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Compound Node–Kayles on paths

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A B S T R A C T

In his celebrated book [J.H. Conway, On Numbers and Games, Academic Press, New-York, 1976, Second edition (2001), A.K. Peters, Wellesley, MA], J.H. Conway introduced twelve versions of compound games. We analyze these twelve versions for the Node–Kayles game on paths. For usual disjunctive compound, Node–Kayles has been solved for a long time under normal play, while it is still unsolved under misère play. We thus focus on the ten remaining versions, leaving only one of them unsolved.

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1. Introduction

An *impartial combinatorial game* involves two players, say A and B, who play alternately, A having the first move, starting from some starting position G_0 [3,5]. When no confusion may arise, a game with starting position G_0 is itself denoted by G_0. A *move* from a given position G consists in selecting the next position within the finite set $O(G) = \{G_1, G_2, \ldots, G_k\}$ of the *options* of G ($O(G)$ corresponds to the set of *legal moves* from G). Such a game is *impartial* since the set $O(G)$ is the same for each player playing on G (otherwise, we speak about *partizan* games, that we do not consider in this paper). A common assumption is that the game finishes after a finite number of moves and the result is a unique winner. In *normal play*, the last player able to move (to a position G with $O(G) = \emptyset$) wins the game. Conversely, in *misère play*, the first player unable to move (from a position G with $O(G) = \emptyset$) wins the game. A fundamental property of finite impartial combinatorial games is that the *outcome* of any such game (that is which of the two players has a winning strategy) is completely determined by its starting position or, in other words, by the game itself.

The main questions we consider when analyzing an impartial combinatorial game are (i) to determine the outcome $o(G)$ of a game G and (ii) to determine which *strategy* the winner has to use. We set $o(G) = \mathcal{N}$ (resp. $o(G) = \mathcal{P}$) when the first player (resp. second player), that is the Next player (resp. the Previous player), has a winning strategy, and, in that case, G is called a $\mathcal{N}$-position (resp. $\mathcal{P}$-position).

For impartial combinatorial games under normal play, these questions can be answered using the Sprague–Grundy Theory [3,5], independently discovered by Sprague [20] and Grundy [12]: each game G is equivalent to an instance of the game of Nim on a heap of size n, for some $n \geq 0$. We then define the *Sprague–Grundy number* $\rho(G)$ of such a game G by $\rho(G) = n$. Therefore, in normal play, $o(G) = \mathcal{P}$ if and only if $\rho(G) = 0$. For any game G, the value of $\rho(G)$ can be computed as the least non negative integer which does not appear in the set $\{\rho(G_i), G_i \in O(G)\}$, denoted by *mex* $(\{\rho(G_i), G_i \in O(G)\})$ (minimum excluded value). The strategy is then the following: when playing on a game G with $o(G) = \mathcal{N}$ (which implies $\rho(G) > 0$), choose an option G_i in $O(G)$ with $\rho(G_i) = 0$ (such an option exists by definition of ρ).

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The *disjunctive sum* of two impartial combinatorial games G and H, denoted by $G + H$, is the game inductively defined by $O(G + H) = \{G_i + H, G_i \in O(G)\} \cup \{G + H_j, H_j \in O(H)\}$ (in other words, a move in $G + H$ consists in either playing on G or playing on H). The Sprague–Grundy value of $G + H$ is obtained as $\rho(G + H) = \rho(G) \oplus \rho(H)$, where $\oplus$ stands for the binary XOR operation (called *Nim-sum* in this context). The disjunctive sum of combinatorial games is the most common way of playing the so-called *compound games*, that is games made of several separated components. (The main subject of this paper is to consider other ways of playing such compound games).

Following an inspiring paper by Smith [19], Conway proposed in [5, Chapter 14] twelve ways of playing compound games, according to the rule deciding the end of the game, to the normal or misère play, and to the possibility of playing on one or more components during the same move.

Node–Kayles is an impartial combinatorial game played on undirected graphs. A move consists in choosing a vertex and deleting this vertex together with its neighbors. If we denote by $N^+(v)$ the set containing the vertex v together with its neighbors, we then have $O(G) = \{G \setminus N^+(v), v \in V(G)\}$ for every graph (or, equivalently, game) G. If G is a non-connected graph with k components, say $C_1, C_2, \ldots, C_k$, playing on G is equivalent to playing on the disjunctive sum $C_1 + C_2 + \cdots + C_k$ of its components (since a move consists in choosing a vertex in exactly one of the components of G).

Node–Kayles is a generalization of Kayles [3, Chapter 4], independently introduced by Dudeney [9] and Loyd [14]. This original game is played on a row of pins by two skilful players who could knock down either one or two adjacent pins.

Playing Node–Kayles on a path is equivalent to a particular *Take-and-Break* game introduced by Dawson [6], and now known as *Dawson's chess*, which corresponds to the octal game **0.137** (see [3, Chapter 4], [5, Chapter 11], or [10] for more details). This game has been completely solved by using Sprague–Grundy Theory (see Section 3.1).

Node–Kayles has been considered by several authors. Schaeffer [17] proved that deciding the outcome of Node–Kayles is PSPACE-complete for general graphs. In [4], Bodlaender and Kratsch proved that this question is polynomial time solvable for graphs with bounded asteroidal number. (This class contains several well-known graph classes such as cographs, cocomparability graphs or interval graphs for instance.) Bodlaender and Kratsch proposed the problem of determining the complexity of Node–Kayles on trees. To our best knowledge, this problem is still unsolved. In 1978 already, Schaeffer mentionned as an open problem to determine the complexity of Node–Kayles on *stars*, that is trees having exactly one vertex of degree at least three. Fleischer and Trippen proved in [11] that this problem is polynomial time solvable.

In this paper, we investigate Conway's twelve versions of compound games for Node–Kayles on paths. Let P_n denote the path with n vertices and, for any i and j, $P_i \cup P_j$ denote the *disjoint union* of P_i and P_j. As observed before, we have $O(P_1) = O(P_2) = P_0$, $O(P_3) = \{P_0, P_1\}$ and $O(P_n) = \{P_{n-2}, P_{n-3}\} \cup \{P_i \cup P_j, j \geq i \geq 1, i + j = n - 3\}$ (and, of course, $O(P_0) = \emptyset$). With initial position P_n, any further position will thus be made of k disjoint paths, $P_{i_1} \cup P_{i_2} \cup \cdots \cup P_{i_k}$, with $i_1 + i_2 + \cdots + i_k \leq n - 3(k-1)$ (since the only way to break a path into two separated paths is to delete three "non-extremal" vertices), which corresponds to a compound game. Different rules for playing on this set of paths will lead to (very) different situations.

This paper is organized as follows. In Section 2, we present in more details Conway's twelve versions of compound games together with the tools available for analyzing them, as introduced in Conways's book [5, Chapter 14]. We then consider these twelve versions of Node–Kayles on paths in Section 3 and discuss some possible extensions in Section 4.

2. Conway's twelve versions of compound games

We recall in this section the twelve versions of compound games introduced by Conway [5, Chapter 14]. Let G be a game made of several independent games $G_1, G_2, \ldots, G_k$ (imagine for instance that we are playing Node–Kayles on a graph G with connected components $G_1, G_2, \ldots, G_k$). As we have seen in the previous section, the game $G = G_1 + G_2 + \cdots + G_k$ is the *disjunctive compound* game obtained as the disjunctive sum of its components. In this situation, a *compound move* consists of making one legal move in exactly one of the components. By modifying this moving rule, we define a *conjunctive compound* game (a move consists in playing in *all* components simultaneously) and a *selective compound* game (a move consists in playing in any number ℓ of components, $1 \leq \ell \leq k$).

We can also distinguish two rules for ending such a compound game: the game ends either when *all* the components have ended (*long rule*) or as soon as one of the components has ended (*short rule*).

Finally, we have already seen that there are two different ways of deciding who is the winner of a game, according to the *normal* or *misère* rule.

Combining these different rules, we get twelve different versions of compound games. Considering that the long rule is more natural for selective and conjunctive compounds, while the short rule is more natural for conjunctive compound, Conway proposed the following terminology:

disjunctive compound	long ending rule, normal or misère play
diminished disjunctive compound	short ending rule, normal or misère play
conjunctive compound	short ending rule, normal or misère play
continued conjunctive compound	long ending rule, normal or misère play
selective compound	long ending rule, normal or misère play
shortened selective compound	short ending rule, normal or misère play

We now recall how one can determine the outcome of these various compound games (more details can be found in [3, Chapter 9] for conjunctive compounds and in [3, Chapter 10] for selective compounds).

Disjunctive compound. Under normal play, the main tool is the Sprague–Grundy Theory introduced in the previous section. The *normal* Sprague–Grundy number $\rho(G)$ is computed as the Nim-sum $\rho(G_1) \oplus \rho(G_2) \oplus \cdots \oplus \rho(G_k)$ (with $\rho(E) = 0$ for any ended position E) and $o(G) = \mathcal{P}$ if and only if $\rho(G) = 0$.

The situation for misère play is more complicated and the most useful features of the Sprague–Grundy Theory for normal play have no natural counterpart in misère play [3, Chapter 13]. For instance, Kayles has been solved under normal play in 1956, independently by Guy and Smith [13] and by Adams and Benson [1] (the Sprague–Grundy sequence has a period of length 12 after a preperiod of length 70) while a solution of Kayles under misère play was only given by Sibert in 1973 (and published in 1992 [18]). Three main approaches have been used in the literature to solve misère impartial games: *genus theory* [2,3], *Sibert–Conway decomposition* [18] and *misère quotient semigroup* [16]. These techniques cannot be summarized in a few lines and, since we will not use them in this paper, we refer the interested reader to the corresponding references (see also [15]).

Diminished disjunctive compound. Under both normal and misère play, we use the *foreclosed Sprague–Grundy number*, denoted by $F^+(G)$ (resp. $F^-(G)$) in normal (resp. misère) play, and defined as follows. Let us declare a position to be *illegal* if the game has just ended or can be ended in a single *winning* move (note here that winning moves are not the same under normal and misère play). If a position is illegal, its foreclosed Sprague–Grundy number is *undefined*, otherwise its foreclosed Sprague–Grundy number is simply its usual Sprague–Grundy number. The foreclosed Sprague–Grundy number of G is then defined if and only if those of $G_1, G_2, \ldots, G_k$ are all defined and, in that case, is computed as their Nim-sum. Now, the outcome of G is $\mathcal{P}$ if its foreclosed Sprague–Grundy number is 0 or some component has outcome $\mathcal{P}$ but undefined foreclosed Sprague–Grundy number.

Conjunctive compound. In that case, the game ends as soon as one of the components ends. Therefore, "small" components (that can be ended in a small number of moves) must be played carefully: a player has interest in winning quickly on winning components and postponing defeat as long as possible on losing ones. Considering this strategy, a game lasts for a number of moves than can be easily computed. This number of moves is called the *remoteness* of the game. Under normal play, the remoteness $R^+(G)$ is computed as follows: (i) if G has an option of even remoteness, $R^+(G)$ is one more the *minimal even* remoteness of any option of G, (ii) if not, the remoteness of G is one more than the *maximal odd* remoteness of any option of G. Moreover, the remoteness of an ended position is 0. A game G will then have outcome $\mathcal{P}$ if and only if $R^+(G)$ is *even* (the second player will play the last move).

Under misère play, the remoteness $R^-(G)$ is computed similarly, except that we interchange the words *odd* and *even* in the above rules. A game G will now have outcome $\mathcal{P}$ if and only if $R^-(G)$ is *odd*.

Continued conjunctive compound. Now, the best strategy is to win slowly on winning components and to lose quickly on losing components. The number of moves of a game under such a strategy is called the *suspense* number of a game, denoted either $S^+(G)$ or $S^-(G)$. The rules for computing this number in normal play are the following: (i) if G has an option of even suspense number, $S^+(G)$ is one more the *maximal even* suspense number of any option of G, (ii) if not, the suspense number of G is one more than the *minimal odd* suspense number of any option of G. Moreover, the suspense number of an ended position is 0. As before, for computing the suspense number under misère play, we interchange the words *odd* and *even* in the above rules.

A game G will have outcome $\mathcal{P}$ under normal play (resp. misère play) if and only if $S^+(G)$ is *odd* (resp. $S^-(G)$ is *even*).

Selective compound. The strategy here is quite obvious: to win the game under normal play, a player has to play on all winning components. Therefore, the outcome of G is $\mathcal{P}$ if and only if the outcomes of $G_1, G_2, \ldots, G_k$ are all $\mathcal{P}$. Under misère play, the winning strategy is the same, except when all the remaining components are losing. If there is only one such component, the player will lose the game. Otherwise, he can win the game by playing on all but one of these losing components. Therefore, unless all but one of the components of G have ended, the outcome of G is the same as in normal play. Otherwise, its outcome is $\mathcal{P}$ if and only if the outcome of the only remaining component is $\mathcal{P}$.

Shortened selective compound. Again, to win the game, a player has to play on all winning components. But when all components are losing, the player will lose the game (even under misère play, since he will necessary reach some configuration in which he cannot play on all but one component without ending one of these components). Hence, the rule here is even simpler than the previous one: under both normal play and misère play, the outcome of G is $\mathcal{P}$ if and only if the outcomes of $G_1, G_2, \ldots, G_k$ are all $\mathcal{P}$. Note that under normal play, all positions have the same outcome in selective compound and in shortened selective compound.

3. Compound Node–Kayles on paths

Recall that for every path P_n of order $n \geq 3$, the set of options of P_n in Node–Kayles is given by

$$O(P_n) = \{P_{n-2}, P_{n-3}\} \cup \{P_i \cup P_j, j \geq i \geq 1, i + j = n - 3\}. \tag{1}$$

In this section, we recall what is known for the usual disjunctive compound Node–Kayles and analyze the ten other versions of compound Node–Kayles introduced in the previous section. In each case, we will first try to characterize the set $\mathcal{L} = \{i \in \mathbb{N}, o(P_i) = \mathcal{P}\}$ of *losing paths* and then consider the complexity of determining the outcome of any position (disjoint union of paths). Finally, we will study the complexity of the *winning strategy* which consists in finding, for any position with outcome $\mathcal{N}$, an option with outcome $\mathcal{P}$.

3.1. Disjunctive compound

Disjunctive composition is the most common way of considering compound games. We recall here what is known (and unknown) for disjunctive compound Node–Kayles on paths.

NORMAL PLAY

This game has been solved using the Sprague–Grundy Theory [3, Chapter 4]. The sequence $\rho(P_0)\rho(P_1)\rho(P_2)\dots\rho(P_{n-1})\rho(P_n)$ ... is called the *Sprague–Grundy sequence* of Node–Kayles. It turns out that this sequence is *periodic*, with period 34, after a preperiod of size 51. We then have:

$$\mathcal{L} = \{0, 4, 8, 14, 19, 24, 28, 34, 38, 42\} \cup \{54 + 34i, 58 + 34i, 62 + 34i, 72 + 34i, 76 + 34i, i \geq 0\}.$$

Determining the outcome of a path can thus be done in constant time. For a disjoint union of paths, we need to compute the Nim-sum of the Sprague–Grundy numbers of its components, which can be done in linear time. Let now $G = P_{i_1} \cup P_{i_2} \cup \dots \cup P_{i_\ell}$ be any $\mathcal{N}$-position and assume $\rho(P_{i_1}) \leq \rho(P_{i_2}) \leq \dots \leq \rho(P_{i_\ell})$. Let $i_j \in \{1, 2, \dots, \ell\}$ be the largest index such that (i) the number of components with Sprague–Grundy number $\rho(P_{i_j})$ is odd and (ii) for every $r > \rho(P_{i_j})$, the number of components with Sprague–Grundy number r is even. Thanks to the properties of the operator $\oplus$, we have $\rho(P_{i_j}) > \oplus_{k\in\{1,\dots,\ell\}\setminus\{j\}}\{P_{i_k}\}$. Therefore, by choosing an option H of P_{i_j} with $\rho(H) = \oplus_{k\in\{1,\dots,\ell\}\setminus\{j\}}\{P_{i_k}\}$, we get an option of G with Sprague–Grundy number 0. Such a "winning move" can thus be found in linear time.

MISÈRE PLAY

On the other hand, the problem is still open for Node–Kayles on paths under misère play [3, Chapter 13].

3.2. Diminished disjunctive compound

Recall that in this version of disjunctive compound, the game ends as soon as one of the components has ended.

We shall compute the foreclosed Sprague–Grundy number of paths. Under normal play, we shall prove that the corresponding sequence is periodic and that the set of losing positions is finite. On the other hand, we are unable to characterize the set of losing positions under misère play.

NORMAL PLAY

Recall that the foreclosed Sprague–Grundy number of illegal positions (that is ended positions or positions that can be won in one move) is undefined. Hence, we will note $F^+(P_0) = F^+(P_1) = F^+(P_2) = F^+(P_3) = *$. The foreclosed Sprague–Grundy number of other positions is computed as the usual Sprague–Grundy number, using the *mex* operator. Hence, from (1), we get for every $n \geq 4$:

$$F^+(P_n) = mex(\{F^+(P_{n-2}), F^+(P_{n-3})\} \cup \{F^+(P_i \cup P_j), j \geq i \geq 1, i + j = n - 3\}),$$

with $F^+(P_i \cup P_j) = F^+(P_i) \oplus F^+(P_j)$.

Using that formula, and the fact that $x \oplus * = * \oplus x = *$ for every x, we can compute the *foreclosed Sprague–Grundy sequence*, given as $F^+(P_0)F^+(P_1)F^+(P_2)\dots F^+(P_{n-1})F^+(P_n)\dots$.

In [13], Guy and Smith proved a useful *periodicity theorem* for octal games (recall that Node–Kayles on paths is the octal game **0.137**), which allows to ensure the periodicity of the usual Sprague–Grundy sequence whenever two occurrences of the period have been computed. This theorem can easily be extended to the foreclosed Sprague–Grundy sequence in our context and we have:

Theorem 1. *Suppose that for some $p > 0$ and $q > 0$ we have*

$$F^+(P_{n+p}) = F^+(P_n) \quad \textit{for every } n \textit{ with } q \leq n \leq 2q + p + 2.$$

Then

$$F^+(P_{n+p}) = F^+(P_n) \quad \textit{for every } n \geq q.$$

Proof. We proceed by induction on n. If $n \leq 2q + p + 2$, the equality holds. Assume now that $n \geq 2q + p + 3$. Recall that

$$O(P_{n+p}) = \{P_{n+p-2}, P_{n+p-3}\} \cup \{P_i \cup P_j, j \geq i \geq 1, i + j = n + p - 3\}.$$

Hence, we have

$$F^+(P_{n+p}) = mex\,(\{F^+(P_{n+p-2}), F^+(P_{n+p-3})\} \cup \{F^+(P_i) \oplus F^+(P_j), j \geq i \geq 1, i + j = n + p - 3\}).$$

Since $n - 2 < n$ and $n - 3 < n$, we get by induction hypothesis $F^+(P_{n-2}) = F^+(P_{n+p-2})$ and $F^+(P_{n-3}) = F^+(P_{n+p-3})$. Similarly, since $q + p \leq \left\lfloor \frac{n+p-3}{2} \right\rfloor - p \leq j - p < n - 3$, we get $F^+(P_{j-p}) = F^+(P_j)$ and thus $F^+(P_{n+p}) = F^+(P_n)$. ■

Table 1
The foreclosed Sprague–Grundy sequence under normal play.

n	$F^+(P_n)$				
0–49	∗∗∗∗001120	0112031122	3112334105	3415534255	3225532255
50–99	0225042253	4423344253	4455341553	4285322853	4285442804
100–149	4283442234	4253345533	1253322533	2253422534	2253422334
150–199	2233425334	4533425532	2553425544	2554425344	2234425334
200–249	5533125342	2533225342	2534225342	2334223342	5334453342
250–299	5532255342	5344255442	5344253442	5334553342	5342253322
300–349	5342253422	5342233422	3342533425	3342553225	...

Table 2
Statistics on the misère foreclosed Sprague–Grundy sequence.

n	*NbZ*	*Max*	*Mean*	*Deviation*	*FreqV*	*%FreqV*	*MaxZ*	*PosMax*
10	3	4	1.4	1.08	0	30	8	9
10^2	8	11	4.23	2.4114	2	15	98	61
10^3	11	43	13.629	7.537448	16	6.8	148	999
10^4	12	163	58.5556	30.621093	33	2.73	1 526	9 977
10^5	13	907	275.95915	177.355129	128	0.795	12 758	94 680
10^6	16	4600	1357.37834	780.786047	4096	0.256	235 086	979 501

By computing the foreclosed Sprague–Grundy sequence, we find a finite number of losing positions and, thanks to Theorem 1, we get that this sequence is periodic, with period 84, after a preperiod of length 245 (see Table 1, the period is underlined).

Hence we have:

Corollary 2. $\mathcal{L} = \{0, 4, 5, 9, 10, 14, 28, 50, 54, 98\}$.

Determining the outcome of any disjoint union of paths or finding a winning move from any $\mathcal{N}$-position can be done in linear time, using the same technique as in the previous subsection.

MISÈRE PLAY

In that case, we have $F^-(P_0) = *, F^-(P_1) = F^-(P_2) = 0, F^-(P_3) = F^-(P_4) = 1$ and, for every $n \geq 5$:

$$F^-(P_n) = mex(\{F^-(P_{n-2}), F^-(P_{n-3})\} \cup \{F^-(P_i \cup P_j), j \geq i \geq 1, i + j = n - 3\}),$$

with $F^-(P_i \cup P_j) = F^-(P_i) \oplus F^-(P_j)$.

Using that formula, and the fact that $x \oplus * = * \oplus x = x$ for every x, we have computed the misère foreclosed Sprague–Grundy number of paths up to $n = 10^6$, without being able to discover any period. Some statistics on the corresponding sequence are summarized in Table 2, where:

- n is the upper bound of the considered interval $I = [1, n]$,
- *NbZ* is the number of paths in I with foreclosed Sprague–Grundy number 0,
- *Max* is the maximal foreclosed Sprague–Grundy number on I,
- *Mean* is the mean of the foreclosed Sprague–Grundy numbers on I,
- *Deviation* is the standard deviation of the foreclosed Sprague–Grundy numbers on I,
- *FreqV* is the most frequently encountered foreclosed Sprague–Grundy number on I,
- *%FreqV* is the percentage of apparition of FreqV on I,
- *MaxZ* is the largest index of a path in I with foreclosed Sprague–Grundy number 0,
- *PosMax* is the index of the largest foreclosed Sprague–Grundy number on I.

Note that the growth of the mean of the foreclosed Sprague–Grundy numbers is approximately logarithmic, which shows that even an *arithmetic period* [3, Chapter 4] cannot be expected on the considered interval. Observe also the intriguing fact that the most frequently encountered foreclosed Sprague–Grundy number on the considered intervals is always of the form 2^k or $2^k + 1$ (which seems to be true for every interval of type $[1, n]$).

In fact, it appears that this foreclosed Sprague–Grundy sequence is related to the Sprague–Grundy sequence of the octal game **0.13337** under normal play by the relation $F^-(P_n) = \rho_{0.13337}(H_{n-2})$, for every n, $n \geq 2$, where H_{n-2} denotes the heap of size $n - 2$. It is easy to check that this relation holds for paths P_2, P_3 and P_4. Now, let us write the options of P_n, $n \geq 5$, which corresponds to H_{n-2}, as follows: (i) P_{n-2}, which corresponds to H_{n-4}, (ii) P_{n-3}, which corresponds to H_{n-5}, (iii) $P_{n-4} \cup P_1 \simeq P_{n-4}$ (since P_1 is losing in one move), which corresponds to H_{n-6}, (iv) $P_{n-5} \cup P_2 \simeq P_{n-5}$ (since P_2 is losing in one move), which corresponds to H_{n-7}, and (v) $\{P_{n-5-j} \cup P_{2+j}, 1 \leq j \leq n-8\}$, which corresponds to $\{H_{n-7-j} \cup H_j, 1 \leq j \leq n-8\}$. Therefore, in terms of heaps, we get: (i) we can remove 2 elements in a heap, leaving 1 or 0 heaps, (ii) we can remove 3 elements in a heap, leaving 1 or 0 heaps, (iii) we can remove 4 elements in a heap, leaving 1 or 0 heaps, (iv) we can remove 5 elements in a heap, leaving 1 or 0 heaps, and (v) we can remove 5 elements in a heap, leaving 2 heaps. Since we can remove 1 element only from a heap of size one, we get exactly the rules of the octal game **0.13337**.

Up to now, it is not known whether the Sprague–Grundy sequence of this octal game is periodic or not [10].

3.3. Conjunctive compound

Recall that if $G = P_{i_1} \cup P_{i_2} \cup \cdots \cup P_{i_k}$ is a graph made of k disjoint paths, we then have $O(G) = \{G_{i_1}, G_{i_2}, \ldots, G_{i_k}\}$ with $G_{i_j} \in O(P_{i_j})$ for every j, $1 \le j \le k$.

This version of our game is easy to solve. In both normal and misère play, it can be checked that there is only a finite number of (small) losing paths. Therefore, we can easily determine the remoteness $R^+(P)$ (resp. $R^-(P)$) of any path P.

NORMAL PLAY

Recall that if $O(G) = \{G_1, G_2, \ldots, G_k\}$, the normal remoteness $R^+(G)$ of G is given by:

$$\begin{cases} R^+(G) = 0 & \text{if } O(G) = \emptyset \\ R^+(G) = 1 + \min_{even}\{R^+(G_1), R^+(G_2), \ldots, R^+(G_k)\} & \text{if } \exists j \in [1,k] \text{ s.t. } R^+(G_j) \text{ is even,} \\ R^+(G) = 1 + \max_{odd}\{R^+(G_1), R^+(G_2), \ldots, R^+(G_k)\} & \text{otherwise.} \end{cases}$$

We prove the following:

Theorem 3. *The normal remoteness R^+ of paths satisfies:*

1. $R^+(P_1) = R^+(P_2) = R^+(P_3) = 1$,
2. $R^+(P_4) = R^+(P_5) = 2$,
3. $R^+(P_6) = R^+(P_7) = R^+(P_8) = 3$,
4. $R^+(P_9) = R^+(P_{10}) = 4$,
5. $R^+(P_n) = 3$, *for every* $n \ge 11$.

Proof. The first four points can easily be checked. Let now $n \ge 11$. Observe that $P_{n-7} \cup P_4 \in O(P_n)$. By induction on n, and thanks to the remoteness of small paths, we have $R^+(P_{n-7} \cup P_4) = \min_{even}\{R^+(P_{n-7}), R^+(P_4)\} = \min_{even}\{R^+(P_{n-7}), 2\} = 2$ (since $n - 7 \ge 4$ we have $R^+(P_{n-7}) \ge 2$). Therefore, we get $R^+(P_n) = 1 + 2 = 3$. ∎

We thus obtain:

Corollary 4. $\mathcal{L} = \{0, 4, 5, 9, 10\}$.

Let now $G = P_{i_1} \cup P_{i_2} \cup \cdots \cup P_{i_\ell}$ be any disjoint union of paths and assume $i_1 \le i_2 \le \cdots \le i_\ell$. Clearly, the outcome of G is $\mathcal{P}$ if and only if $i_1 \in \{4, 5, 9, 10\}$, which can be decided in linear time. Suppose now that G is a $\mathcal{N}$-position. If $i_1 \le 3$, one can win in one move. If $6 \le i_1 \le 8$, one can play in such a way that P_{i_1} gives a path of length 4 or 5 and any other component gives a path of length at least 4. Finally, if $i_1 \ge 11$, one can play in such a way that each component of order p gives rise to $P_4 \cup P_{p-7}$. Finding such a winning move can thus be done in linear time.

MISÈRE PLAY

Similarly, if $O(G) = \{G_1, G_2, \ldots, G_k\}$, the misère remoteness $R^-(G)$ of G is given by:

$$\begin{cases} R^-(G) = 0 & \text{if } O(G) = \emptyset \\ R^-(G) = 1 + \min_{odd}\{R^-(G_1), R^-(G_2), \ldots, R^-(G_k)\} & \text{if } \exists j \in [1,k] \text{ s.t. } R^-(G_j) \text{ is odd,} \\ R^-(G) = 1 + \max_{even}\{R^-(G_1), R^-(G_2), \ldots, R^-(G_k)\} & \text{otherwise.} \end{cases}$$

We prove the following:

Theorem 5. *The misère remoteness R^- of paths satisfies:*

1. $R^-(P_1) = R^-(P_2) = 1$,
2. $R^-(P_n) = 2$ *for every* $n \ge 2$.

Proof. The first point is obvious. Similarly, we can easily check that $R^-(P_3) = R^-(P_4) = 2$. Let now $n \ge 5$. Observe that $P_1 \cup P_{n-4} \in O(P_n)$. By induction on n, and thanks to the remoteness of small paths, we have $R^-(P_1 \cup P_{n-4}) = \min_{odd}\{R^-(P_1), R^-(P_{n-4})\} = \min_{odd}\{1, R^-(P_{n-4})\} = 1$ (since $n - 4 > 0$). Thus, we get $R^-(P_n) = 1 + 1 = 2$. ∎

And therefore:

Corollary 6. $\mathcal{L} = \{1, 2\}$.

Hence, if G is a disjoint union of paths, the outcome of G is $\mathcal{P}$ if and only if the shortest component in G has order 1 or 2, which can be decided in linear time. If G is a $\mathcal{N}$-position, a winning move can be obtained, again in linear time, by playing for instance in such a way that each component gives rise to a path of order 1.

3.4. Continued conjunctive compound

In this section, we will compute the suspense number $S^+(P_n)$ under normal play (resp. $S^-(P_n)$ under misère play) for each path P_n. Note that these two functions are *additive* [5, p. 177] and we have $S^+(P_i \cup P_j) = \max\{S^+(P_i), S^+(P_j)\}$ (resp. $S^-(P_i \cup P_j) = \max\{S^-(P_i), S^-(P_j)\}$) for every two paths P_i and P_j.

NORMAL PLAY

Recall that if $O(G) = \{G_1, G_2, \dots, G_k\}$, the normal suspense number $S^+(G)$ of G is given by:

$$\begin{cases} S^+(G) = 0 & \text{if } O(G) = \emptyset \\ S^+(G) = 1 + \max_{even}\{S^+(G_1), S^+(G_2), \dots, S^+(G_k)\} & \text{if } \exists\, j \in [1, k] \text{ s.t. } S^+(G_j) \text{ is even,} \\ S^+(G) = 1 + \min_{odd}\{S^+(G_1), S^+(G_2), \dots, S^+(G_k)\} & \text{otherwise.} \end{cases}$$

Then we prove the following:

Theorem 7. *The normal suspense number S^+ of paths is an increasing function and satisfies for every $n \geq 0$:*

1. $S^+(P_{5(2^n-1)}) = 2n$,
2. $S^+(P_k) = 2n + 1$, *for every* $k \in [5(2^n - 1) + 1; 5(2^{n+1} - 1) - 2]$,
3. $S^+(P_{5(2^{n+1}-1)-1}) = 2n + 2$.

Proof. We proceed by induction on n. For $n = 0$, we can easily check that $S^+(P_0) = 0$, $S^+(P_1) = S^+(P_2) = S^+(P_3) = 1$ and that $S^+(P_4) = S^+(P_5) = 2$.

Assume now that the result holds for every p, $0 \leq p < n$ and let $k \in [5(2^n - 1); 5(2^{n+1} - 1) - 1]$. We consider three cases.

1. $k = 5(2^n - 1)$.
   Since $\lceil \frac{k-3}{2} \rceil = 5.2^{n-1} - 4 > 5(2^{n-1} - 1)$, using induction hypothesis, we get $S^+(P_j) = 2n - 1$ for every j, $\lceil \frac{k-3}{2} \rceil \leq j \leq k - 4$, and thus $\max(S^+(P_i), S^+(P_j)) = 2n - 1$ for every i, j, $j \geq i \geq 1$, $i + j = k - 3$. Therefore, since $S^+(P_{k-2}) = S^+(P_{k-3}) = 2n - 1$, P_k has no option with even suspense number and thus:

$$\begin{aligned} S^+(P_k) &= 1 + \min_{odd} (\{S^+(P_{k-2}), S^+(P_{k-3})\} \cup \{\max(S^+(P_i), S^+(P_j)), j \geq i \geq 1, i + j = k - 3\}) \\ &= 1 + \min_{odd} (\{2n - 1\} \cup \{2n - 1\}) \\ &= 2n. \end{aligned}$$

2. $k \in [5(2^n - 1) + 1; 5(2^{n+1} - 1) - 2]$.
   Note first that for every such k, $P_{5(2^n-1)} \cup P_{k-3-5(2^n-1)}$ is an option of P_k with even suspense number, since $k - 3 - 5(2^n - 1) \leq 5(2^{n+1} - 1) - 2 - 3 - 5(2^n - 1) = 5(2^n - 1) - 10 < 5(2^n - 1)$ and, thus, $\max(S^+(P_{5(2^n-1)}), S^+(P_{k-3-5(2^n-1)})) = S^+(P_{5(2^n-1)}) = 2n$ (thanks to the induction hypothesis and Case 1 above). Therefore:

$$S^+(P_k) = 1 + \max_{even} (\{S^+(P_{k-2}), S^+(P_{k-3})\} \cup \{\max(S^+(P_i), S^+(P_j)), j \geq i \geq 1, i + j = k - 3\}).$$

   We now proceed by induction on k. We have

$$\begin{aligned} S^+(P_{5(2^n-1)+1}) &= 1 + \max_{even} (\{S^+(P_{5(2^n-1)-1}), S^+(P_{5(2^n-1)-2})\} \\ &\qquad \cup \{\max(S^+(P_i), S^+(P_j)), j \geq i \geq 1, i + j = 5(2^n - 1) - 2\}) \\ &= 1 + \max_{even} (\{2n, 2n - 1\} \cup \{2n - 1, 2n\}) \\ &= 2n + 1 \end{aligned}$$

   and, similarly, $S^+(P_{5(2^n-1)+2}) = S^+(P_{5(2^n-1)+3}) = 2n + 1$. Then, using induction hypothesis, we get

$$\begin{aligned} S^+(P_k) &= 1 + \max_{even} (\{S^+(P_{k-2}), S^+(P_{k-3})\} \cup \{\max(S^+(P_i), S^+(P_j)), j \geq i \geq 1, i + j = k - 3\}) \\ &= 1 + \max_{even} (\{2n - 1\} \cup \{2n - 1, 2n\}) \\ &= 2n + 1. \end{aligned}$$

3. $k = 5(2^{n+1} - 1) - 1$.
Thanks to Case 2 above, we have $S^+(P_{k-2}) = S^+(P_{k-3}) = 2n + 1$. Moreover, since $\lceil \frac{k-3}{2} \rceil = 5.2^n - 3 > 5(2^n - 1)$, using induction hypothesis and Case 2 above, we get $S^+(P_j) = 2n + 1$ for every j, $\lceil \frac{k-3}{2} \rceil \le j \le k - 4$, and thus $\max(S^+(P_i), S^+(P_j)) = 2n + 1$ for every $i, j, j \ge i \ge 1, i + j = k - 3$. Hence, P_k has no option with even suspense number and thus:

$$\begin{aligned} S^+(P_k) &= 1 + \min_{odd} (\{S^+(P_{k-2}), S^+(P_{k-3})\} \cup \{\max(S^+(P_i), S^+(P_j)), j \ge i \ge 1, i + j = k - 3\}) \\ &= 1 + \min_{odd} (\{2n + 1\} \cup \{2n + 1\}) \\ &= 2n + 2. \quad \blacksquare \end{aligned}$$

And therefore:

Corollary 8. $\mathcal{L} = \{5(2^n - 1), n \ge 0\} \cup \{5(2^{n+1} - 1) - 1, n \ge 0\}$.

Note that Theorem 7 shows that the normal suspense sequence of paths has a *geometric period* with geometric ratio 2.

Let $G = P_{i_1} \cup P_{i_2} \cup \cdots \cup P_{i_\ell}$ be a disjoint union of paths and assume $i_1 \le i_2 \le \cdots \le i_\ell$. The position G has outcome $\mathcal{P}$ if and only if $i_\ell \in \mathcal{L}$, which can be decided in linear time. Now, if G is a $\mathcal{N}$-position, let r be the greatest integer such that $t = 5(2^r - 1) < i_\ell$. A winning move can be obtained by playing in such a way that each component of order $p > t$ gives rise to P_{t-1} (if $p = t + 1$), to P_t (if $p = t + 2$) or to $P_t \cup P_{p-t-3}$ (otherwise). Such a move clearly leads to a $\mathcal{P}$-position and can be found in linear time.

Misère play

Recall that if $O(G) = \{G_1, G_2, \ldots, G_k\}$, the misère suspense number $S^-(G)$ of G is given by:

$$\begin{cases} S^-(G) = 0 & \text{if } O(G) = \emptyset \\ S^-(G) = 1 + \max_{odd}\{S^-(G_1), S^-(G_2), \ldots, S^-(G_k)\} & \text{if } \exists j \in [1, k] \text{ s.t. } S^-(G_j) \text{ is odd,} \\ S^-(G) = 1 + \min_{even}\{S^-(G_1), S^-(G_2), \ldots, S^-(G_k)\} & \text{otherwise.} \end{cases}$$

Then we prove the following:

Theorem 9. *The misère suspense number S^- of paths is an increasing function and satisfies for every $n \ge 0$:*

1. $S^-(P_{7.2^n-6)}) = 2n + 1$,
2. $S^-(P_{7.2^n-5)}) = 2n + 1$,
3. $S^-(P_k) = 2n + 2$ *for every k, $7.2^n - 4 \le k \le 7.2^{n+1} - 7$.*

Proof. The proof is very similar to that of Theorem 7 and we thus omit it. ■

And therefore:

Corollary 10. $\mathcal{L} = \{7.2^n - 6, n \ge 0\} \cup \{7.2^n - 5, n \ge 0\}$.

As in normal play, determining the outcome of a disjoint union of paths or finding a winning move from a $\mathcal{N}$-position can be done in linear time.

3.5. Selective compound

With selective compound, each player may play on any number of components (at least one). As seen in Section 2, it is enough to know the outcome of each component to decide the outcome of their (disjoint) union. Therefore, we shall simply compute a boolean function σ, defined by $\sigma(P) = 1$ (resp. $\sigma(P) = 0$) if and only if $o(P) = \mathcal{N}$ (resp. $o(P) = \mathcal{P}$) for every path P.

Then we have:

$$\begin{cases} \sigma(G) = 0 \text{ (normal) or } 1 \text{ (misère)} & \text{if } O(G) = \emptyset, \\ \sigma(G) = 1 - \min\{\sigma(G'), G' \in O(G)\} & \text{otherwise.} \end{cases}$$

The function σ is additive, under both normal and misère play, and we have $\sigma(P_i \cup P_j) = \sigma(P_i) \vee \sigma(P_j)$ (boolean disjunction) for any two non-empty paths P_i and P_j.

We shall prove that the sequence $\sigma(P_0)\sigma(P_1)\sigma(P_2)\ldots\sigma(P_{n-1})\sigma(P_n)\ldots$ has period 5 under normal play and period 7 under misère play.

Normal play

We prove the following:

Theorem 11. *For every $n \geq 0$, we have:*

1. $\sigma(P_{5n}) = \sigma(P_{5n+4}) = 0$,
2. $\sigma(P_{5n+1}) = \sigma(P_{5n+2}) = \sigma(P_{5n+3}) = 1$.

Proof. We proceed by induction on n. For $n = 0$, the result clearly holds. Assume now that the result holds up to $n - 1$. Then we have:

1. Recall that $O(P_{5n}) = \{P_{5n-2}, P_{5n-3}\} \cup \{P_i \cup P_j, j \geq i \geq 1, i + j = 5n - 3\}$. Hence:

$$\begin{aligned}\sigma(P_{5n}) &= 1 - \min\{\sigma(P'), P' \in O(P_{5n})\} \\ &= 1 - \min\{1, 1, \min_{j \geq i \geq 1, i+j=5n-3}\{\sigma(P_i) \vee \sigma(P_j)\}\} \\ &= 1 - \min\{1, 1, \min_{j=5n-8,\dots,5n-4}\{\sigma(P_{5n-3-j}) \vee \sigma(P_j)\}\} \\ &= 1 - \min\{1, 1, \min\{0 \vee 1, 0 \vee 1, 1 \vee 0, 1 \vee 0, 1 \vee 1\}\} \\ &= 1 - 1 \\ &= 0.\end{aligned}$$

   We can check in a similar way that $\sigma(P_{5n+4}) = 0$.
2. Since $\sigma(P_{5n}) = \sigma(P_{5n-1}) = 0$, $P_{5n-1} \in O(P_{5n+1})$, $P_{5n} \in O(P_{5n+2})$ and $P_{5n} \in O(P_{5n+3})$, we have $\sigma(P_{5n+1}) = \sigma(P_{5n+2}) = \sigma(P_{5n+3}) = 1$. ■

And therefore:

Corollary 12. $\mathcal{L} = \{5n, n \geq 0\} \cup \{5n + 4, n \geq 0\}$.

Now, the outcome of a disjoint union of paths if $\mathcal{P}$ if and only if each component P is such that $\sigma(P) = 0$, which can be decided in linear time. A winning move from a $\mathcal{N}$-position can be obtained by playing on each component P with $\sigma(P) = 1$ in such a way that this component gives rise to a path P' with $\sigma(P') = 0$, as explained in the proof of Theorem 11. Here again, such a move can be found in linear time.

Misère play

We prove the following:

Theorem 13. *For every $n \geq 0$, we have:*

1. $\sigma(P_{7n+1}) = \sigma(P_{7n+2}) = 0$,
2. $\sigma(P_{7n+a}) = 1$, *for every a, $3 \leq a \leq 7$.*

Proof. We proceed by induction on n. For $n = 0$, the result clearly holds. Assume now that the result holds up to $n - 1$. Then we have:

1. Recall that $O(P_{7n+1}) = \{P_{7n-1}, P_{7n-2}\} \cup \{P_i \cup P_j, j \geq i \geq 1, i + j = 7n - 2\}$. Hence:

$$\begin{aligned}\sigma(P_{7n+1}) &= 1 - \min\{\sigma(P'), P' \in O(P_{7n+1})\} \\ &= 1 - \min\{1, 1, \min_{j \geq i \geq 1, i+j=7n-2}\{\sigma(P_i) \vee \sigma(P_j)\}\} \\ &= 1 - \min\{1, 1, \min_{j=7n-9,\dots,7n-3}\{\sigma(P_{7n-2-j}) \vee \sigma(P_j)\}\} \\ &= 1 - \min\{1, 1, \min\{1 \vee 1, 1 \vee 1, 1 \vee 1, 0 \vee 1, 0 \vee 1, 1 \vee 0, 1 \vee 0\}\} \\ &= 1 - 1 \\ &= 0.\end{aligned}$$

   We can check in a similar way that $\sigma(P_{7n+2}) = 0$.
2. Since $\sigma(P_{7n+1}) = \sigma(P_{7n+2}) = 0$, $P_{7n+1} \in O(P_{7n+3})$, $P_{7n+1} \in O(P_{7n+4})$ and $P_{7n+2} \in O(P_{7n+5})$, we have $\sigma(P_{7n+3}) = \sigma(P_{7n+4}) = \sigma(P_{7n+5}) = 1$.

   Now, observe that $P_{7n+2} \cup P_1 \in O(P_{7n+6})$. Since $\sigma(P_{7n+2}) = \sigma(P_1) = 0$, we have $\sigma(P_{7n+2} \cup P_1) = \sigma(P_{7n+2}) \vee \sigma(P_1) = 0 \vee 0 = 0$, which implies $\sigma(P_{7n+6}) = 1$.

   Similarly, since $P_{7n+2} \cup P_2 \in O(P_{7n+7})$, we get $\sigma(P_{7n+7}) = 1$. ■

And therefore:

Corollary 14. $\mathcal{L} = \{7n + 1, n \geq 0\} \cup \{7n + 2, n \geq 0\}$.

As in normal play, determining the outcome of a disjoint union of paths or finding a winning move from a $\mathcal{N}$-position can be done in linear time.

3.6. Shortened selective compound

We will use the same boolean function σ as in the previous subsection. In both normal and misère play, we prove that the corresponding sequence is periodic with period 5.

NORMAL PLAY

As we have noted in Section 2 all positions have the same outcome as in the selective compound. Therefore, we get from the previous subsection:

Corollary 15. $\mathcal{L} = \{5n, n \geq 0\} \cup \{5n + 4, n \geq 0\}$.

The outcome of disjoint union of paths and winning moves are also similar.

MISÈRE PLAY

On the other hand, selective compound and a shortened selective compound behave differently under misère play. For instance, if G is made of k isolated vertices ($G = P_1 \cup P_1 \cup \cdots \cup P_1$), with $k \geq 2$, then G is a $\mathcal{P}$-position in selective compound and a $\mathcal{N}$-position in a shortened selective compound.

As observed in [5, Chapter 14] the function σ is not additive under misère play. For instance, $\sigma(P_1) = 0$ while $\sigma(P_1 \cup \cdots \cup P_1) = 1$, and $\sigma(P_4) = \sigma(P_5) = \sigma(P_8) = 1$ while $\sigma(P_5 \cup P_4) = 0$ and $\sigma(P_8 \cup P_4) = 1$.

We first prove the following lemma which allows us to determine $\sigma(G)$ for every position G made of at least two components (paths).

Lemma 16. *Let $G = P_{i_1} \cup P_{i_2} \cup \cdots \cup P_{i_\ell}$, with $\ell \geq 2$, and let $\lambda_i(G)$, $0 \leq i \leq 4$, be the number of paths in G whose order is congruent to i, modulo 5. Then,*

$$\sigma(G) = 0 \text{ if and only if } \lambda_1(G) + \lambda_2(G) + \lambda_3(G) = 0.$$

Proof. We proceed by induction on the order n of G. The result clearly holds for $n = 2$ (in that case, $G = P_1 \cup P_1$ and $\sigma(G) = 1$). Suppose now that the result holds for every $p < n$.

Recall that $O(P_k) = \{P_{k-2}, P_{k-3}\} \cup \{P_i \cup P_j, j \geq i \geq 1, i + j = k - 3\}$ for every path with k vertices. Hence, if $k \equiv 0$ or 4 (mod 5), then every option of P_k contains a path with order $m \equiv 1, 2$ or 3 (mod 5). Therefore, if $\lambda_1(G) + \lambda_2(G) + \lambda_3(G) = 0$ then for every option G' of G we get $\lambda_1(G') + \lambda_2(G') + \lambda_3(G') \neq 0$. By induction hypothesis, that means $\sigma(G') = 1$ for every option G' of G, and thus $\sigma(G) = 0$.

Suppose now that $\lambda_1(G) + \lambda_2(G) + \lambda_3(G) > 0$. Note that every path P_k with $k \equiv 1, 2$ or 3 (mod 5), has either an empty option (if $k \leq 3$) or an option $P_{k'}$ with $k' \equiv 0$ or 4 (mod 5) (by deleting 2 or 3 vertices on one extremity of P_k). Therefore, by choosing such a move for every path of G of order $k \equiv 1, 2$ or 3 (mod 5), we get an option G' of G with $\sigma(G') = 0$ (by induction hypothesis) and thus $\sigma(G) = 1$. ■

We can now prove the following:

Theorem 17. *The boolean function σ satisfies:*

1. $\sigma(P_1) = \sigma(P_2) = \sigma(P_8) = \sigma(P_9) = 0$,
2. $\sigma(P_i) = 1$ *for every* $i \in \{3, 4, 5, 6, 7, 10, 11, 12, 13, 14\}$,
3. $\sigma(P_{5n}) = \sigma(P_{5n+4}) = 0$ *for every* $n \geq 3$,
4. $\sigma(P_{5n+1}) = \sigma(P_{5n+2}) = \sigma(P_{5n+3}) = 1$ *for every* $n \geq 3$.

Proof. The first values can easily by checked. For cases 3 and 4 we proceed by induction on n.

Since $P_{5n-1} \in O(P_{5n+1})$, $P_{5n} \in O(P_{5n+2})$, $P_{5n} \in O(P_{5n+3})$ and, by induction hypothesis, $\sigma(P_{5n-1}) = \sigma(P_{5n}) = 0$, we get $\sigma(P_{5n+1}) = \sigma(P_{5n+2}) = \sigma(P_{5n+3}) = 1$.

Observe (as in the proof of Lemma 16) that every option of P_{5n} or P_{5n+4} contains a path of order $m \equiv 1, 2$ or 3 (mod 5). Therefore, by Lemma 16, every such option is a winning position, and thus $\sigma(P_{5n}) = \sigma(P_{5n+4}) = 0$. ■

And therefore:

Corollary 18. $\mathcal{L} = \{1, 2, 8, 9\} \cup \{5n, n \geq 3\} \cup \{5n + 4, n \geq 3\}$.

Now, the outcome of a disjoint union of paths has outcome $\mathcal{P}$ if and only if the order of every component belongs to the set $\mathcal{L}$, which can be decided in linear time. A winning move from a $\mathcal{N}$-position can be obtained by playing on every component of order $p \notin \mathcal{L}$ as indicated in the proof of Theorem 17. Such a winning move can be found in linear time.

It is worth noting here that the set of losing paths is the same as under normal play (and, thus, as in the selective compound game under normal play), except for a few small paths, namely $P_0, P_1, P_2, P_4, P_5, P_8, P_9, P_{10}$ and P_{14}. We do not have any explanation of this fact.

Table 3
Losing positions for compound Node–Kayles on paths.

Compound version	Losing set $\mathcal{L}$
disj. comp., normal play	$\{0, 4, 8, 14, 19, 24, 28, 34, 38, 42\} \cup \{54 + 34i, 58 + 34i, 62 + 34i, 72 + 34i, 76 + 34i, i \geq 0\}$
disj. comp., misère play	*unsolved*
dim. disj. comp., normal play	$\{0, 4, 5, 9, 10, 14, 28, 50, 54, 98\}$
dim. disj. comp., misère play	*unsolved*
conj. comp., normal play	$\{0, 4, 5, 9, 10\}$
conj. comp., misère play	$\{1, 2\}$
cont. conj. comp., normal play	$\{5(2^n - 1), n \geq 0\} \cup \{5(2^{n+1} - 1) - 1, n \geq 0\}$
cont. conj. comp., misère play	$\{7.2^n - 6, n \geq 0\} \cup \{7.2^n - 5, n \geq 0\}$
sel. comp., normal play	$\{5n, n \geq 0\} \cup \{5n + 4, n \geq 0\}$
sel. comp., misère play	$\{7n + 1, n \geq 0\} \cup \{7n + 2, n \geq 0\}$
short. sel. comp., normal play	$\{5n, n \geq 0\} \cup \{5n + 4, n \geq 0\}$
short. sel. comp., misère play	$\{1, 2, 8, 9\} \cup \{5n, n \geq 3\} \cup \{5n + 4, n \geq 3\}$

4. Discussion

In this paper, we have solved ten versions of Conway's compound Node–Kayles on paths by providing the set of losing positions of every such game (see Table 3 for a summary of these results). In each case, the outcome of any position can be computed in linear time. The question of finding a losing option from any winning position (which gives the winning strategy) can as well be solved in linear time.

The first natural question is to complete our analysis, by solving the diminished disjunctive compound under misère play and, of course, the longstanding open problem of disjunctive compound under misère play.

It would also be interesting to extend our results to other graph families, such as stars, trees or outerplanar graphs (we can solve for instance continued conjunctive compound Node–Kayles on stars). Note here that all our results trivially extend to cycles since we have $O(C_n) = \{P_{n-3}\}$ for every cycle length $n \geq 3$.

Stromquist and Ullman studied in [21] the notion of *sequential compounds* of games. In such a compound game $G \rightarrow H$, no player can play on H while G has not ended. They proposed as an open question to consider the following compound game. Let $<$ be a partial order on games and $G = G_1 \cup G_2 \cup \cdots \cup G_k$ be a compound game. Then, a player can play on component G_i if and only if there is no other component G_j in G with $G_j > G_i$. This idea can be applied to Node–Kayles on paths by ordering the components according to their length. (Note that this new rule makes sense only for disjunctive and selective compounds).

Another variation could be to study Node–Kayles on *directed* paths (paths with directed edges), where each player deletes a vertex together to its out-neighbors. Such a directed version of Node–Kayles on general graphs has been considered in [8] (see also [7]), under the name of *universal domination game*.

Finally, inspired by the selective rule, we could also consider a *selective* Node–Kayles game, where each player deletes a vertex together with *some* of its neighbors. Restricted to paths, this game corresponds to the octal game **0.777**, still unsolved, and lies in some sense between *Kayles* and *Dawson's chess*.

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