3 - b_3^2 - 2b_2b_4 + b_6) - 7(b_2^2b_3 - b_3b_4 - b_2b_5 + b_7)$
- $F_8 = b_1^8 - 8b_1^6b_2 + 8b_1^5b_3 + 4b_1^4(5b_2^2 - 2b_4) + 8b_1^3(-4b_2b_3 + b_5)$ $- 4b_1^2(4b_2^3 - 3b_3^2 - 6b_2b_4 + 2b_6) + 8b_1(3b_2^2b_3 - 2b_3b_4 - 2b_2b_5 + b_7)$ $+ 2(b_2^4 - 4b_2b_3^2 - 4b_2^2b_4 + 2b_4^2 + 4b_3b_5 + 4b_2b_6 - 4b_8)$
- $F_9 = -b_1^9 + 9b_1^7b_2 - 9b_1^6b_3 + 9b_2^3b_3 - 3b_3^3 + 9b_1^5(-3b_2^2 + b_4) + 9b_1^4(5b_2b_3 - b_5)$ $- 9b_2^2b_5 + 9b_4b_5 + 9b_3b_6 + 3b_1^3(10b_2^3 - 6b_3^2 - 12b_2b_4 + 3b_6) + 9b_2(-2b_3b_4 + b_7)$ $- 9b_1^2(6b_2^2b_3 - 3b_3b_4 - 3b_2b_5 + b_7) - 9b_1(b_2^4 - 3b_2^2b_4 + b_4^2 + 2b_3b_5$ $+ b_2(-3b_3^2 + 2b_6) - b_8) - 9b_9$
- $F_{10} = b_1^{10} - 10b_1^8b_2 - 2b_2^5 + 10b_1^7b_3 + 5b_1^6(7b_2^2 - 2b_4) + 10b_2^3b_4$ $+ 10b_1^5(-6b_2b_3 + b_5) + 5b_2^2(3b_3^2 - 2b_6) - 5b_1^4(10b_2^3 - 5b_3^2 - 10b_2b_4 + 2b_6)$ $+ 10b_1^3(10b_2^2b_3 - 4b_3b_4 - 4b_2b_5 + b_7)$ $+ 5b_1^2(5b_2^4 - 12b_2^2b_4 + 3b_4^2 + 6b_3b_5 + 6b_2(-2b_3^2 + b_6) - 2b_8)$ $- 10b_2(b_4^2 + 2b_3b_5 - b_8)$ $- 10b_1(4b_2^3b_3 - b_3^3 - 3b_2^2b_5 + 2b_4b_5 + 2b_3b_6 + b_2(-6b_3b_4 + 2b_7) - b_9)$ $+ 5(-2b_3^2b_4 + b_5^2 + 2b_4b_6 + 2b_3b_7 - 2b_{10})$
- $F_{11} = -b_1^{11} + 11b_1^9b_2 - 11b_1^8b_3 + 11b_1^7(-4b_2^2 + b_4) + 11b_1^6(7b_2b_3 - b_5)$ $+ 11b_1^5(7b_2^3 - 3b_3^2 - 6b_2b_4 + b_6) - 11b_1^4(15b_2^2b_3 - 5b_3b_4 - 5b_2b_5 + b_7)$ $- 11b_1^3(5b_2^4 - 10b_2^2b_4 + 2b_4^2 + 4b_3b_5 + b_2(-10b_3^2 + 4b_6) - b_8)$ $+ 11b_1^2(10b_2^3b_3 - 2b_3^3 - 6b_2^2b_5 + 3b_4b_5 + 3b_3b_6 + 3b_2(-4b_3b_4 + b_7) - b_9)$ $+ 11b_1(b_2^5 - 4b_2^3b_4 + 3b_3^2b_4 - b_5^2 - 2b_4b_6 + b_2^2(-6b_3^2 + 3b_6) - 2b_3b_7$

$$+ b_2(3b_4^2 + 6b_3b_5 - 2b_8) + b_{10}) - 11(b_2^4b_3 - b_2^3b_5 + b_3^2b_5 - b_5b_6 - b_4b_7$$
$$+ b_2^2(-3b_3b_4 + b_7) + b_3(b_4^2 - b_8) - b_2(b_3^3 - 2b_4b_5 - 2b_3b_6 + b_9) + b_{11}).$$

From Eq. (1.4), we deduce that

$$1 + b_1w + b_2w^2 + \cdots + b_kw^k + \cdots = \exp\left(-\sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \ldots, b_k)}{k} w^k\right).$$

We differentiate this equation with respect to w, we obtain

$$b_1 + 2b_2w + \cdots + kb_kw^{k-1} + \cdots = \bigl(1 + b_1w + b_2w^2 + \cdots + b_pw^p + \cdots\bigr) \times \left(-\sum_{k=1}^{+\infty} F_k w^{k-1}\right)$$

and equal coefficients of equal powers of w, it gives

$$F_{n+1} = -b_1F_n - \sum_{k=1}^{n-1} b_{n+1-k}F_k - (n+1)b_{n+1}. \tag{1.11}$$

Remark 1.2. Consider the differential operator

$$L(b_1, b_2, \ldots, b_n, \ldots) = \sum_{k=1}^{\infty} kb_k \frac{\partial}{\partial b_k}.$$

From (1.11), we deduce that

$$-kb_k = \sum_{1 \leqslant j \leqslant k} F_j b_{k-j}.$$

Thus $L = \sum_{k \geqslant 1} kb_k \frac{\partial}{\partial b_k} = -\sum_{j \geqslant 1} F_j (\sum_{k \geqslant j} b_{k-j} \frac{\partial}{\partial b_k})$ which gives

$$L = \sum_{j \geqslant 1} F_j W_j. \tag{1.12}$$

From (1.7), we have $W_jG_m = G_{m-j}$. Since G_m is homogeneous of degree m, we have $LG_m = mG_m$. If we compare with (1.12), it gives that

$$kG_k = \sum_{1 \leqslant j \leqslant k} F_j G_{k-j}.$$

Now, differentiating (1.4) with respect to b_1 and Eq. (1.5) with respect to w we obtain for $k \geqslant 1$:

$$G_k(b_1, b_2, \ldots, b_k) = -\frac{1}{k+1} \frac{\partial F_{k+1}}{\partial b_1}(b_1, b_2, \ldots, b_{k+1}). \tag{1.13}$$

We remark that $\frac{1}{k+1} \frac{\partial F_{k+1}}{\partial b_1}(b_1, b_2, \ldots, b_{k+1})$ is independent of b_{k+1} since F_{k+1} is a homogeneous polynomial of degree $k+1$ and b_{k+1} of weight $k+1$.

The first Faber polynomials G_n are given by:

- $G_1 = -b_1$
- $G_2 = b_1^2 - b_2$
- $G_3 = -b_1^3 + 2b_1b_2 - b_3$
- $G_4 = b_1^4 - 3b_1^2b_2 + 2b_1b_3 + b_2^2 - b_4$
- $G_5 = -b_1^5 + 4b_1^3b_2 - 3b_1^2b_3 - b_1(3b_2^2 - 2b_4) + 2b_2b_3 - b_5$
- $G_6 = b_1^6 - 5b_1^4b_2 + 4b_1^3b_3 - 3b_1^2(-2b_2^2 + b_4) - 2b_1(3b_2b_3 - b_5)$
  $- (b_2^3 - b_3^2 - 2b_2b_4 + b_6)$
- $G_7 = -b_1^7 + 6b_1^5b_2 - 5b_1^4b_3 - 2b_1^3(5b_2^2 - 2b_4) - 3b_1^2(-4b_2b_3 + b_5)$
  $+ b_1(4b_2^3 - 3b_3^2 - 6b_2b_4 + 2b_6) - (3b_2^2b_3 - 2b_3b_4 - 2b_2b_5 + b_7)$.

On the other hand, from Eq. (1.5) we obtain:

$$1 = \left(1 + \frac{G_1}{z} + \frac{G_2}{z^2} + \cdots + \frac{G_m}{z^m} + \cdots\right) \times \left(1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots + \frac{b_m}{z^m} + \cdots\right).$$

We observe that $b_1, b_2, \ldots,$ and $G_1, G_2, \ldots,$ play a symmetric role and they verify the following recurrence formula:

$$\begin{aligned}
&G_1 + b_1 = 0,\\
&G_2 + b_1G_1 + b_2 = 0,\\
&G_3 + b_1G_2 + b_2G_1 + b_3 = 0,\\
&\quad\vdots\\
&G_n + b_1G_{n-1} + b_2G_{n-2} + \cdots + b_{n-1}G_1 + b_n = 0. \qquad (1.14)
\end{aligned}$$

Corollary 1.4. *Let $\psi(z) = z + b_1 + \frac{b_m}{z^{m-1}}$, then the Faber polynomials $F_k(b_1, 0, \ldots, b_m, 0, \ldots)$ are given by*:

$$F_k = \sum_{i=0}^{j} (-1)^{k+(m-1)i} \frac{(k-(m-1)i-1)!k}{(k-mi)!i!} b_m^i b_1^{k-mi}$$

with $k = jm + p$ and $0 \leqslant p < m$.

For example, if $\psi(z) = z + b_1 + \frac{b_3}{z^2}$, one has:

$$\begin{aligned}
&F_1 = -b_1, && F_2 = b_1^2, && F_3 = -b_1^3 - 3b_3,\\
&F_4 = b_1^4 + 4b_1b_3, && F_5 = -b_1^5 - 5b_1^2b_3, && F_6 = b_1^6 + 6b_1^3b_3 + 3b_3^2.
\end{aligned}$$

Let us consider as an other example the mapping ψ defined by

$$\psi(z) = z + \frac{1}{(m-1)z^{m-1}}$$

which is conformal in the exterior of the unit circle. The boundary of the associated compact set

$$K = \widehat{\mathbb{C}} \setminus \{z \in \mathbb{C}\colon\ z = \psi(w),\ |w| > 1\}$$

is called a m-cusped hypocycloid.

As consequence of Corollary 1.4, we have the following result proved by Mathew X. He in [4]:

Corollary 1.5. *The Faber polynomials F_k associated to an m-cusped hypocycloid are given by*:

$$F_k(z) = \left(\frac{1}{m-1}\right)^j z^p \sum_{i=0}^{j} \frac{(k-(m-1)i-1)!k}{(k-mi)!i!}(-1)^i\big((m-1)z^m\big)^{j-i},$$

with $k = jm + p$ and $0 \leqslant p < m$.

In [1], the authors introduced the generalized Faber polynomials (H_j^k) associated to the function $\psi(w) = w + \sum_{k=0}^{\infty} \frac{b_{k+1}}{w^k}$ by:

$$w\frac{\psi'(w)}{\psi(w)}\left(\frac{\psi(w)}{w}\right)^k = 1 + \sum_{j=1}^{\infty} H_j^{k-j} w^{-j} \tag{1.15}$$

and the generalized Faber polynomials (F_j^k) associated to the univalent function $f(z) = z(1 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots)$ by

$$z\frac{f'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^k = 1 - \sum_{j=1}^{\infty} F_j^{k+j} z^j.$$

They showed that those Faber polynomials are linked to the coefficients in the asymptotic expansion of the function $(\frac{\psi(w)}{w})^p$:

$$\left(\frac{\psi(w)}{w}\right)^p = 1 + \sum_{n=1}^{\infty} K_n^p w^{-n}. \tag{1.16}$$

When $\frac{\psi(z)}{z} = zf(\frac{1}{z})$, then

$$K_j^k = \frac{1}{2}\big(H_j^{k-j} - F_j^{k+j}\big). \tag{1.17}$$

In this paper, we give explicit formulas for the polynomials (H_j^k), (F_j^k) and (K_j^k).

For an indeterminate u, we set

$$s(u) = \sum_{j=1}^{+\infty} b_j u^j$$

and

$$a_1 = -b_1, \quad a_j = \frac{1}{1-j}\big(1 + s(u)\big)_j^{j-1}, \quad \text{for } j = 2, 3, \ldots$$

such that $(1+s(u))_j^{j-1}$ stands for the coefficient of u^j in the Taylor expansion of $(1+s(u))^{j-1}$. For simplicity, we set $\lambda = i_2 + i_3 + \cdots + i_{m+n}$ and $\gamma = 2i_2 + 3i_3 + \cdots + (n+m)i_{n+m}$. Using the explicit formula of the Faber polynomials, we will show that:

Theorem 1.6. *The coefficients K_n^m are given by*:

If $n \leqslant m$, we have

$$K_n^m = \sum_{n+i_1+2i_2+3i_3+\cdots+mi_m=m} L_{(n,i_1,i_2,i_3,\ldots,i_k)}(-1)^{i_1+i_2+i_3+\cdots+i_k} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}$$

and if $n > m$ we get:

$$K_n^m = \sum_{m=1}^{\infty} \sum_{\gamma=m+n} H_{(i_2,i_3,\ldots,i_{n+m})} a_2^{i_2} \cdots a_{n+m}^{i_{n+m}} \sum_{\mathbf{n_m}=n} \prod_{i=1}^{m} G_{n_i}(b_1, b_2, \ldots, b_{n_i}),$$

where

$$L_{(n,i_1,i_2,i_3,\ldots,i_m)} := \frac{(n+i_1+i_2+\cdots+i_m-1)!m}{n!i_1!\cdots i_m!},$$

$$H_{(i_2,i_3,\ldots,i_{n+m})} := -n\frac{(\lambda-1)!}{i_2!\cdots i_{n+m}!}\left[\prod_{j=2}^{n+m}\left(\frac{x-x^j}{1-x}\right)^{i_j}\right]_n$$

such that $[\ldots]_n$ denotes the coefficient of x^n in the expansion of the expression in the bracket in power of x and

$$\left[\left(\frac{x-x^j}{1-x}\right)^{i_j}\right]_n = \sum_{k=0}^{i_j}(-1)^k\binom{i_j}{k}\binom{i_j-1+n-jk}{i_j-1}.$$

Combining (1.4) with (1.15), we get

$$\left(1+\sum_{n=1}^{\infty} F_n(b_1, b_2, \ldots, b_n)w^{-n}\right)\left(1+\sum_{n=1}^{\infty} K_n^k w^{-n}\right) = 1+\sum_{j=1}^{\infty} H_j^{k-j} w^{-j}$$

thus

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} F_{n-i}(b_1, b_2, \ldots, b_n)K_i^k\right)w^{-n} = \sum_{j=0}^{\infty} H_j^{k-j} w^{-j}$$

with

$$F_0(b_1, b_2, \ldots, b_n) := K_0^k := H_0^k := 1.$$

It follows that:

$$H_n^{k-n} = \sum_{i=0}^{n} F_{n-i}(b_1, b_2, \ldots, b_n)K_i^k. \tag{1.18}$$

From (1.9) we get an explicit formula of $F_{n-i}(b_1, b_2, \dots, b_n)$ and if we combine Theorem 1.6 with (1.18) we get an explicit expression from H_n^{k-n}. When $\frac{\psi(z)}{z} = zf(\frac{1}{z})$ and from (1.17), the polynomials F_j^{k+j} are expressed by:

$$F_j^{k+j} = H_j^{k-j} - 2K_j^k$$

which gives also an explicit formula for F_j^{k+j}.

In the sequel, we give a generalized Cayley–Hamilton equation. To motivate our results, we consider the case $A \in SL_2(\mathbb{C}) := \{X \in \mathcal{M}_2(\mathbb{C}),\ \det X = 1\}$. The Cayley–Hamilton equation can be written as:

$$A^2 - tA + I = 0, \qquad t = \operatorname{trace}(A).$$

The function $\psi(z) := z + \frac{1}{z}$ is the conformal map from the exterior of the unit disk onto the exterior of $[-2, 2]$. Eq. (1.2) become:

$$\frac{w}{w^2 - wz + 1} = \sum_{n=0}^{\infty} G_n(z) w^{-n-1}. \tag{1.19}$$

We will show that for any $n \geqslant 1$, we have

$$A^n = G_{n-1}(t)A - G_{n-2}(t)I \tag{1.20}$$

which gives the following:

Proposition 1.7. *If $A \in SL_2(\mathbb{C})$, then for any $m_1, m_2 \geqslant 1$, we have*:

$$G_{m_2-1}A^{m_1} - G_{m_1-1}A^{m_2} = (G_{m_1}G_{m_2-1} - G_{m_1-1}G_{m_2})I.$$

Now, if we set

$$\mathcal{G}_{m_1,m_2} := G_{m_1}G_{m_2-1} - G_{m_1-1}G_{m_2}, \tag{1.21}$$

Proposition 1.7 can be written as:

$$\mathcal{G}_{0,m_2}A^{m_1} - \mathcal{G}_{0,m_1}A^{m_2} = \mathcal{G}_{m_1,m_2}I.$$

Using Eq. (1.19), the polynomials $\mathcal{G}_{m_1,m_2}$ verify:

$$\frac{\xi_1 - \xi_2}{P_A(\xi_1)P_A(\xi_2)} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \mathcal{G}_{m_1,m_2}(t)\xi_1^{-m_1-1}\xi_2^{-m_2-1}.$$

More generally, we give a general Cayley–Hamilton equation with the help of the polynomials $\mathcal{G}_{m_1,m_2,\dots,m_{p+1}}$ defined as follows: let $A \in SL_{p+1}(\mathbb{C}) := \{X \in \mathcal{M}_{P+1}(\mathbb{C}),\ \det X = 1\}$ and $P_A(\xi) = \det(\xi I - A)$ its characteristic polynomial. We define these polynomials by

$$\frac{a_p(\xi)}{\prod_{i=1}^{p+1} P_A(\xi_i)} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_{p+1}=0}^{\infty} \mathcal{G}_{m_1,m_2,\dots,m_{p+1}}(t)\prod_{i=1}^{p+1}\xi_i^{-m_i-1} \tag{1.22}$$

with

$$a_p(\xi)=\prod_{1\leqslant i<j\leqslant p+1}(\xi_i-\xi_j), \qquad t=\mathrm{trace}(A).$$

We will show that:

Theorem 1.8. *Let $A\in SL_{p+1}(\mathbb{C})$, then for every no-negative integer m_i, $1\leqslant i\leqslant p+1$, we get the following generalized Cayley–Hamilton equation*:

$$\sum_{i=1}^{p+1}(-1)^{i-1}\boldsymbol{\mathcal{G}}_{0,m_1,m_2,\ldots,\tilde{m}_i,\ldots,m_{p+1}}A^{m_i}=\boldsymbol{\mathcal{G}}_{m_1,m_2,\ldots,m_{p+1}}I,$$

where $\boldsymbol{\mathcal{G}}_{0,m_1,m_2,\ldots,\tilde{m}_i,\ldots,m_{p+1}}$ is defined by removing the indice m_i of the polynomials $\boldsymbol{\mathcal{G}}_{0,m_1,m_2,\ldots,m_i,\ldots,m_{p+1}}$.

In order to give a simplified expression of the polynomials $\boldsymbol{\mathcal{G}}_{m_1,m_2,\ldots,m_{p+1}}$, we associated to the matrix $A\in SL_{p+1}(\mathbb{C})$ the polynomials $\mathcal{G}_k^p(t)$ defined by the generating function:

$$\frac{\xi^p}{P_A(\xi)}=\sum_{k=0}^{\infty}\mathcal{G}_k^p(t)\xi^{-k-1}. \tag{1.23}$$

If we consider the case $p=1$ in (1.23) that is $A\in SL_2(\mathbb{C})$, putting $P_A(\xi)=\xi^2-t\xi+1$ into the expression (1.23) and comparing with (1.19), we obtain:

$$\mathcal{G}_n^1=G_n. \tag{1.24}$$

Combining this with Eq. (1.21), we have:

$$\boldsymbol{\mathcal{G}}_{0,m+1}=\boldsymbol{\mathcal{G}}_m^1, \tag{1.25}$$

$$\boldsymbol{\mathcal{G}}_{m_1,m_2}=\boldsymbol{\mathcal{G}}_{m_1}^1\boldsymbol{\mathcal{G}}_{m_2-1}^1-\boldsymbol{\mathcal{G}}_{m_1-1}^1\boldsymbol{\mathcal{G}}_{m_2}^1. \tag{1.26}$$

In general we have:

Theorem 1.9. *The polynomials $\boldsymbol{\mathcal{G}}_{m_1,m_2,\ldots,m_{p+1}}$ are given in terms of the polynomials $\boldsymbol{\mathcal{G}}_{m_i}$, $1\leqslant i\leqslant p+1$, as follows*:

$$\boldsymbol{\mathcal{G}}_{m_1,m_2,\ldots,m_{p+1}}=\begin{vmatrix} \mathcal{G}_{m_1}^p & \mathcal{G}_{m_2}^p & \ldots & \mathcal{G}_{m_{p+1}}^p \\ \mathcal{G}_{m_1-1}^p & \mathcal{G}_{m_2-1}^p & \ldots & \mathcal{G}_{m_{p+1}-1}^p \\ \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \vdots \\ \mathcal{G}_{m_1-p}^p & \mathcal{G}_{m_2-p}^p & \ldots & \mathcal{G}_{m_{p+1}-p}^p \end{vmatrix}. \tag{1.27}$$

Conversely, the polynomials $\boldsymbol{\mathcal{G}}_m^p$ are given by:

$$\boldsymbol{\mathcal{G}}_m^p=\boldsymbol{\mathcal{G}}_{0,1,\ldots,p-1,m+p}. \tag{1.28}$$

2. Preliminary results

Set

$$\psi(w) = w + \sum_{k=0}^{\infty} \frac{b_{k+1}}{w^k},$$

$$\frac{\psi'(w)}{\psi(w)} = \sum_{k=0}^{\infty} F_k(b_1, b_2, \ldots, b_k) w^{-k-1}.$$

Lemma 2.1. *The following holds*:

$$\log\bigl(1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots\bigr) = -\sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \ldots, b_k)}{k} w^k. \tag{2.1}$$

Proof. Let $f(w) = 1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots$. We have

$$f(w) = w\psi\left(\frac{1}{w}\right),$$

$$f'(w) = \psi\left(\frac{1}{w}\right) - \frac{1}{w}\psi'\left(\frac{1}{w}\right)$$

and also

$$\frac{wf'(w)}{f(w)} = 1 - \frac{1}{w}\frac{\psi'(\frac{1}{w})}{\psi(\frac{1}{w})} = 1 - \sum_{k=0}^{+\infty} F_k(b_1, b_2, \ldots, b_k) w^k.$$

It implies the formula:

$$\begin{aligned} \frac{f'(w)}{f(w)} &= \frac{\mathrm{d}}{\mathrm{d}w} \log\bigl(1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots\bigr) \\ &= -\sum_{k=1}^{+\infty} F_k(b_1, b_2, \ldots, b_k) w^{k-1}. \end{aligned} \tag{2.2}$$

Integrating this equation with respect to w, the lemma is proved. □

Theorem 2.2. *Let* $(x_1, \ldots, x_k)$ *be the roots of the polynomial*

$$Q(\xi) = \xi^k + b_1 \xi^{k-1} + \cdots + b_k$$

and

$$\Pi_k = x_1^k + x_2^k + \cdots + x_k^k.$$

Then

$$F_k = \Pi_k.$$

Proof. We have that

$$1+b_1w+b_2w^2+b_3w^3+\cdots+b_kw^k=\prod_{i=1}^{k}\left(\frac{1}{w}-x_i\right)w^k=\prod_{i=1}^{k}(1-x_iw)$$

and also

$$\begin{aligned}&\log\bigl(1+b_1w+b_2w^2+b_3w^3+\cdots+b_kw^k\bigr)\\&\quad=\sum_{i=1}^{k}\log(1-x_iw)=-\sum_{j=1}^{\infty}\sum_{i=1}^{k}\frac{(x_iw)^j}{j}=-\sum_{j=1}^{\infty}\Pi_j\frac{w^j}{j},\end{aligned}$$

with

$$\Pi_j=x_1^j+x_2^j+\cdots+x_k^j.$$

On the other hand, formula (2.1) shows that:

$$\begin{aligned}-\frac{F_k}{k}&=\frac{1}{k!}\frac{\mathrm{d}^k\log(1+\sum_{i=1}^{\infty}b_iw^i)}{\mathrm{d}w^k}(0)\\&=\frac{1}{k!}\frac{\mathrm{d}^k\log(1+\sum_{i=1}^{k}b_iw^i)}{\mathrm{d}w^k}(0)+\frac{1}{k!}\frac{\mathrm{d}^k\log\bigl(1+w^{k+1}\frac{\sum_{i=1}^{\infty}b_{k+i}w^{i-1}}{1+\sum_{i=1}^{k}b_iw^i}\bigr)}{\mathrm{d}w^k}(0)\\&=\frac{1}{k!}\frac{\mathrm{d}^k\log(1+\sum_{i=1}^{k}b_iw^i)}{\mathrm{d}w^k}(0)\\&=-\frac{\Pi_k}{k}\end{aligned}$$

which completes the proof of the theorem. □

Proposition 2.3. *We consider the Faber polynomials $F_n(b_1,b_2,\ldots,b_n)$. If the coefficients b_j of odd index vanish, one has*

$$F_{2n+1}(0,b_2,0,b_4,0,\ldots,b_{2n},0)=0, \tag{2.3}$$

$$F_{2n}(0,b_2,0,b_4,0,\ldots,b_{2n})=2F_n(b_2,b_4,\ldots,b_{2n}). \tag{2.4}$$

Proof. We have:

$$\begin{aligned}\psi(z)&=z+\frac{b_2}{z}+\frac{b_4}{z^3}+\cdots+\frac{b_{2n+2}}{z^{2n+1}}+\cdots\\&=\frac{1}{z}\left(z^2+b_2+\frac{b_4}{z^2}+\cdots+\frac{b_{2n+1}}{z^{2n}}\right)=:\frac{\psi_1(z^2)}{z}.\end{aligned}$$

Then

$$\log\frac{\psi(z)}{z}=\log\frac{\psi_1(z^2)}{z^2}. \tag{2.5}$$

Using the relation

$$\log\frac{\psi(z)}{z}=-\sum_{m=1}^{\infty}\frac{F_m}{m}\times\frac{1}{z^m}$$

we obtain:

$$\sum_{m=1}^{\infty} \frac{F_{2m}(0, b_2, \ldots, 0, b_{2m})}{2m} \times \frac{1}{z^{2m}} = \sum_{m=1}^{\infty} \frac{F_m(b_2, b_4, \ldots, b_{2m})}{m} \times \frac{1}{z^{2m}}$$

which gives the result. □

More generally, if $\psi(z)$ has the form

$$\psi(z) = z + \sum_{k=0}^{\infty} \frac{b_{p(k+1)}}{z^{pk+p-1}}, \quad z \to \infty, \tag{2.6}$$

we get:

Proposition 2.4. *The following holds*:

$$\begin{aligned} &F_k(0, 0, \ldots, 0, b_p, 0, 0, \ldots, 0, b_{2p}, 0, 0, \ldots, 0, b_{mp}, \ldots, 0) = 0 \\ &\quad \textit{if } k \textit{ is not a multiple of } p, \\ &F_{pm}(0, 0, \ldots, 0, b_p, 0, 0, \ldots, 0, b_{2p}, 0, 0, \ldots, 0, b_{mp}) = pF_m(b_p, b_{2p}, \ldots, b_{mp}). \end{aligned}$$

Proof. We have:

$$\psi(z) = z + \sum_{n \geqslant 0} \frac{b_{p(n+1)}}{z^{pn+p-1}} = \frac{1}{z^{p-1}} \left(z^p + \sum_{n \geqslant 0} \frac{b_{p(n+1)}}{z^{pn}} \right) = \frac{1}{z^{p-1}} \psi_1(z^p) \tag{18}$$

with

$$\psi_1(z) = z + c_1 + \sum_{j \geqslant 1}^{\infty} \frac{c_j}{z^j} \quad \text{and} \quad c_j = b_{pj}$$

which gives:

$$\log \frac{\psi(z)}{z} = \log \frac{\psi_1(z^p)}{z^p}.$$

Using formula (2.1), the result follows. □

Remark 2.1. From Eq. (2.2), we get:

$$b_1 + 2b_2 w + 3b_3 w^2 + \cdots = (1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots) \left(-\sum_{k=1}^{+\infty} F_k w^{k-1} \right).$$

By comparing coefficients of like powers of w and using the Cramer formula, the Faber polynomials $F_k(b_1, b_2, \ldots, b_k)$ has the following representation:

$$F_k(b_1, b_2, \ldots, b_k) = (-1)^k \begin{vmatrix} b_1 & 1 & & & \\ 2b_2 & b_1 & 1 & & \\ \vdots & b_2 & \ddots & \ddots & \\ & \vdots & \ddots & & 1 \\ kb_k & b_{k-1} & & b_2 & b_1 \end{vmatrix}.$$

3. Proof of the main results

Proof of Theorem 1.1. From Eq. (1.11), we observe that the polynomials F_k depend only of the coefficients $b_1, b_2, \dots, b_k$. We obtain:

$$W_n F_m = 0 \quad \text{for } m < n, \tag{3.1}$$

$$W_n F_n = n. \tag{3.2}$$

To end the proof, it remains to prove that $W_n F_{n+p} = 0$ for every positive integer p. We do it by induction on p. Combining (1.11), (3.1) and (3.2) we get:

$$W_n F_{n+1} = -nb_1 + F_1 + (n+1)b_1 = 0. \tag{3.3}$$

The proposition is true for $p = 1$, suppose it true for $p - 1$. Eq. (1.11) can be written as:

$$F_{n+p} = -b_1 F_{n+p-1} - \sum_{k=1}^{n+p-2} b_{n+p-k} F_k - (n+p) b_{n+p}. \tag{3.4}$$

So that by the induction hypothesis and (3.1), (3.2), we have:

$$W_n F_{n+p} = -\sum_{k=1}^{n+p-2} W_n(b_{n+p-k} F_k) + (n+p) b_p. \tag{3.5}$$

It follows that

$$W_n F_{n+p} = -\sum_{k=1}^{n+p-2} (b_{n+p-k} k \delta_{n,k} - b_{p-k} F_k) + (n+p) b_p, \tag{3.6}$$

where $b_0 = 1$ and $b_k = 0$ for $k < 0$. Hence

$$W_n F_{n+p} = \sum_{k=1}^{p} b_{p-k} F_k + p b_p. \tag{3.7}$$

And by using the recurrence formula (1.11), we get:

$$W_n F_{n+p} = F_p + b_1 F_{p-1} + \sum_{k=1}^{p-2} b_{p-k} F_k + p b_p = 0. \tag{3.8}$$

Which ends the proof of the first part of this theorem. The proof of (1.7) is similar. From Eq. (1.14) and by using the fact that the polynomials G_k depend only of the coefficients $b_1, b_2, \dots, b_k$, we obtain:

$$W_n G_m = 0 \quad \text{for } m < n, \tag{3.9}$$

$$W_n G_n = 1 = G_0. \tag{3.10}$$

To end the proof, it remains to prove that $W_n G_{n+p} = 0$ for every positive integer p. We prove it again by induction on p. If we combine (1.14), (3.9) with (3.10), we get:

$$W_n G_{n+1} = -b_1 + G_1 + b_1 = G_1.$$

The proposition is true for $p = 1$, suppose it holds for $p - 1$. Eq. (1.14) can be written as:

$$G_{n+p} = -\sum_{k=0}^{n+p-1} b_{k+1} G_{n+p-1-k}. \tag{3.11}$$

So that by the induction assumption and (3.9), (3.10):

$$W_n G_{n+p} = -\sum_{k=0}^{n+p-1} W_n(b_{k+1} G_{n+p-1-k}). \tag{3.12}$$

We then get

$$W_n G_{n+p} = -\sum_{k=0}^{n+p-1} (b_{k+1} G_{p-1-k} - b_{k+1-n} G_{n+p-1-k}), \tag{3.13}$$

with $b_0 = 1$ and $b_k = G_k = 0$ for $k < 0$. Hence

$$\begin{aligned} W_n G_{n+p} &= -\sum_{k=0}^{p-1} b_{k+1} G_{p-1-k} + \sum_{k=n-1}^{n+p-1} b_{k+1-n} G_{n+p-1-k} \\ &= -\sum_{k=0}^{p-1} b_{k+1} G_{p-1-k} + \sum_{k=0}^{p-1} b_{k+1} G_{p-1-k} + G_p. \end{aligned}$$

It follows that

$$W_n G_{n+p} = G_p. \tag{3.14}$$

Then the result is true for all $p \geqslant 1$, which ends the proof. □

Remark 3.1. In [1], H. Airault and J. Ren introduced the family of operators

$$Z_k = -\sum_{n=1}^{\infty} n b_n \frac{\partial}{\partial b_{n+k-1}}$$

and proved the following functional relations:

$$\begin{aligned} &Z_1 F_n(b_1, b_2, \ldots, b_n) = -n F_n(b_1, b_2, \ldots, b_n), \\ &Z_2 F_0(b_1, b_2, \ldots, b_n) = 0, \\ &Z_2 F_1(b_1, b_2, \ldots, b_n) = 0, \\ &Z_2 F_n(b_1, b_2, \ldots, b_n) = -n F_{n-1}(b_1, b_2, \ldots, b_{n-1}) \quad \text{for } n \geqslant 2, \\ &Z_k F_n(b_1, b_2, \ldots, b_n) = 0 \quad \text{for } n \leqslant k-2, \\ &Z_k F_{k-1}(b_1, b_2, \ldots, b_{k-1}) = 0, \\ &Z_k F_n(b_1, b_2, \ldots, b_n) = -n F_{n-k+1}(b_1, b_2, \ldots, b_{n-k+1}) \quad \text{for } n > k-1. \end{aligned}$$

Proof of Corollary 1.2. It follows from Theorem 1.1 and Proposition 2.4. □

Proof of Theorem 1.3. Let $p \geqslant 2$ an integer, Waring's formula relates the kth power sum $x_1^k + \cdots + x_p^k$ to the elementary symmetric functions $s_l(x_1, x_2, \ldots, x_p)$, $1 \leqslant l \leqslant p$, as follows:

$$x_1^k + \cdots + x_p^k = \sum_{i_2=0}^{[\frac{k}{2}]} \cdots \sum_{i_p=0}^{[\frac{k}{p}]} E_{(i_2,i_3,\ldots,i_p)} s_1^{k-2i_2-3i_3-\cdots-pi_p} s_2^{i_2} \cdots s_p^{i_p},$$

where

$$E_{(i_2,i_3,\ldots,i_p)} := (-1)^{i_2+2i_3+\cdots+(p-1)i_p} \times \frac{(k-i_2-2i_3-\cdots-(p-1)i_p-1)!k}{(k-2i_2-3i_3-\cdots-pi_p)!i_2!\cdots i_p!}. \tag{3.15}$$

If we combine it with Theorem 2.2 with $b_l = (-1)^l s_l$, $1 \leqslant l \leqslant k$, we obtain the first part of the theorem. The second part follows by using Eq. (1.13) . □

Proof of Corollary 1.4. It follows immediately from Theorem 1.3. □

Proof of Corollary 1.5. We have $F_n(z) = F_n(b_1 - z, b_2, \ldots, b_n)$ and in the case of the m-cusped hypocycloid, one has: $b_i = 0$ for $i \neq m$ and $b_m = \frac{1}{m-1}$ then by using Corollary 1.4, we obtain the result. □

Proof of Theorem 1.6. The inverse mapping of ψ in (1.3) is given by:

$$\phi(z) = z + \sum_{k=0}^{\infty} \frac{a_{k+1}}{z^k}.$$

Now, we consider the Faber polynomials associated to ϕ:

$$\frac{\phi'(w)}{\phi(w) - z} = \sum_{m=0}^{\infty} F_m(z) w^{-m-1}, \tag{3.16}$$

$$\frac{1}{\phi(w) - z} = \sum_{m=0}^{\infty} G_m(z) w^{-m-1}. \tag{3.17}$$

The Faber polynomials of the first kind verify [2]:

$$F_n\big(\phi(z)\big) = z^n - \sum_{m=1}^{\infty} \frac{\alpha_{n,m}}{m} z^{-m}$$

which gives:

$$\big(\psi(z)\big)^n = F_n(z) + \sum_{m=1}^{\infty} \frac{\alpha_{n,m}}{m} \left(\frac{1}{\psi(z)}\right)^m. \tag{3.18}$$

Put $\lambda = i_2 + i_3 + \cdots + i_{m+n}$ and $\gamma = 2i_2 + 3i_3 + \cdots + (n+m)i_{n+m}$. From [6], we have:

$$\frac{\alpha_{n,m}}{m} = \sum_{\gamma=m+n} H_{(i_2,i_3,\ldots,i_{n+m})} a_2^{i_2} \cdots a_{n+m}^{i_{n+m}},$$

where

$$H_{(i_2,i_3,\ldots,i_{n+m})} := -n \frac{(\lambda-1)!}{i_2! \cdots i_{n+m}!} \left[\prod_{j=2}^{n+m} \left(\frac{x - x^j}{1-x} \right)^{i_j} \right]_n$$

such that $[\ldots]_n$ denotes the coefficient of x^n in the power expansion (with respect to x) of the expression inside the brackets.

Furthermore, using Eq. (1.9), the Faber polynomials $F_m(z) = F_m(b_1 - z, b_2, \ldots, b_m)$ associated to ϕ can be written as:

$$F_m(z) = \sum_{i=0}^{m} z^i \sum_{i+i_1+2i_2+3i_3+\cdots+mi_m=m} L_{(i,i_1,i_2,i_3,\ldots,i_k)} \times (-1)^{i_1+i_2+i_3+\cdots+i_k} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m},$$

where

$$L_{(i,i_1,i_2,i_3,\ldots,i_m)} := \frac{(i + i_1 + i_2 + \cdots + i_m - 1)! m}{i! i_1! \cdots i_m!}.$$

On other hand, from (1.5) we find:

$$\left(\frac{1}{\psi(z)} \right)^m = \sum_{n=0}^{\infty} \left(\sum_{n_1+n_2+\cdots+n_m=n} \prod_{i=1}^{m} G_{n_i}(b_1, b_2, \ldots, b_{n_i}) \right) z^{-n-1}.$$

Then by putting $\mathbf{n_m} = n_1 + n_2 + \cdots + n_m$, Eq. (3.18) rewrites as:

$$\begin{aligned} \big(\psi(z)\big)^m = {} & \sum_{i=0}^{m} z^i \sum_{i+i_1+2i_2+3i_3+\cdots+mi_m=m} L_{(i,i_1,i_2,i_3,\ldots,i_k)} \\ & \times (-1)^{i_1+i_2+i_3+\cdots+i_k} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m} \\ & + \sum_{n=0}^{\infty} \left(\sum_{m=1}^{\infty} \sum_{\gamma=m+n} H_{(i_2,i_3,\ldots,i_{n+m})} a_2^{i_2} \cdots a_{n+m}^{i_{n+m}} \right. \\ & \left. \times \sum_{\mathbf{n_m}=n} \prod_{i=1}^{m} G_{n_i}(b_1, b_2, \ldots, b_{n_i}) \right) z^{-n-1} \end{aligned}$$

which implies that:

$$\left(\frac{\psi(z)}{z} \right)^m = 1 + \sum_{i=0}^{m-1} \frac{1}{z^{m-i}} \sum_{i+i_1+2i_2+3i_3+\cdots+mi_m=m} L_{(i,i_1,i_2,i_3,\ldots,i_k)} \times (-1)^{i_1+i_2+i_3+\cdots+i_k} \prod_{j=1}^{m} a_j^{i_j}$$

$$+\sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty}\sum_{\gamma=m+n}H_{(i_2,i_3,\dots,i_{n+m})}\prod_{j=2}^{m+n}a_j^{i_j}\right.$$
$$\left.\times\sum_{\mathbf{n_m}=n}\prod_{i=1}^{m}G_{n_i}(b_1,b_2,\dots,b_{n_i})\right)\frac{1}{z^{n+m+1}}.$$

If we combine this equation with (1.16), the result is proved. □

Proof of Proposition 1.7. We start by the case $p=1$, that is $A\in SL_2(\mathbb{C})$. The Cayley–Hamilton equation can be written as:

$$A^2-tA+I=0,\qquad t=\mathrm{trace}(A).$$

For any $n\geqslant 1$, we have

$$A^n=l_n(t)A-j_n(t)I.$$

Thus,

$$\begin{aligned}A^{n+1}&=l_n(t)(tA-I)-j_n(t)A\\&=\big(tl_n(t)-j_n(t)\big)A-l_n(t)I\\&=l_{n+1}(t)A-j_{n+1}(t)I.\end{aligned}$$

By identification, we obtain

$$\begin{aligned}&l_{n+1}(t)=tl_n(t)-l_{n-1}(t),\\&l_0=0,\qquad l_1(t)=1\end{aligned}$$

and

$$j_n=l_{n-1}.$$

Let G_n be the Faber polynomials associated to $[-2,2]$. From Eq. (1.19), the polynomials G_n verify the recurrence relations

$$G_{n+1}(t)=tG_n(t)-G_{n-1}(t),\tag{3.19}$$
$$G_{-1}=0,\qquad G_0=1,\qquad G_1(t)=t.\tag{3.20}$$

Hence $l_n=G_{n-1}$ and the Cayley–Hamilton equation in this case can be written for any $n\geqslant 1$ as

$$A^n=G_{n-1}(t)A-G_{n-2}(t)I.$$

Thus,

$$\begin{aligned}&G_{m_2-1}A^{m_1}-G_{m_1-1}A^{m_2}\\&\quad=(G_{m_2-1}G_{m_1-1}-G_{m_1-1}G_{m_2-1})A-(G_{m_2-1}G_{m_1-2}-G_{m_1-1}G_{m_2-2})I\\&\quad=(G_{m_1-1}G_{m_2-2}-G_{m_2-1}G_{m_1-2})I.\end{aligned}$$

Combining this formula with (3.19), we obtain:

$$G_{m_2-1}A^{m_1} - G_{m_1-1}A^{m_2} = \left\{G_{m_1-1}(tG_{m_2-1} - G_{m_2}) - G_{m_2-1}(tG_{m_1-1} - G_{m_1})\right\}I$$
$$= (G_{m_1}G_{m_2-1} - G_{m_2}G_{m_1-1})I$$

which gives the desired result. □

In what follows we set for simplicity $\mathcal{G}_m := \mathcal{G}_m^p$.

Proof of Theorem 1.8. Without loss of generality, we can suppose that A is a diagonal matrix. We start by proving the result for $p = 2$. Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A. We have

$$\begin{aligned}
&\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\left(\mathcal{G}_{0,m_2,m_3}\lambda_1^{m_1} - \mathcal{G}_{0,m_1,m_3}\lambda_1^{m_2} + \mathcal{G}_{0,m_1,m_2}\lambda_1^{m_3}\right)\frac{1}{\xi_1^{m_1+1}}\frac{1}{\xi_2^{m_2+1}}\frac{1}{\xi_3^{m_3+1}} \\
&\quad = \frac{1}{\xi_1-\lambda_1}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\mathcal{G}_{0,m_2,m_3}\frac{1}{\xi_2^{m_2+1}}\frac{1}{\xi_3^{m_3+1}} \\
&\qquad - \frac{1}{\xi_2-\lambda_1}\sum_{m_1=0}^{\infty}\sum_{m_3=0}^{\infty}\mathcal{G}_{0,m_1,m_3}\frac{1}{\xi_1^{m_1+1}}\frac{1}{\xi_3^{m_3+1}} \\
&\qquad + \frac{1}{\xi_3-\lambda_1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\mathcal{G}_{0,m_1,m_2}\frac{1}{\xi_1^{m_1+1}}\frac{1}{\xi_2^{m_2+1}} \\
&\quad = \frac{1}{\xi_1-\lambda_1}\frac{\xi_2-\xi_3}{P_A(\xi_2)P_A(\xi_3)} - \frac{1}{\xi_2-\lambda_1}\frac{\xi_1-\xi_3}{P_A(\xi_1)P_A(\xi_3)} + \frac{1}{\xi_3-\lambda_1}\frac{\xi_1-\xi_2}{P_A(\xi_1)P_A(\xi_2)} \\
&\quad = \big((\xi_1-\lambda_2)(\xi_1-\lambda_3)(\xi_2-\xi_3) - (\xi_2-\lambda_2)(\xi_2-\lambda_3)(\xi_1-\xi_3) \\
&\qquad + (\xi_3-\lambda_2)(\xi_3-\lambda_3)(\xi_1-\xi_2)\big)\big/\big(P_A(\xi_1)P_A(\xi_2)P_A(\xi_3)\big) \\
&\quad = \frac{(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_2-\xi_3)}{P_A(\xi_1)P_A(\xi_2)P_A(\xi_3)} \\
&\quad = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\mathcal{G}_{m_1,m_2,m_3}\frac{1}{\xi_1^{m_1+1}}\frac{1}{\xi_2^{m_2+1}}\frac{1}{\xi_3^{m_3+1}}.
\end{aligned}$$

Which ends the proof in the case of $p = 2$. For the general case and in similar way we get:

$$\begin{aligned}
&\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_{p+1}=0}^{\infty}\left(\sum_{i=1}^{p+1}(-1)^{i+1}\mathcal{G}_{0,m_1,m_2,\ldots,\tilde{m}_i,\ldots,m_{p+1}}\,\lambda_1^{m_i}\right)\prod_{j=1}^{p+1}\xi_j^{-m_j-1} \\
&\quad = \frac{\sum_{n=1}^{p+1}(-1)^{n+1}\prod_{j=2}^{p+1}(\xi_n-\lambda_j)\Delta_n}{\prod_{i=1}^{p+1}P_A(\xi_i)},
\end{aligned} \tag{3.21}$$

where Δ_n is the Vandermond determinant $\det(\xi_i^{p+1-j})$ with $2 \leqslant j \leqslant p+1$, $1 \leqslant i \leqslant p+1$ and $i \neq n$. On other hand, $\prod_{1 \leqslant i<j \leqslant p+1}(\xi_i - \xi_j)$ is the Vandermond determinant

$$\prod_{1 \leqslant i<j \leqslant p+1}(\xi_i - \xi_j) = \begin{vmatrix} \xi_1^p & \xi_1^{p-1} & \dots & 1 \\ \xi_2^p & \xi_2^{p-1} & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ \xi_{p+1}^p & \xi_{p+1}^{p-1} & \dots & 1 \end{vmatrix}.$$

Let s_i be the elementary symmetric polynomials of $\lambda_2, \lambda_3, \dots, \lambda_{p+1}$. If we change the column c_1 by $c_1 - s_1 c_2 + s_2 c_3 - \cdots + (-1)^p s_p c_{p+1}$, we find:

$$\prod_{1 \leqslant i<j \leqslant p+1}(\xi_i - \xi_j) = \begin{vmatrix} \prod_{j=2}^{p+1}(\xi_1 - \lambda_j) & \xi_1^{p-1} & \dots & 1 \\ \prod_{j=2}^{p+1}(\xi_2 - \lambda_j) & \xi_2^{p-1} & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ \prod_{j=2}^{p+1}(\xi_{p+1} - \lambda_j) & \xi_{p+1}^{p-1} & \dots & 1 \end{vmatrix}.$$

Expanding with respect to the first column, we find that

$$\prod_{1 \leqslant i<j \leqslant p+1}(\xi_i - \xi_j) = \sum_{n=1}^{p+1} (-1)^{n+1} \prod_{j=2}^{p+1} (\xi_n - \lambda_j) \Delta_n. \tag{3.22}$$

Combining it with Eq. (3.21), we get:

$$\frac{\prod_{1 \leqslant i<j \leqslant p+1}(\xi_i - \xi_j)}{\prod_{i=1}^{p+1} P_A(\xi_i)} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{p+1}=0}^{\infty} \left(\sum_{i=1}^{p+1} (-1)^{i+1} \mathcal{G}_{0,m_1,m_2,\dots,\tilde{m}_i,\dots,m_{p+1}} \lambda_1^{m_i} \right) \prod_{j=1}^{p+1} \xi_j^{-m_j-1}.$$

If is now clear in view of formula (1.22) that it implies the result. $\square$

Proof of Theorem 1.9. From Eq. (1.23), we have:

$$\frac{\prod_{i=1}^{p+1} \xi_i^p}{\prod_{i=1}^{p+1} P_A(\xi_i)} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{p+1}=0}^{\infty} \mathcal{G}_{m_1} \mathcal{G}_{m_2} \cdots \mathcal{G}_{m_{p+1}} \prod_{i=1}^{p+1} \xi_i^{-m_i-1}$$

and

$$\frac{1}{\prod_{i=1}^{p+1} P_A(\xi_i)} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{p+1}=0}^{\infty} \mathcal{G}_{m_1} \mathcal{G}_{m_2} \cdots \mathcal{G}_{m_{p+1}} \prod_{i=1}^{p+1} \xi_i^{-m_i-p-1}.$$

Since $\prod_{1 \leqslant i<j \leqslant p+1}(\xi_i - \xi_j) = \det(\xi_i^{p+1-j})_{1 \leqslant i,j \leqslant p+1}$ is the Vandermond determinant, we get

$$\frac{\prod_{1\leqslant i<j\leqslant p+1}(\xi_i-\xi_j)}{\prod_{i=1}^{p+1}P_A(\xi_i)}$$
$$=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_{p+1}=0}^{\infty}\sum_{\sigma\in S_{p+1}}\epsilon(\sigma)\boldsymbol{\mathcal{G}}_{m_1+1-\sigma(1)}\boldsymbol{\mathcal{G}}_{m_2+1-\sigma(2)}\cdots\boldsymbol{\mathcal{G}}_{m_{p+1}+1-\sigma(p+1)}$$
$$\times\prod_{i=1}^{p+1}\xi_i^{-m_i-1}$$

and by comparing it with Eq. (1.22) we connect the polynomials $\boldsymbol{\mathcal{G}}_{m_1,m_2,\ldots,m_{p+1}}$ and $\boldsymbol{\mathcal{G}}_{m_i}$ for $1\leqslant i\leqslant p+1$ as follows

$$\boldsymbol{\mathcal{G}}_{m_1,m_2,\ldots,m_{p+1}}=\sum_{\sigma\in S_{p+1}}\epsilon(\sigma)\boldsymbol{\mathcal{G}}_{m_1+1-\sigma(1)}\boldsymbol{\mathcal{G}}_{m_2+1-\sigma(2)}\cdots\boldsymbol{\mathcal{G}}_{m_{p+1}+1-\sigma(p+1)}. \tag{3.23}$$

Which gives the first part of the theorem. To prove the second part, we will need to show the following:

Lemma 3.1. *For $m\geqslant 0$, the following holds*:

$$\sum_{i=0}^{p+1}(-1)^i s_i\boldsymbol{\mathcal{G}}_{0,1,\ldots,p-2,p-1,m+p+1-i}=0, \tag{3.24}$$

with the starting value $\boldsymbol{\mathcal{G}}_{0,1,\ldots,p-2,p-1,p-i}=\delta_{i,0}$ for $0\leqslant i\leqslant p$ and $s_0:=s_{p+1}=1$.

Proof. It enough to observe that for $1\leqslant i\leqslant p$ the power of ξ_i in the nominator of the left side of Eq. (1.22) is $p+1-i$. Thus by multiplying both sides of (1.22) by ξ_i^i and letting $\xi_i\to\infty$ we get that

$$\frac{1}{P_A(\xi_{p+1})}=\sum_{m_{p+1}=0}^{\infty}\boldsymbol{\mathcal{G}}_{0,1,\ldots,p-2,p-1,m_{p+1}}\xi^{-m_{p+1}-1}. \tag{3.25}$$

Multiplying both sides of this equation by $P_A(\xi_{p+1})$ and comparing the powers of ξ_{p+1} we get Eq. (3.24). □

In the other hand, the polynomials $\mathcal{G}_m^p$ satisfy for $m\geqslant p+2$

$$\mathcal{G}_m-t\mathcal{G}_{m-1}+s_2\mathcal{G}_{m-2}+\cdots+(-1)^{p+1}s_{p+1}\mathcal{G}_{m-p-1}=0 \tag{3.26}$$

with the initial conditions

$$\mathcal{G}_m-t\mathcal{G}_{m-1}+s_2\mathcal{G}_{m-2}+\cdots+(-1)^{m-1}s_{m-1}\mathcal{G}_1=(-1)^m s_m\mathcal{G}_0$$
$$(1\leqslant m\leqslant p+1). \tag{3.27}$$

Now, by comparing Eq. (3.26) with (3.24) the result is proved. □

Remark 3.2. From (1.22), we find that the polynomials $\mathcal{G}_{m_1,m_2,\ldots,m_{p+1}}$ verify the following properties:

$$\mathcal{G}_{m_1,m_2,\ldots,m_i,\ldots,m_j,\ldots,m_{p+1}} = -\mathcal{G}_{m_1,m_2,\ldots,m_j,\ldots,m_i,\ldots,m_{p+1}}, \tag{3.28}$$

$$\mathcal{G}_{0,1,\ldots,p} = 1. \tag{3.29}$$

To prove (3.29), it suffices to observe that for $1 \leqslant i \leqslant p+1$ the power of ξ_i in the nominator of the left side of Eq. (1.22) is $p+1-i$. Thus by multiplying both sides of (1.22) by ξ_i^i and letting $\xi_i \to \infty$ we get the result.

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