Faber polynomials, Cayley–Hamilton equation and Newton symmetric functions

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Abstract


Résumé


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1. Introduction

The Faber polynomials introduced by Faber [3] play an important role in different areas of mathematics and there is a rich literature [2,7,8,5] describing their properties and their applications. In this paper, our goal is to show that elementary linear algebra techniques can provide new tools for the analysis of Faber polynomials. Let us briefly recall the basic definitions. Let $K$ be a compact set in $\mathbb{C}$, not a single point, whose complement $\hat{\mathbb{C}} \setminus K$ (with respect to the extended plane) is simply connected. By the Riemann theorem on conformal mapping there exists a unique function $z = \psi(w)$, meromorphic for $|w| > 1$, which maps the domain $|w| > 1$ onto $\hat{\mathbb{C}} \setminus K$ and satisfies the conditions

$$\psi(\infty) = \infty, \quad \psi'(\infty) > 0.$$ 

This condition implies that the function $z = \psi(w)$, being analytic in the domain $|w| > 1$ without the point $w = \infty$, has a simple pole at the point $w = \infty$.

The $n$th Faber polynomials of the first kind $F_n(z)$ and of the second kind $G_n(z)$ associated to $\psi$ can be given from the following generating function [3,7]

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{m=0}^{\infty} F_m(z) w^{-m-1}, \quad \text{(1.1)}$$

and

$$\frac{1}{\psi(w) - z} = \sum_{m=0}^{\infty} G_m(z) w^{-m-1}. \quad \text{(1.2)}$$

In this paper, the Laurent expansion of the mapping $\psi$ is given by:

$$\psi(w) = w + \sum_{k=0}^{\infty} \frac{b_{k+1}}{w^k}, \quad w \to \infty. \quad \text{(1.3)}$$

The variables $(b_1, b_2, \ldots, b_n, \ldots)$ are in the subset $\mathcal{M}$ of $\mathbb{C}^\mathbb{N}$ such that $\psi$ is univalent outside of the unit disk. From (1.1) and (1.2), we observe that the Faber polynomials $F_n(z)$ and $G_n(z)$ depend on the parameters $b_1, b_2, \ldots, b_n$. We write

$$F_n(z) =: F_n(b_1 - z, b_2, \ldots, b_n),$$
$$G_n(z) =: G_n(b_1 - z, b_2, \ldots, b_n)$$

and in particular we have:

$$\frac{\psi'(w)}{\psi(w)} = \sum_{n=0}^{\infty} F_n(b_1, b_2, \ldots, b_n) w^{-n-1}, \quad \text{(1.4)}$$

and

$$\frac{1}{\psi(w)} = \sum_{n=0}^{\infty} G_n(b_1, b_2, \ldots, b_n) w^{-n-1}. \quad \text{(1.5)}$$

with

$$F_0 = G_0 = 1 \quad \text{and} \quad F_1 = G_1 = -b_1.$$
On the submanifold $\mathcal{M}$, we introduce the family of the partial differential operators $(W_n)_{n \geq 1}$, the variables are $b_1, b_2, \ldots, b_n, \ldots$, and $\frac{\partial}{\partial b_n}$ denotes the partial derivative with respect to the $n$th variable $b_n$,

$$W_n(b_1, b_2, \ldots, b_n, \ldots) = -\frac{\partial}{\partial b_n} - \sum_{i=1}^{\infty} b_i \frac{\partial}{\partial b_{n+i}}.$$

We prove that:

**Theorem 1.1.** The Faber polynomials $F_n(b_1, b_2, \ldots, b_n)$ and $G_n(b_1, b_2, \ldots, b_n)$ verify the following differential equations for any $n \geq 1$ and $m \geq 0$:

$$W_n(b_1, b_2, \ldots, b_n, \ldots)F_m = n\delta_{n,m}, \quad (1.6)$$

$$W_n(b_1, b_2, \ldots, b_n, \ldots)G_m = G_m - n. \quad (1.7)$$

If $\psi(z)$ has the form

$$\psi(z) = z + \sum_{k=0}^{\infty} b_{p(k+1)}z^{pk+p-1}, \quad z \to \infty,$$

we obtain

**Corollary 1.2.** The following holds:

$$W_n(b_p, b_{2p}, \ldots, b_{np}, \ldots)F_m = m\delta_{n,m}. \quad (1.8)$$

The solutions of this system of differential equations are:

**Theorem 1.3.** For any $k \geq 2$, the polynomials $F_k(b_1, b_2, \ldots, b_k)$ and $G_k(b_1, b_2, \ldots, b_k)$ are given by:

$$F_k(b_1, b_2, \ldots, b_k) = \sum_{i_2=0}^{\lfloor \frac{k}{2} \rfloor} \ldots \sum_{i_k=0}^{\lfloor \frac{k}{k} \rfloor} A(i_2, i_3, \ldots, i_k) b_2^{i_2} b_3^{i_3} b_k^{k-2i_2-3i_3-\cdots-ki_k}, \quad (1.9)$$

$$G_k(b_1, b_2, \ldots, b_k) = \sum_{i_2=0}^{\lfloor \frac{k+1}{2} \rfloor} \ldots \sum_{i_k=0}^{\lfloor \frac{k+1}{k} \rfloor} B(i_2, i_3, \ldots, i_k) b_2^{i_2} b_3^{i_3} b_k^{k-2i_2-3i_3-\cdots-ki_k}, \quad (1.10)$$

where

$$A(i_2, i_3, \ldots, i_k) := (-1)^{k+3i_2+4i_3+\cdots+(k+1)i_k} \frac{(k-i_2-2i_3-\cdots-(k-1)i_k-1)!k}{(k-2i_2-3i_3-\cdots-ki_k)!i_2!\cdots i_k!},$$

$$B(i_2, i_3, \ldots, i_k) := (-1)^{k+3i_2+4i_3+\cdots+(k+1)i_k} \frac{(k-i_2-2i_3-\cdots-(k-1)i_k)!}{(k-2i_2-3i_3-\cdots-ki_k)!i_2!\cdots i_k!}.$$
Remark 1.1. The Faber polynomials $F_k(b_1, b_2, \ldots, b_k)$ can be written as:

$$F_k(b_1, b_2, \ldots, b_k) = \sum_{i_1 \geq i_2 \geq \cdots \geq i_k \geq 0} \mathcal{A}(i_1, i_2, i_3, \ldots, i_k) b_1^{i_1} b_2^{i_2} \cdots b_k^{i_k},$$

where

$$\mathcal{A}(i_1, i_2, i_3, \ldots, i_k) := (-1)^{k+\sum_{j=1}^{k+1} i_j} (i_1 + i_2 + \cdots + i_k - 1)!k \prod_{j=1}^{i_1} (i_2! \cdots i_k!).$$

The first Faber polynomials $F_n$ are given by:

- $F_2 = b_1^2 - 2b_2$
- $F_3 = -b_1^3 + 3b_1b_2 - 3b_3$
- $F_4 = b_1^4 - 4b_1^2b_2 + 2b_2^2 + 4b_1b_3 - 4b_4$
- $F_5 = -b_1^5 + 5b_1^3b_2 - 5b_1^2b_3 + 5b_2b_3 - 5b_1(b_2^2 - b_4) - 5b_5$
- $F_6 = b_1^6 - 6b_1^4b_2 - 2b_1^3 + 6b_1^2b_3 + 3b_2^3 + b_1^3(9b_2^2 - 6b_4) + 6b_2b_4 + 6b_1(-2b_2b_3 + b_5) - 6b_6$
- $F_7 = -b_1^7 + 7b_1^5b_2 - 7b_1^4b_3 - b_1^3(7b_2^2 - 2b_4) + 7b_1^2(3b_2b_3 - b_5) + 7b_1(b_3^3 - 2b_2b_4 + b_6) - 7(b_2b_3 - b_3b_4 - b_2b_5 + b_7)$
- $F_8 = b_1^8 - 8b_1^6b_2 + 8b_1^5b_3 + 4b_1^4(5b_2^2 - 2b_4) + 8b_1^3(-4b_2b_3 + b_5) - 4b_1^2(4b_2^2 - 3b_2 - 6b_2b_4 + 2b_6) + 2b_1(3b_2b_3 - 2b_3b_4 - 2b_2b_5 + b_7) + 2b_2^2 - 4b_2b_3 - 4b_2^2b_4 + 2b_2^4 + 4b_3b_5 + 4b_2b_6 - 6b_8$
- $F_9 = -b_1^9 + 9b_1^6b_2 - 9b_1^5b_3 + 9b_1^4b_3 - 3b_1^3 + 9b_1^2(5b_2b_3 - b_5) - 9b_2^2b_3 + 9b_4b_5 - 12b_2b_4 - 3b_6) + 9b_2(-2b_3b_4 + b_7) - 9b_1^2(6b_2b_3 - 3b_3b_4 - 3b_2b_5 + b_7) - 9b_1(b_2^4 - 3b_2b_4 + 2b_2^3b_5 + 2b_3b_5 + 2b_3b_5 - 3b_2b_3 + 2b_6) - b_8 - 9b_9$
- $F_{10} = b_1^{10} - 10b_1^8b_2 - 2b_1^7 + 10b_1^6b_3 + 5b_1^5(7b_2^2 - 2b_4) + 10b_1^4b_4 + 10b_1^3(6b_2b_3 + b_5) + 2b_1^2(3b_2^2 - 2b_6) - 5b_1^1(10b_3 - 5b_2^2 - 10b_2b_4 + 2b_6) + 10b_1^2 + 10b_1(-2b_2b_3 - 4b_3b_4 - 4b_2b_5 + b_7) + 5b_1^2(5b_2^2 - 12b_2b_4 + 3b_3 + 6b_3b_5 + 6b_2(-2b_2^2 + 2b_3b_5) - 10b_2^2b_3^2 + 2b_3b_5 - b_8) - 10b_1(4b_2b_3 - 3b_2^2b_5 + 2b_4b_5 + 2b_1b_6 + b_2(-2b_3b_4 + b_7) - b_9) + 5(-2b_2^2b_4 + b_2^3 + 2b_4b_6 + 2b_3b_7 - 2b_{10})$
- $F_{11} = -b_1^{11} + 11b_1^9b_2 - 11b_1^8b_3 + 11b_1^7(-4b_2^2 + b_4) + 11b_1^6(7b_2b_3 - b_5) + 11b_1^5(7b_2^2 - 3b_3^2 - 6b_2b_4 + b_6) - 11b_1^4(15b_2^2b_3 - 5b_3b_4 - 5b_2b_5 + b_7) - 11b_1^3(5b_2^4 - 10b_4b_5 + 2b_4^2 + 4b_3b_5 + b_2(-10b_2^2 + 4b_6) - b_8) + 11b_1^2(10b_2^3b_3 - 2b_1^2 - 3b_2^2b_5 + 5b_2b_4 + b_6) + 11b_1b_2(2b_2^3 - 4b_2b_4 + 3b_3b_4 - 2b_2^2b_6 + b_2(-6b_3b_4 + b_7) - b_9) + 11b_1(-b_2^4 - 4b_2b_4 + 3b_3b_4 - b_2 - 2b_4b_6 + b_2(-6b_3^2 + 3b_6) - 2b_3b_7$
+ b_2(3b_3^2 + 6b_2b_5 - 2b_8) + b_{10}) - 11(b_2^3b_3 - b_2^2b_5 + b_2b_6 - b_4b_7 \\
+ b_2^2(-3b_3b_4 + b_7) + b_3(b_3^2 - b_8) - b_2(b_3^2 - 2b_4b_5 - 2b_3b_6 + b_9) + b_{11}).

From Eq. (1.4), we deduce that

$$1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots = \exp \left( -\sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \ldots, b_k)}{k} w^k \right).$$

We differentiate this equation with respect to $w$, we obtain

$$b_1 + 2b_2 w + \cdots + kb_k w^{k-1} + \cdots = (1 + b_1 w + b_2 w^2 + \cdots + b_p w^p + \cdots) \times \left( -\sum_{k=1}^{+\infty} F_k w^{k-1} \right)$$

and equal coefficients of equal powers of $w$, it gives

$$F_{n+1} = -b_1 F_n - \sum_{k=1}^{n-1} b_{n+1-k} F_k - (n+1)b_{n+1}.$$  \hspace{1cm} (1.11)

**Remark 1.2.** Consider the differential operator

$$L(b_1, b_2, \ldots, b_n, \ldots) = \sum_{k=1}^{\infty} k b_k \frac{\partial}{\partial b_k}.$$ 

From (1.11), we deduce that

$$-k b_k = \sum_{1 \leq j \leq k} F_j b_{k-j}.$$ 

Thus $L = \sum_{k \geq 1} k b_k \frac{\partial}{\partial b_k} = \sum_{j \geq 1} F_j (\sum_{k \geq j} b_{k-j} \frac{\partial}{\partial b_k})$ which gives

$$L = \sum_{j \geq 1} F_j W_j. \hspace{1cm} (1.12)$$

From (1.7), we have $W_j G_m = G_{m-j}$. Since $G_m$ is homogeneous of degree $m$, we have $L G_m = m G_m$. If we compare with (1.12), it gives that

$$k G_k = \sum_{1 \leq j \leq k} F_j G_{k-j}.$$ 

Now, differentiating (1.4) with respect to $b_1$ and Eq. (1.5) with respect to $w$ we obtain for $k \geq 1$:

$$G_k(b_1, b_2, \ldots, b_k) = -\frac{1}{k+1} \frac{\partial F_{k+1}}{\partial b_1}(b_1, b_2, \ldots, b_{k+1}).$$  \hspace{1cm} (1.13)

We remark that $\frac{1}{k+1} \frac{\partial F_{k+1}}{\partial b_1}(b_1, b_2, \ldots, b_{k+1})$ is independent of $b_{k+1}$ since $F_{k+1}$ is a homogeneous polynomial of degree $k + 1$ and $b_{k+1}$ of weight $k + 1$.

The first Faber polynomials $G_n$ are given by:
• \( G_1 = -b_1 \)
• \( G_2 = b_1^2 - b_2 \)
• \( G_3 = -b_1^3 + 2b_1b_2 - b_3 \)
• \( G_4 = b_1^4 - 3b_1^2b_2 + 2b_1b_3 + b_2^2 - b_4 \)
• \( G_5 = -b_1^5 + 4b_1^3b_2 - 3b_1^2b_3 - b_1(3b_2^2 - 2b_4) + 2b_2b_3 - b_5 \)
• \( G_6 = b_1^6 - 5b_1^4b_2 + 4b_1^3b_3 - 3b_1^2(-2b_2^2 + b_4) - 2b_1(3b_2b_3 - b_5) \)
\[ - (b_3^2 - b_2^2 - 2b_2b_4 + b_6) \]
• \( G_7 = -b_1^7 + 6b_1^5b_2 - 5b_1^4b_3 - 2b_1^3(5b_2^2 - 2b_4) - 3b_1^2(-4b_2b_3 + b_5) \)
\[ + b_1(4b_3^2 - 3b_3^2 - 6b_2b_4 + 2b_6) - (3b_2^2b_3 - 2b_3b_4 - 2b_2b_5 - b_7). \]

On the other hand, from Eq. (1.5) we obtain:

\[
1 = \left(1 + \frac{G_1}{z} + \frac{G_2}{z^2} + \cdots + \frac{G_m}{z^m} + \cdots\right) \times \left(1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots + \frac{b_m}{z^m} + \cdots\right).
\]

We observe that \( b_1, b_2, \ldots, \) and \( G_1, G_2, \ldots, \) play a symmetric role and they verify the following recurrence formula:

\[
G_1 + b_1 = 0,
G_2 + b_1G_1 + b_2 = 0,
G_3 + b_1G_2 + b_2G_1 + b_3 = 0,
\]
\[
\vdots
\]
\[
G_n + b_1G_{n-1} + b_2G_{n-2} + \cdots + b_{n-1}G_1 + b_n = 0.
\]

(1.14)

**Corollary 1.4.** Let \( \psi(z) = z + b_1 + \frac{b_m}{z^{m-1}} \), then the Faber polynomials \( F_k(b_1, 0, \ldots, b_m, 0, \ldots) \) are given by:

\[
F_k = \sum_{i=0}^{j} (-1)^{(k+i)(m-1)} \frac{(k-(m-1)i-1)!k^{i}b_{m}^{i}b_{1}^{k-m}}{(k-mi)!i!}.
\]

with \( k = jm + p \) and \( 0 \leq p < m \).

For example, if \( \psi(z) = z + b_1 + \frac{b_3}{z^2} \), one has:

\[
F_1 = -b_1, \quad F_2 = b_1^2, \quad F_3 = -b_1^3 - 3b_3,
F_4 = b_1^4 + 4b_1b_3, \quad F_5 = -b_1^5 - 5b_1^2b_3, \quad F_6 = b_1^6 + 6b_1^3b_3 + 3b_3^2.
\]

Let us consider as an other example the mapping \( \psi \) defined by

\[
\psi(z) = z + \frac{1}{(m-1)z^{m-1}}.
\]
which is conformal in the exterior of the unit circle. The boundary of the associated compact set
\[ K = \hat{\mathbb{C}} \setminus \{ z \in \mathbb{C} : z = \psi(w), \ |w| > 1 \} \]
is called a \( m \)-cusped hypocycloid.

As consequence of Corollary 1.4, we have the following result proved by Mathew X. He in [4]:

**Corollary 1.5.** The Faber polynomials \( F_k \) associated to an \( m \)-cusped hypocycloid are given by:
\[
F_k(z) = \left( \frac{1}{m-1} \right)^j z^p \sum_{i=0}^j \frac{(k-(m-1)i-1)!k}{(k-mi)!i!} \frac{(-1)^i ((m-1)z^m)^{j-i}}{j-i},
\]
with \( k = jm + p \) and \( 0 \leq p < m \).

In [1], the authors introduced the generalized Faber polynomials \( (H^k_j) \) associated to the function \( \psi(w) = w + \sum_{k=0}^{\infty} \frac{b_k+1}{w^k} \) by:
\[
w \frac{\psi'(w)}{\psi(w)} \left( \frac{\psi(w)}{w} \right)^k = 1 + \sum_{j=1}^{\infty} H^{k-j}_j w^{-j}
= 1 - \sum_{j=1}^{\infty} F^{k+j}_j z^j.
\]
They showed that those Faber polynomials are linked to the coefficients in the asymptotic expansion of the function \( \left( \frac{\psi(w)}{w} \right)^p \):
\[
\left( \frac{\psi(w)}{w} \right)^p = 1 + \sum_{n=1}^{\infty} K^p_n w^{-n}.
\]
When \( \frac{\psi(z)}{z} = zf\left( \frac{1}{z} \right) \), then
\[
K^p_j = \frac{1}{2} (H^{k-j}_j - F^{k+j}_j).
\]

In this paper, we give explicit formulas for the polynomials \( (H^k_j) \), \( (F^k_j) \) and \( (K^p_j) \).

For an indeterminate \( u \), we set
\[
s(u) = \sum_{j=1}^{+\infty} b_j u^j
\]
and
\[
a_1 = -b_1, \quad a_j = \frac{1}{1 - j} \left( 1 + s(u) \right)^j, \quad \text{for } j = 2, 3, \ldots
\]
such that \((1+s(u))^{j-1}\) stands for the coefficient of \(u^j\) in the Taylor expansion of \((1+s(u))^{j-1}\). For simplicity, we set \(\lambda = i_2 + i_3 + \cdots + i_{m+n}\) and \(\gamma = 2i_2 + 3i_3 + \cdots + (n + m)i_{n+m}\). Using the explicit formula of the Faber polynomials, we will show that:

**Theorem 1.6.** The coefficients \(K_n^m\) are given by:

If \(n \leq m\), we have

\[
K_n^m = \sum_{n+i_1+2i_2+3i_3+\cdots+mi_m=m} L(n,i_1,i_2,i_3,\ldots,i_k)(-1)^{i_1+i_2+i_3+\cdots+i_k}a_1^{i_1}a_2^{i_2}\cdots a_m^{i_m}
\]

and if \(n > m\) we get:

\[
K_n^m = \sum_{m=1}^{\infty} \sum_{y=m+n} H(i_2,i_3,\ldots,i_{n+m})a_2^{i_2}\cdots a_{n+m}^{i_{n+m}} \sum_{m_m=n} \prod_{i=1}^{m} G_n_i (b_1, b_2, \ldots, b_{n_i}),
\]

where

\[
L(n,i_1,i_2,i_3,\ldots,i_m) := \frac{(n + i_1 + i_2 + \cdots + i_m - 1)!m}{n!i_1!\cdots i_m!},
\]

\[
H(i_2,i_3,\ldots,i_{n+m}) := -n \frac{(\lambda - 1)!}{i_2!\cdots i_{n+m}!} \left[ \prod_{j=2}^{n+m} \left( \frac{x - x_j}{1-x} \right)^{i_j} \right]_n
\]

such that \([\ldots]_n\) denotes the coefficient of \(x^n\) in the expansion of the expression in the bracket in power of \(x\) and

\[
\left[ \frac{x - x_j}{1-x} \right]_n^{i_j} = \sum_{k=0}^{ij} (-1)^k \binom{i_j}{k} \binom{i_j - 1 + n - jk}{i_j - 1}.
\]

Combining (1.4) with (1.15), we get

\[
\left( 1 + \sum_{n=1}^{\infty} F_n(b_1, b_2, \ldots, b_n)w^{-n} \right) \left( 1 + \sum_{n=1}^{\infty} K_n^k w^{-n} \right) = 1 + \sum_{j=1}^{\infty} H_j^{k-j} w^{-j}
\]

thus

\[
\sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} F_{n-i}(b_1, b_2, \ldots, b_n)K_i^k \right) w^{-n} = \sum_{j=0}^{\infty} H_j^{k-j} w^{-j}
\]

with

\[
F_0(b_1, b_2, \ldots, b_n) := K_0^k := H_0^k := 1.
\]

It follows that:

\[
H_n^{k-n} = \sum_{i=0}^{n} F_{n-i}(b_1, b_2, \ldots, b_n)K_i^k.
\]
From (1.9) we get an explicit formula of $F_{n-i}(b_1, b_2, \ldots, b_n)$ and if we combine Theorem 1.6 with (1.18) we get an explicit expression from $H_{n}^{k-n}$. When $\frac{\psi(z)}{z} = zf(\frac{1}{z})$ and from (1.17), the polynomials $F_{j}^{k+j}$ are expressed by:

$$F_{j}^{k+j} = H_{j}^{k-j} - 2K_{j}^{k}$$

which gives also an explicit formula for $F_{j}^{k+j}$.

In the sequel, we give a generalized Cayley–Hamilton equation. To motivate our results, we consider the case $A \in SL_{2}(\mathbb{C}) := \{X \in M_{2}(\mathbb{C}), \det X = 1\}$. The Cayley–Hamilton equation can be written as:

$$A^{2} - tA + I = 0, \quad t = \text{trace}(A).$$

The function $\psi(z) := z + \frac{1}{z}$ is the conformal map from the exterior of the unit disk onto the exterior of $[-2, 2]$. Eq. (1.2) become:

$$w = \sum_{n=0}^{\infty} G_{n}(z)w^{-n-1}. \quad (1.19)$$

We will show that for any $n \geq 1$, we have

$$A^{n} = G_{n-1}(t)A - G_{n-2}(t)I \quad (1.20)$$

which gives the following:

**Proposition 1.7.** If $A \in SL_{2}(\mathbb{C})$, then for any $m_1, m_2 \geq 1$, we have:

$$G_{m_2-1}A^{m_1} - G_{m_1-1}A^{m_2} = (G_{m_1}G_{m_2-1} - G_{m_1-1}G_{m_2})I.$$

Now, if we set

$$G_{m_1, m_2} := G_{m_1}G_{m_2-1} - G_{m_1-1}G_{m_2}, \quad (1.21)$$

Proposition 1.7 can be written as:

$$G_{0, m_2}A^{m_1} - G_{0, m_1}A^{m_2} = G_{m_1, m_2}I.$$

Using Eq. (1.19), the polynomials $G_{m_1, m_2}$ verify:

$$\frac{\xi_1 - \xi_2}{P_{A}(\xi_1)P_{A}(\xi_2)} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} G_{m_1, m_2}(t)\xi_1^{-m_1-1}\xi_2^{-m_2-1}. \quad (1.22)$$

More generally, we give a general Cayley–Hamilton equation with the help of the polynomials $G_{m_1, m_2, \ldots, m_{p+1}}$ defined as follows: let $A \in SL_{p+1}(\mathbb{C}) := \{X \in M_{p+1}(\mathbb{C}), \det X = 1\}$ and $P_{A}(\xi) = \det(\xi I - A)$ its characteristic polynomial. We define these polynomials by

$$\frac{a_{p}(\xi)}{\prod_{i=1}^{p+1} P_{A}(\xi_{i})} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{p+1}=0}^{\infty} G_{m_1, m_2, \ldots, m_{p+1}}(t)\prod_{i=1}^{p+1} \xi_{i}^{-m_{i}-1} \quad (1.22)$$
with
\[ a_p(\xi) = \prod_{1 \leq i < j \leq p+1} (\xi_i - \xi_j), \quad t = \text{trace}(A). \]

We will show that:

**Theorem 1.8.** Let \( A \in SL_{p+1}(\mathbb{C}) \), then for every non-negative integer \( m_i, 1 \leq i \leq p+1 \), we get the following generalized Cayley–Hamilton equation:

\[
\sum_{i=1}^{p+1} (-1)^{i-1} G_{0,m_1,m_2,\ldots,m_i,\ldots,m_{p+1}} A^{m_i} = G_{m_1,m_2,\ldots,m_{p+1}} I,
\]

where \( G_{0,m_1,m_2,\ldots,m_i,\ldots,m_{p+1}} \) is defined by removing the indice \( m_i \) of the polynomials \( G_{0,m_1,m_2,\ldots,m_i,\ldots,m_{p+1}} \).

In order to give a simplified expression of the polynomials \( G_{m_1,m_2,\ldots,m_{p+1}} \), we associated to the matrix \( A \in SL_{p+1}(\mathbb{C}) \) the polynomials \( G^p_k(t) \) defined by the generating function:

\[
\frac{\xi^p}{P_A(\xi)} = \sum_{k=0}^{\infty} G^p_k(t) \xi^{-k-1}. \tag{1.23}
\]

If we consider the case \( p = 1 \) in (1.23) that is \( A \in SL_2(\mathbb{C}) \), putting \( P_A(\xi) = \xi^2 - t\xi + 1 \) into the expression (1.23) and comparing with (1.19), we obtain:

\[
G^1_1 = G_n. \tag{1.24}
\]

Combining this with Eq. (1.21), we have:

\[
G_{0,m+1} = G^1_m, \tag{1.25}
\]

\[
G_{m_1,m_2} = G^1_{m_1} G^1_{m_2-1} - G^1_{m_1-1} G^1_{m_2}. \tag{1.26}
\]

In general we have:

**Theorem 1.9.** The polynomials \( G_{m_1,m_2,\ldots,m_{p+1}} \) are given in terms of the polynomials \( G_{m_i}, 1 \leq i \leq p+1 \), as follows:

\[
G_{m_1,m_2,\ldots,m_{p+1}} = \begin{vmatrix}
G^p_{m_1} & G^p_{m_2} & \cdots & G^p_{m_{p+1}} \\
G^p_{m_1-1} & G^p_{m_2-1} & \cdots & G^p_{m_{p+1}-1} \\
\vdots & \vdots & \ddots & \vdots \\
G^p_{m_1-p} & G^p_{m_2-p} & \cdots & G^p_{m_{p+1}-p}
\end{vmatrix}. \tag{1.27}
\]

Conversely, the polynomials \( G^p_m \) are given by:

\[
G^p_m = G_{0,1,\ldots,p-1,m+p}. \tag{1.28}
\]
2. Preliminary results

Set
\[ \psi(w) = w + \sum_{k=0}^{\infty} \frac{b_{k+1}}{w^k}, \]
\[ \frac{\psi'(w)}{\psi(w)} = \sum_{k=0}^{\infty} F_k(b_1, b_2, \ldots, b_k) w^{-k-1}. \]

Lemma 2.1. The following holds:
\[ \log\left(1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots\right) = -\sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \ldots, b_k)}{k} w^k. \]  \hspace{1cm} (2.1)

Proof. Let \( f(w) = 1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots \). We have
\[ f(w) = w \psi\left(\frac{1}{w}\right), \]
\[ f'(w) = \psi\left(\frac{1}{w}\right) - \frac{1}{w} \psi'\left(\frac{1}{w}\right) \]
and also
\[ \frac{w f'(w)}{f(w)} = 1 - \frac{1}{w} \frac{\psi'(1/w)}{\psi(1/w)} = 1 - \sum_{k=0}^{+\infty} F_k(b_1, b_2, \ldots, b_k) w^k. \]

It implies the formula:
\[ \frac{f'(w)}{f(w)} = \frac{d}{dw} \log\left(1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots\right) \]
\[ = -\sum_{k=1}^{+\infty} F_k(b_1, b_2, \ldots, b_k) w^{k-1}. \]  \hspace{1cm} (2.2)

Integrating this equation with respect to \( w \), the lemma is proved. \( \square \)

Theorem 2.2. Let \((x_1, \ldots, x_k)\) be the roots of the polynomial
\[ Q(\xi) = \xi^k + b_1 \xi^{k-1} + \cdots + b_k \]
and
\[ \Pi_k = x_1^k + x_2^k + \cdots + x_k^k. \]
Then
\[ F_k = \Pi_k. \]
Proof. We have that

\[ 1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots + b_k w^k = \prod_{i=1}^{k} \left( \frac{1}{w} - x_i \right) w^k = \prod_{i=1}^{k} (1 - x_i w) \]

and also

\[
\log(1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots + b_k w^k) = \sum_{i=1}^{k} \log(1 - x_i w) = - \sum_{j=1}^{\infty} \sum_{i=1}^{k} (x_i w)^j = - \sum_{j=1}^{\infty} \prod_{j=1}^{i} \frac{w^j}{j},
\]

with

\[ \prod_j = x_1^j + x_2^j + \cdots + x_k^j. \]

On the other hand, formula (2.1) shows that:

\[
-\frac{F_k}{k} = \frac{d^k \log(1 + \sum_{i=1}^{\infty} b_i w^i)}{d w^k}(0)
\]

\[
= \frac{1}{k!} \frac{d^k \log(1 + \sum_{i=1}^{k} b_i w^i)}{d w^k}(0) + \frac{1}{k!} \frac{d^k \log(1 + \sum_{i=1}^{\infty} b_{k+1} w^{i-1})}{d w^k}(0)
\]

\[
= \frac{1}{k!} \frac{d^k \log(1 + \sum_{i=1}^{k} b_i w^i)}{d w^k}(0)
\]

\[
= -\frac{\Pi_k}{k}
\]

which completes the proof of the theorem. \( \square \)

Proposition 2.3. We consider the Faber polynomials \( F_n(b_1, b_2, \ldots, b_n) \). If the coefficients \( b_j \) of odd index vanish, one has

\[ F_{2n+1}(0, b_2, 0, b_4, 0, \ldots, b_{2n}, 0) = 0, \quad (2.3) \]

\[ F_{2n}(0, b_2, 0, b_4, 0, \ldots, b_{2n}) = 2 F_n(b_2, b_4, \ldots, b_{2n}). \quad (2.4) \]

Proof. We have:

\[ \psi(z) = z + \frac{b_2}{z} + \frac{b_4}{z^3} + \cdots + \frac{b_{2n+2}}{z^{2n+4}} + \cdots \]

\[ = \frac{1}{z} \left( z^2 + b_2 + \frac{b_4}{z^2} + \cdots + \frac{b_{2n+1}}{z^{2n}} \right) =: \psi_1(z^2) / z. \]

Then

\[ \log \frac{\psi(z)}{z} = \log \frac{\psi_1(z^2)}{z^2}. \quad (2.5) \]

Using the relation

\[ \log \frac{\psi(z)}{z} = - \sum_{m=1}^{\infty} \frac{F_m}{m} \times \frac{1}{z^m} \]
we obtain:

\[ \sum_{m=1}^{\infty} \frac{F_{2m}(0, b_2, \ldots, 0, b_{2m})}{2^m} \times \frac{1}{z^{2m}} = \sum_{m=1}^{\infty} \frac{F_m(b_2, b_4, \ldots, b_{2m})}{m} \times \frac{1}{z^{2m}} \]

which gives the result. \( \square \)

More generally, if \( \psi(z) \) has the form

\[ \psi(z) = z + \sum_{k=0}^{\infty} \frac{b_{p(k+1)}}{z^{pk+p-1}}, \quad z \to \infty, \quad (2.6) \]

we get:

**Proposition 2.4.** The following holds:

- \( F_k(0, 0, \ldots, 0, b_p, 0, 0, \ldots, 0, b_{2p}, 0, 0, \ldots, 0, b_{mp}, \ldots, 0) = 0 \)
  - if \( k \) is not a multiple of \( p \),
- \( F_{pm}(0, 0, \ldots, 0, b_p, 0, 0, \ldots, 0, b_{2p}, 0, 0, \ldots, 0, b_{mp}) = p F_m(b_p, b_{2p}, \ldots, b_{mp}) \).

**Proof.** We have:

\[ \psi(z) = z + \sum_{n \geq 0} \frac{b_{p(n+1)}}{z^{pn+p-1}} = \frac{1}{z^{p-1}} \left( z^p + \sum_{n \geq 0} \frac{b_{p(n+1)}}{z^{pn}} \right) = \frac{1}{z^{p-1}} \psi_1(z^p) \]

with

\[ \psi_1(z) = z + c_1 + \sum_{j \geq 1} \frac{c_j}{z^j} \quad \text{and} \quad c_j = b_{pj} \]

which gives:

\[ \log \frac{\psi(z)}{z} = \log \frac{\psi_1(z^p)}{z^p}. \]

Using formula (2.1), the result follows. \( \square \)

**Remark 2.1.** From Eq. (2.2), we get:

\[ b_1 + 2b_2 w + 3b_3 w^2 + \cdots = (1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots) \left( -\sum_{k=1}^{+\infty} F_k w^{k-1} \right). \]

By comparing coefficients of like powers of \( w \) and using the Cramer formula, the Faber polynomials \( F_k(b_1, b_2, \ldots, b_k) \) has the following representation:

\[
F_k(b_1, b_2, \ldots, b_k) = (-1)^k \begin{vmatrix}
  b_1 & 1 \\
  2b_2 & b_1 & 1 \\
  \vdots & b_2 & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots \\
  kb_k & b_{k-1} & b_2 & b_1 \\
\end{vmatrix}
\]
3. Proof of the main results

Proof of Theorem 1.1. From Eq. (1.11), we observe that the polynomials $F_k$ depend only of the coefficients $b_1, b_2, \ldots, b_k$. We obtain:

\[ W_n F_m = 0 \quad \text{for } m < n, \quad (3.1) \]
\[ W_n F_n = n. \quad (3.2) \]

To end the proof, it remains to prove that $W_n F_{n+p} = 0$ for every positive integer $p$. We do it by induction on $p$. Combining (1.11), (3.1) and (3.2) we get:

\[ W_n F_{n+1} = -nb_1 + F_1 + (n + 1)b_1 = 0. \quad (3.3) \]

The proposition is true for $p = 1$, suppose it true for $p - 1$. Eq. (1.11) can be written as:

\[ F_{n+p} = -b_1 F_{n+p-1} - \sum_{k=1}^{n+p-2} b_{n+p-k} F_k - (n + p)b_{n+p}. \quad (3.4) \]

So that by the induction hypothesis and (3.1), (3.2), we have:

\[ W_n F_{n+p} = -\sum_{k=1}^{n+p-2} W_n (b_{n+p-k} F_k) + (n + p)b_p. \quad (3.5) \]

It follows that

\[ W_n F_{n+p} = -\sum_{k=1}^{n+p-2} (b_{n+p-k} \delta_{n,k} - b_{p-k} F_k) + (n + p)b_p, \quad (3.6) \]

where $b_0 = 1$ and $b_k = 0$ for $k < 0$. Hence

\[ W_n F_{n+p} = \sum_{k=1}^{p} b_{p-k} F_k + pb_p. \quad (3.7) \]

And by using the recurrence formula (1.11), we get:

\[ W_n F_{n+p} = F_p + b_1 F_{p-1} + \sum_{k=1}^{p-2} b_{p-k} F_k + pb_p = 0. \quad (3.8) \]

Which ends the proof of the first part of this theorem. The proof of (1.7) is similar. From Eq. (1.14) and by using the fact that the polynomials $G_k$ depend only of the coefficients $b_1, b_2, \ldots, b_k$, we obtain:

\[ W_n G_m = 0 \quad \text{for } m < n, \quad (3.9) \]
\[ W_n G_n = 1 = G_0. \quad (3.10) \]

To end the proof, it remains to prove that $W_n G_{n+p} = 0$ for every positive integer $p$. We prove it again by induction on $p$. If we combine (1.14), (3.9) with (3.10), we get:

\[ W_n G_{n+1} = -b_1 + G_1 + b_1 = G_1. \]
The proposition is true for \( p = 1 \), suppose it holds for \( p - 1 \). Eq. (1.14) can be written as:

\[
G_{n+p} = - \sum_{k=0}^{n+p-1} b_{k+1} G_{n+p-1-k}.
\] (3.11)

So that by the induction assumption and (3.9), (3.10):

\[
W_n G_{n+p} = - \sum_{k=0}^{n+p-1} W_n (b_{k+1} G_{n+p-1-k}).
\] (3.12)

We then get

\[
W_n G_{n+p} = - \sum_{k=0}^{n+p-1} (b_{k+1} G_{p-1-k} - b_{k+1-n} G_{n+p-1-k}),
\] (3.13)

with \( b_0 = 1 \) and \( b_k = G_k = 0 \) for \( k < 0 \). Hence

\[
W_n G_{n+p} = - \sum_{k=0}^{p-1} b_{k+1} G_{p-1-k} + \sum_{k=n-1}^{n+p-1} b_{k+1-n} G_{n+p-1-k}
\]

\[
= - \sum_{k=0}^{p-1} b_{k+1} G_{p-1-k} + \sum_{k=0}^{p-1} b_{k+1} G_{p-1-k} + G_p.
\]

It follows that

\[
W_n G_{n+p} = G_p.
\] (3.14)

Then the result is true for all \( p \geq 1 \), which ends the proof. \( \square \)

**Remark 3.1.** In [1], H. Airault and J. Ren introduced the family of operators

\[
Z_k = - \sum_{n=1}^{\infty} nb_n \frac{\partial}{\partial b_{n+k-1}}
\]

and proved the following functional relations:

\[
Z_1 F_n(b_1, b_2, \ldots, b_n) = -n F_n(b_1, b_2, \ldots, b_n),
\]

\[
Z_2 F_0(b_1, b_2, \ldots, b_n) = 0,
\]

\[
Z_2 F_1(b_1, b_2, \ldots, b_n) = 0,
\]

\[
Z_2 F_n(b_1, b_2, \ldots, b_n) = -n F_{n-1}(b_1, b_2, \ldots, b_{n-1}) \quad \text{for } n \geq 2,
\]

\[
Z_k F_n(b_1, b_2, \ldots, b_n) = 0 \quad \text{for } n \leq k - 2,
\]

\[
Z_k F_{k-1}(b_1, b_2, \ldots, b_{k-1}) = 0,
\]

\[
Z_k F_n(b_1, b_2, \ldots, b_n) = -n F_{n+k-1}(b_1, b_2, \ldots, b_{n+k-1}) \quad \text{for } n > k - 1.
\]
Proof of Corollary 1.2. It follows from Theorem 1.1 and Proposition 2.4. □

Proof of Theorem 1.3. Let \( p \geq 2 \) an integer, Waring’s formula relates the \( k \)th power sum \( x_1^k + \cdots + x_p^k \) to the elementary symmetric functions \( s_l(x_1, x_2, \ldots, x_p), 1 \leq l \leq p \), as follows:

\[
x_1^k + \cdots + x_p^k = \sum_{i_2=0}^{[\frac{k}{2}]} \sum_{i_p=0}^{[\frac{k}{p}]} E(i_2, i_3, \ldots, i_p) s_1^{k-2i_2-3i_3-\cdots-pi_p} s_2 i_2 \cdots s_p i_p,
\]

where

\[
E(i_2, i_3, \ldots, i_p) := (-1)^{i_2+2i_3+\cdots+(p-1)i_p} \frac{(k-i_2-2i_3-\cdots-(p-1)i_p-1)!}{(k-2i_2-3i_3-\cdots-pi_p)!i_2!\cdots i_p!}.
\]

(3.15)

If we combine it with Theorem 2.2 with \( b_l = (-1)^l s_l, 1 \leq l \leq k \), we obtain the first part of the theorem. The second part follows by using Eq. (1.13). □

Proof of Corollary 1.4. It follows immediately from Theorem 1.3. □

Proof of Corollary 1.5. We have \( F_n(z) = F_n(b_1-z, b_2, \ldots, b_n) \) and in the case of the \( m \)-cusped hypocycloid, one has: \( b_i = 0 \) for \( i \neq m \) and \( b_m = \frac{1}{m-1} \) then by using Corollary 1.4, we obtain the result. □

Proof of Theorem 1.6. The inverse mapping of \( \psi \) in (1.3) is given by:

\[
\phi(z) = z + \sum_{k=0}^{\infty} \frac{a_{k+1}}{z^k}.
\]

Now, we consider the Faber polynomials associated to \( \phi \):

\[
\frac{\phi'(w)}{\phi(w) - z} = \sum_{m=0}^{\infty} F_m(z) w^{-m-1}, \quad \text{(3.16)}
\]

\[
\frac{1}{\phi(w) - z} = \sum_{m=0}^{\infty} G_m(z) w^{-m-1}. \quad \text{(3.17)}
\]

The Faber polynomials of the first kind verify [2]:

\[
F_n(\phi(z)) = z^n - \sum_{m=1}^{\infty} \frac{\alpha_{n,m}}{m} z^{-m}
\]

which gives:

\[
(\psi(z))^n = F_n(z) + \sum_{m=1}^{\infty} \frac{\alpha_{n,m}}{m} \left( \frac{1}{\psi(z)} \right)^m. \quad \text{(3.18)}
\]
Put \( \lambda = i_2 + i_3 + \cdots + i_{m+n} \) and \( \gamma = 2i_2 + 3i_3 + \cdots + (n + m)i_{n+m} \). From [6], we have:

\[
\frac{a_{n,m}}{m} = \sum_{y=m+n} \frac{\lambda}{i_2! \cdots i_{n+m}!} \prod_{j=2}^{n+m} \left( \frac{x - x_j}{1 - x} \right) \]

where

\[
H(i_2, i_3, \ldots, i_{m+n}) := -n \left( \lambda - 1 \right)! \frac{1}{i_2! \cdots i_{n+m}!} \left( \prod_{j=2}^{n+m} \left( \frac{x - x_j}{1 - x} \right) \right)_{n} \]

such that \([\ldots]_n\) denotes the coefficient of \(x^n\) in the power expansion (with respect to \(x\)) of the expression inside the brackets.

Furthermore, using Eq. (1.9), the Faber polynomials \(F_m(z) = F_m(b_1 - z, b_2, \ldots, b_m)\) associated to \(\phi\) can be written as:

\[
F_m(z) = \sum_{i=0}^{m} z^i \sum_{i_1 + i_2 + \cdots + i_m = m} L(i, i_1, i_2, \ldots, i_k) \times (-1)^{i_1 + i_2 + \cdots + i_k} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m},
\]

where

\[
L(i, i_1, i_2, \ldots, i_m) := \frac{(i + i_1 + i_2 + \cdots + i_m - 1)!m}{i! i_1! \cdots i_m!}
\]

On the other hand, from (1.5) we find:

\[
\left( \frac{1}{\psi(z)} \right)^m = \sum_{n=0}^{\infty} \left( \sum_{n_1 + n_2 + \cdots + n_m = n} \prod_{i=1}^{m} G_{n_i}(b_1, b_2, \ldots, b_{n_i}) \right) z^{-n-1}
\]

Then by putting \(n_m = n_1 + n_2 + \cdots + n_m\), Eq. (3.18) rewrites as:

\[
\left( \psi(z) \right)^m = \sum_{i=0}^{m} z^i \sum_{i_1 + i_2 + \cdots + i_m = m} L(i, i_1, i_2, \ldots, i_k) \times (-1)^{i_1 + i_2 + \cdots + i_k} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m} \\
+ \sum_{n=0}^{\infty} \left( \sum_{m=1}^{n} \sum_{\gamma=m+n} H(i_2, i_3, \ldots, i_{n+m}) \frac{i_2^{i_2} \cdots a_{n+m}^{i_{n+m}}}{i_1! \cdots i_m!} \right) \prod_{i=1}^{m} G_{n_i}(b_1, b_2, \ldots, b_{n_i}) z^{-n-1}
\]

which implies that:

\[
\left( \frac{\psi(z)}{z} \right)^m = 1 + \sum_{i=0}^{m-1} \frac{1}{z^{m-i}} \sum_{i_1 + i_2 + \cdots + i_m = m} L(i, i_1, i_2, \ldots, i_k) \times (-1)^{i_1 + i_2 + \cdots + i_k} \prod_{j=1}^{m} a_j^{i_j}
\]

\[
+ \sum_{n=0}^{\infty} \left( \sum_{m=1}^{n} \sum_{\gamma=m+n} H(i_2, i_3, \ldots, i_{n+m}) \frac{i_2^{i_2} \cdots a_{n+m}^{i_{n+m}}}{i_1! \cdots i_m!} \right) \prod_{i=1}^{m} G_{n_i}(b_1, b_2, \ldots, b_{n_i}) z^{-n-1}
\]

\[
\times (-1)^{i_1 + i_2 + \cdots + i_k} \prod_{j=1}^{m} a_j^{i_j}
\]
\[
+ \sum_{n=0}^{\infty} \left( \sum_{m=1}^{\infty} \sum_{\gamma=m+n} H_{(i_2, i_3, \ldots, i_{n+m})} \prod_{j=2}^{m+n} a_{ij} \right) \prod_{j=2}^{m+n} a_{ij} \times \sum_{n_m=n} \prod_{i=1}^{m} G_{n_i}(b_1, b_2, \ldots, b_n) \frac{1}{z^{n+m+1}}.
\]

If we combine this equation with (1.16), the result is proved. \qed

**Proof of Proposition 1.7.** We start by the case \( p = 1 \), that is \( A \in SL_2(\mathbb{C}) \). The Cayley–Hamilton equation can be written as:

\[
A^2 - tA + I = 0, \quad t = \text{trace}(A).
\]

For any \( n \geq 1 \), we have

\[
A^n = l_n(t)A - j_n(t)I.
\]

Thus,

\[
A^{n+1} = l_n(t)(tA - I) - j_n(t)A = (tl_n(t) - j_n(t))A - l_n(t)I = l_{n+1}(t)A - j_{n+1}(t)I.
\]

By identification, we obtain

\[
l_{n+1}(t) = tl_n(t) - l_{n-1}(t),
\]

\[
l_0 = 0, \quad l_1(t) = 1
\]

and

\[
j_n = l_{n-1}.
\]

Let \( G_n \) be the Faber polynomials associated to \([-2, 2]\). From Eq. (1.19), the polynomials \( G_n \) verify the recurrence relations

\[
G_{n+1}(t) = tG_n(t) - G_{n-1}(t), \quad (3.19)
\]

\[
G_{-1} = 0, \quad G_0 = 1, \quad G_1(t) = t. \quad (3.20)
\]

Hence \( l_n = G_{n-1} \) and the Cayley–Hamilton equation in this case can be written for any \( n \geq 1 \) as

\[
A^n = G_{n-1}(t)A - G_{n-2}(t)I.
\]

Thus,

\[
G_{m_2-1}A^{m_1} - G_{m_1-1}A^{m_2} = (G_{m_2-1}G_{m_1-1} - G_{m_1-1}G_{m_2-1})A - (G_{m_2-1}G_{m_1-2} - G_{m_1-1}G_{m_2-2})I
\]

\[
= (G_{m_1-1}G_{m_2-2} - G_{m_2-1}G_{m_1-2})I.
\]

Combining this formula with (3.19), we obtain:
Without loss of generality, we can suppose that

\[ A_{m_2-1}A^{m_1} - A_{m_1-1}A^{m_2} = \left\{ G_{m_1-1}(tG_{m_2-1} - G_{m_2}) - G_{m_2-1}(tG_{m_1-1} - G_{m_1}) \right\} I \]

\[ = (G_{m_1}G_{m_2-1} - G_{m_2}G_{m_1-1}) I \]

which gives the desired result. \( \square \)

In what follows we set for simplicity \( G_m := G_m^p \).

**Proof of Theorem 1.8.** Without loss of generality, we can suppose that \( A \) is a diagonal matrix. We start by proving the result for \( p = 2 \). Let \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) be the eigenvalues of the matrix \( A \). We have

\[ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} (G_{0,m_2,m_3} \lambda_1^{m_1} - G_{0,m_1,m_3} \lambda_1^{m_2} + G_{0,m_1,m_2} \lambda_1^{m_3}) \frac{1}{\xi_1^{m_1+1}} \frac{1}{\xi_2^{m_2+1}} \frac{1}{\xi_3^{m_3+1}} \]

\[ = - \frac{1}{\xi_2 - \lambda_1} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} G_{0,m_1,m_3} \frac{1}{\xi_1^{m_1+1}} \frac{1}{\xi_3^{m_3+1}} \]

\[ + \frac{1}{\xi_3 - \lambda_1} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} G_{0,m_1,m_2} \frac{1}{\xi_1^{m_1+1}} \frac{1}{\xi_2^{m_2+1}} \]

\[ = \frac{1}{\xi_1 - \lambda_1} \frac{\xi_2 - \xi_3}{P_A(\xi_2)P_A(\xi_3)} - \frac{1}{\xi_2 - \lambda_1} \frac{\xi_1 - \xi_3}{P_A(\xi_1)P_A(\xi_3)} + \frac{1}{\xi_3 - \lambda_1} \frac{\xi_1 - \xi_2}{P_A(\xi_1)P_A(\xi_2)} \]

\[ = \frac{(\xi_1 - \lambda_2)(\xi_1 - \lambda_3)(\xi_2 - \xi_3) - (\xi_2 - \lambda_2)(\xi_2 - \lambda_3)(\xi_1 - \xi_3) + (\xi_3 - \lambda_2)(\xi_3 - \lambda_3)(\xi_1 - \xi_2)}{(P_A(\xi_1)P_A(\xi_2)P_A(\xi_3))} \]

\[ = \frac{(\xi_1 - \xi_2)(\xi_1 - \xi_3)(\xi_2 - \xi_3)}{P_A(\xi_1)P_A(\xi_2)P_A(\xi_3)} \]

Which ends the proof in the case of \( p = 2 \). For the general case and in similar way we get:

\[ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p+1=0}^{\infty} \left( \sum_{i=1}^{p+1} (-1)^{i+1} G_{0,m_1,m_2,\ldots,m_i,\ldots,m_{p+1}} \lambda_1^{m_i} \right) \prod_{j=1}^{p+1} \frac{\xi_j^{-m_j-1}}{\xi_j} \]

\[ = \frac{\sum_{n=1}^{p+1} (-1)^{n+1} \prod_{j=2}^{p+1} (\xi_n - \lambda_j) \Delta_n}{\prod_{i=1}^{p+1} P_A(\xi_i)}, \quad (3.21) \]
where $\Delta_n$ is the Vandermonde determinant $\det(\xi_i^{p+1-j})$ with $2 \leq j \leq p + 1$, $1 \leq i \leq p + 1$ and $i \neq n$. On other hand, $\prod_{1 \leq i < j \leq p+1} (\xi_i - \xi_j)$ is the Vandermonde determinant

$$
\prod_{1 \leq i < j \leq p+1} (\xi_i - \xi_j) = 
\begin{vmatrix}
\xi_1^p & \xi_1^{p-1} & \cdots & 1 \\
\xi_2^p & \xi_2^{p-1} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{p+1}^p & \xi_{p+1}^{p-1} & \cdots & 1
\end{vmatrix}
$$

Let $s_i$ be the elementary symmetric polynomials of $\lambda_2, \lambda_3, \ldots, \lambda_{p+1}$. If we change the column $c_1$ by $c_1 - s_1 c_2 + s_2 c_3 - \cdots + (-1)^p s_p c_{p+1}$, we find:

$$
\prod_{1 \leq i < j \leq p+1} (\xi_i - \xi_j) = 
\begin{vmatrix}
\prod_{j=2}^{p+1} (\xi_1 - \lambda_j) & \xi_1^{p-1} & \cdots & 1 \\
\prod_{j=2}^{p+1} (\xi_2 - \lambda_j) & \xi_2^{p-1} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\prod_{j=2}^{p+1} (\xi_{p+1} - \lambda_j) & \xi_{p+1}^{p-1} & \cdots & 1
\end{vmatrix}
$$

Expanding with respect to the first column, we find that

$$
\prod_{1 \leq i < j \leq p+1} (\xi_i - \xi_j) = \sum_{n=1}^{p+1} (-1)^n \prod_{j=2}^{p+1} (\xi_n - \lambda_j) \Delta_n. 
$$

Combining it with Eq. (3.21), we get:

$$
\frac{\prod_{1 \leq i < j \leq p+1} (\xi_i - \xi_j)}{\prod_{i=1}^{p+1} P_A(\xi_i)} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{p+1}=0}^{\infty} \left( \sum_{i=1}^{p+1} (-1)^i \mathcal{G}_{0,m_1,m_2,\ldots,m_{i+1},m_{p+1}}^{m_i,\lambda_i} \right) \prod_{j=1}^{p+1} \xi_j^{-m_j-1}.
$$

If is now clear in view of formula (1.22) that it implies the result. □

**Proof of Theorem 1.9.** From Eq. (1.23), we have:

$$
\frac{\prod_{i=1}^{p+1} \xi_i^p}{\prod_{i=1}^{p+1} P_A(\xi_i)} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{p+1}=0}^{\infty} \mathcal{G}_{m_1,m_2,\ldots,m_{p+1}}^{m_{p+1}} \prod_{i=1}^{p+1} \xi_i^{-m_i-1}
$$

and

$$
\frac{1}{\prod_{i=1}^{p+1} P_A(\xi_i)} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{p+1}=0}^{\infty} \mathcal{G}_{m_1,m_2,\ldots,m_{p+1}}^{m_{p+1}} \prod_{i=1}^{p+1} \xi_i^{-m_i-p-1}.
$$

Since $\prod_{1 \leq i < j \leq p+1} (\xi_i - \xi_j) = \det(\xi_i^{p+1-j})_{1 \leq i,j \leq p+1}$ is the Vandermonde determinant, we get
\[
\frac{\prod_{1 \leq i < j \leq p+1} (\xi_i - \xi_j)}{\prod_{i=1}^{p+1} P_A(\xi_i)} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{p+1}=0}^{\infty} \sum_{\sigma \in S_{p+1}} \epsilon(\sigma) G_{m_1+1-\sigma(1)} G_{m_2+1-\sigma(2)} \cdots G_{m_{p+1}+1-\sigma(p+1)} \\
\times \prod_{i=1}^{p+1} \xi_i^{-m_i-1}
\]

and by comparing it with Eq. (1.22) we connect the polynomials \(G_{m_1,m_2,...,m_{p+1}}\) and \(G_{m_i}\) for \(1 \leq i \leq p + 1\) as follows

\[
G_{m_1,m_2,...,m_{p+1}} = \sum_{\sigma \in S_{p+1}} \epsilon(\sigma) G_{m_1+1-\sigma(1)} G_{m_2+1-\sigma(2)} \cdots G_{m_{p+1}+1-\sigma(p+1)}. \tag{3.23}
\]

Which gives the first part of the theorem. To prove the second part, we will need to show the following:

**Lemma 3.1.** For \(m \geq 0\), the following holds:

\[
\sum_{i=0}^{p+1} (-1)^i s_i s_{0,1,...,p-2,p-1,m+p+1-i} = 0, \tag{3.24}
\]

with the starting value \(G_{0,1,...,p-2,p-1,p-i} = \delta_{i,0}\) for \(0 \leq i \leq p\) and \(s_0 := s_{p+1} = 1\).

**Proof.** It enough to observe that for \(1 \leq i \leq p\) the power of \(\xi_i\) in the nominator of the left side of Eq. (1.22) is \(p+1-i\). Thus by multiplying both sides of (1.22) by \(\xi_i^i\) and letting \(\xi_i \to \infty\) we get that

\[
\frac{1}{P_A(\xi_{p+1})} = \sum_{m_{p+1}=0}^{\infty} G_{0,1,...,p-2,p-1,m_{p+1}-p-1} \xi^{-m_{p+1}-1}. \tag{3.25}
\]

Multiplying both sides of this equation by \(P_A(\xi_{p+1})\) and comparing the powers of \(\xi_{p+1}\) we get Eq. (3.24). \(\square\)

In the other hand, the polynomials \(G_m^p\) satisfy for \(m \geq p + 2\)

\[
G_m - t G_{m-1} + s_2 G_{m-2} + \cdots + (-1)^{p+1} s_{p+1} G_{m-p-1} = 0 \tag{3.26}
\]

with the initial conditions

\[
G_m - t G_{m-1} + s_2 G_{m-2} + \cdots + (-1)^{m-1} s_{m-1} G_1 = (-1)^m s_m G_0 \quad (1 \leq m \leq p+1). \tag{3.27}
\]

Now, by comparing Eq. (3.26) with (3.24) the result is proved. \(\square\)
Remark 3.2. From (1.22), we find that the polynomials $\mathcal{G}_{m_1, m_2, \ldots, m_{p+1}}$ verify the following properties:

\begin{align*}
\mathcal{G}_{m_1, m_2, \ldots, m_j, \ldots, m_{p+1}} &= -\mathcal{G}_{m_1, m_2, \ldots, m_j, \ldots, m_{p+1}}, \\
\mathcal{G}_{0, 1, \ldots, p} &= 1.
\end{align*}

(3.28)  
(3.29)

To prove (3.29), it suffices to observe that for $1 \leq i \leq p + 1$ the power of $\xi_i$ in the nominator of the left side of Eq. (1.22) is $p + 1 - i$. Thus by multiplying both sides of (1.22) by $\xi_i$ and letting $\xi_i \to \infty$ we get the result.

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References