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Thermoelasticity of bodies with microstructure and microtemperatures

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Abstract

This paper is concerned with a linear theory of thermodynamics for elastic materials with microstructure, whose microelements possess microtemperatures. It is shown that there exists the coupling of microrotation vector field with the microtemperatures even for isotropic bodies. Uniqueness and continuous dependence results are presented. The theory is used to establish the solution corresponding to a concentrated heat source acting in an unbounded continuum.

Keywords: Elastic bodies with microstructure; Thermoelasticity with microtemperatures; Continuous dependence results; Microstretch continua; Concentrated loads

1. Introduction

The origin of the modern theories of a continuum with microstructure goes back to papers by Ericksen and Truesdell (1958), Mindlin (1964), Eringen and Suhubi (1964) and Green and Rivlin (1964). Green (1965) has established the connection of the theory of multipolar continuum mechanics and the other theories. Much of the theoretical progress in the field is discussed in the books of Kunin (1983), Ciarletta and Ieşan (1993) and Eringen (1999). In the theory of micromorphic bodies formulated by Eringen and Suhubi (1964, 1999) the material particle is endowed with three deformable directors and the theory introduces nine extra degrees of freedom over the classical theory. On the basis of the theory of bodies with inner structure, Grot (1969) has established a theory of thermodynamics of elastic bodies with microstructure whose microelements possess microtemperatures. The Clausius–Duhem inequality is modified to include microtemperatures, and the first-order moment of the energy equations are added to the usual balance laws of a continuum with microstructure. The theory of micromorphic fluids with microtemperatures has been studied in various papers (see, e.g., Koh, 1973; Riha, 1975, 1977; Verma et al., 1979). Riha (1976) has presented a study of heat conduction in materials with microtemperatures. Experimental data for the silicone rubber containing spherical aluminium particles and for human blood were found to conform closely to predicted theoretical thermal conductivity.

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A theory of thermoelasticity with microtemperatures, in which the microelements can stretch and contract independently of their translations has been studied by Ieşan (2001). This is the simplest thermomechanical theory of elastic bodies that takes into account the microtemperatures and the inner structure of the materials. The theory introduces one mechanical extra degree of freedom over the classical theory. The theory of thermoelasticity with microtemperatures has been investigated by various authors. Casas and Quintanilla (2005) have studied the problem of stability. The theory of steady vibrations has been investigated by Scalia and Svanadze (2006) and Svanadze (2003, 2004).

Eringen (1999) has defined a class of micromorphic solids called microstretch solids. The material particles of these materials have seven degrees of freedom: three displacements, three microrotations and one microstretch. The microstretch continuum can model various porous media filled with gas or inviscid fluids, composite materials reinforced with chopped elastic fibers, mixtures with breathing elements and biological fluids.

In the present paper we use the results established by Grot (1969) to derive a linear theory of microstretch elastic solids with microtemperatures. This theory introduces three extra degrees of freedom over the theory presented by Ieşan (2001). A material particle is then equipped with the degrees of freedom for rigid rotations, in addition to the classical translation degrees of freedom and the microstretch. An interesting aspect in this theory is the coupling of microrotation vector with the microtemperatures even for isotropic bodies. We note that in the classical theory of Cosserat thermoelasticity for isotropic bodies, the microrotation vector is independent of the thermal field. In Section 2, we establish the field equations of the linear theory of thermoelasticity with microtemperatures. A uniqueness theorem in the dynamical theory of anisotropic bodies is presented in Section 3. In Section 4, we study the continuous dependence of solutions upon initial data and body loads. Section 5 is concerned with the effects of a concentrated heat source in a body that occupies the entire three-dimensional euclidean space.

2. Field equations

In the first part of this section we present the general balance laws of a continuum with microstructure in the form given by Grot (1969) and Eringen (1999). Then we derive the field equations of the linear theory of microstretch thermoelastic bodies with microtemperatures. The second moment of stress tensor and the microstress moment average are neglected in the balance laws since these functions appear only nonlinearly in the field equations (cf. Grot, 1969).

We consider a body that at some instant occupies the region B of the euclidean three-dimensional space and is bounded by the piecewise smooth surface ∂B. The motion of the body is referred to a fixed system of rectangular cartesian axes Ox_i $(i = 1, 2, 3)$. We denote by $\mathbf{n}$ the outward unit normal of ∂B. Boldface characters stand for tensors of an order $p \geqslant 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij\ldots s}$ (p subscripts) for the components of $\mathbf{v}$ in the cartesian coordinate frame. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. In what follows we use a superposed dot to denote partial differentiation with respect to the time t.

Let $\mathbf{u}$ be a displacement vector field over B. The balance of linear momentum can be written in the form

$$t_{ji,j} + \rho f_i = \rho \ddot{u}_i, \tag{2.1}$$

where t_{ij} is the stress tensor, ρ is the reference mass density, and f_i is the body force. We denote by m_{ijk} the first stress moment tensor. The balance of first stress moments is given by

$$m_{kij,k} + t_{ji} - s_{ji} + \rho \ell_{ij} = \rho \dot{\sigma}_{ij}, \tag{2.2}$$

where s_{ij} is the microstress tensor, ℓ_{ij} is the first body moment density and σ_{ij} is the inertia per unit mass. Let e be the internal energy density per unit mass, and let ε_i denote the first moment of energy vector. The balance of energy and the balance of first moment of energy can be expressed as

$$\rho \dot{e} = t_{ij} v_{j,i} + (s_{ij} - t_{ij}) v_{ji} + m_{kij} v_{ij,k} + q_{j,j} + \rho S \tag{2.3}$$

and

$$\rho\dot{\varepsilon}_i = m_{rsi}(v_{s,r} - v_{sr}) + q_{ji,j} + q_i - Q_i + \rho G_i, \tag{2.4}$$

respectively. Here v_i is the velocity vector, v_{ij} is the microgyration tensor, q_i is the heat flux vector, S is the heat supply per unit mass, q_{ij} is the first heat flux moment tensor, Q_i is the microheat flux average, and G_i is the first heat supply moment vector. The local from of the second law of thermodynamics is given by

$$\rho\dot{\eta} - \left(\frac{1}{T}q_k + \frac{1}{T}q_{km}T_m\right)_{,k} - \frac{1}{T}\rho(S + G_iT_i) \geqslant 0, \tag{2.5}$$

where η is the entropy density per unit mass, T is the absolute temperature, and T_i is the microtemperature vector.

The linear strain tensors are defined by (cf. Eringen, 1999)

$$\varepsilon_{ij} = u_{j,i} - \varphi_{ji}, \quad 2\gamma_{ij} = \varphi_{ij} + \varphi_{ji}, \quad \gamma_{ijk} = \varphi_{ij,k}, \tag{2.6}$$

where φ_{ij} is the microdeformation tensor. The spin inertia per unit mass can be expressed as

$$\sigma_{rs} = i_{ms}(\dot{v}_{rm} + v_{rk}v_{km}), \tag{2.7}$$

where $i_{rs}(=i_{sr})$ is the microinertia tensor. We note that in the linear theory we have

$$v_{ij} = \dot{\varphi}_{ij}, \quad \sigma_{rs} = i_{ms}\ddot{\varphi}_{rm}. \tag{2.8}$$

The components of surface traction t_i, the components of surface moments μ_{ij}, the heat flux q and the heat flux moment vector Λ_i at regular points of ∂B are defined by

$$t_i = t_{ji}n_j, \quad \mu_{ij} = m_{kij}n_k, \quad q = q_jn_j, \quad \Lambda_i = q_{ji}n_j, \tag{2.9}$$

respectively.

Eringen (1999) introduced a class of bodies with microstructure called microstretch continua, which is characterized by

$$\varphi_{ij} = \varphi\delta_{ij} - \varepsilon_{ijk}\varphi_k, \quad m_{kij} = \frac{1}{3}h_k\delta_{ij} - \frac{1}{2}\varepsilon_{ijr}m_{kr}, \quad \ell_{ij} = \frac{1}{3}\ell\delta_{ij} - \frac{1}{2}\varepsilon_{ijr}g_r, \tag{2.10}$$

where φ is the microdilatation function, δ_{ij} is Kronecker's delta, ε_{ijk} is the alternating symbol, φ_i is the microrotation vector, h_j is the microstretch vector, m_{ij} is the couple stress tensor, ℓ is the external microstretch body load, and g_r is the body couple density. We recall that the microstress tensor s_{ij} is symmetric. In the context of the linear theory of microstretch continua the balance of first stress moments (2.2) reduces to

$$h_{k,k} + g + \rho\ell = J\ddot{\varphi} \tag{2.11}$$

and

$$m_{ji,j} + \varepsilon_{irs}t_{rs} + \rho g_i = I_{ij}\ddot{\varphi}_j, \tag{2.12}$$

where $J = \rho i_{ss}$, $I_{rs} = \rho(i_{mm}\delta_{rs} - i_{rs})$, and $g = t_{jj} - s_{jj}$ is the internal body force which will be defined by a constitutive relation. In the linear theory of microstretch continua, the strain tensors are

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \kappa_{ij} = \varphi_{j,i}, \quad \zeta_i = \varphi_{,i}. \tag{2.13}$$

In view of (2.8), (2.10), and (2.13), the balance of energy (2.3) can be expressed as

$$\rho\dot{e} = t_{ij}\dot{e}_{ij} + m_{ij}\dot{\kappa}_{ij} + h_j\dot{\zeta}_j - g\dot{\varphi} + q_{j,j} + \rho S. \tag{2.14}$$

Eq. (2.4) can be written in the form

$$\rho\dot{\varepsilon}_i = m_{rsi}(\dot{e}_{rs} - \dot{\varphi}\delta_{rs}) + q_{ji,j} + q_i - Q_i + \rho G_i. \tag{2.15}$$

With the help of (2.14) and (2.15), the inequality (2.5) becomes

$$\begin{aligned}&\rho(T\dot{\eta} - \dot{e} - T_i\dot{\varepsilon}_i) + t_{ij}\dot{e}_{ij} + m_{ij}\dot{\kappa}_{ij} + h_i\dot{\zeta}_i - g\dot{\varphi} + m_{rsi}(\dot{e}_{rs} - \dot{\varphi}\delta_{rs})T_i + \frac{1}{T}q_iT_{,i} + \frac{1}{T}T_{,i}q_{ij}T_j - q_{km}T_{m,k}\\ &\quad + (q_j - Q_j)T_j \geqslant 0.\end{aligned} \tag{2.16}$$

Let us introduce the function ψ by

$$\psi = e + T_i \varepsilon_i - T\eta. \tag{2.17}$$

Then the inequality (2.16) can be written in the form

$$-\rho(\dot{\psi} + \dot{T}\eta - \dot{T}_i\varepsilon_i) + (t_{ij} + m_{ijs}T_s)\dot{e}_{ij} + m_{ij}\dot{\kappa}_{ij} + h_i\dot{\zeta}_i - (g + m_{ssi}T_i)\dot{\varphi} + \frac{1}{T}q_iT_{,i} + \frac{1}{T}T_{,i}q_{ij}T_j - q_{km}T_{m,k} + (q_j - Q_j)T_j \geqslant 0. \tag{2.18}$$

We introduce the notation

$$\theta = T - T_0, \tag{2.19}$$

where T_0 is the absolute temperature in the reference configuration. We assume that T_0 is a given positive constant.

In the linear theory we assume that $u_i = \epsilon u_i'$, $\varphi = \epsilon\varphi'$, $\varphi_i = \epsilon\varphi_i'$, $\theta = \epsilon\theta'$, $T_i = \epsilon T_i'$ where ϵ is a constant small enough for squares and higher powers to be neglected, and u_i', φ', φ_i', θ' and T_j' are independent of ϵ. We assume that in the undeformed state the functions t_{ij}, m_{ij}, h_i, g, q_i, q_{ij} and Q_i all vanish. Without loss of generality, we can also assume that η and ε_i vanish in this state. In the context of the linear theory the inequality (2.18) becomes

$$-\rho(\dot{\psi} + \dot{\theta}\eta - \dot{T}_i\varepsilon_i) + t_{ij}\dot{e}_{ij} + m_{ij}\dot{\kappa}_{ij} + h_i\dot{\zeta}_i - g\dot{\varphi} + \frac{1}{T_0}q_i\theta_{,i} - q_{ij}T_{j,i} + (q_i - Q_i)T_i \geqslant 0. \tag{2.20}$$

In what follows we consider the linear theory of thermoelastic materials with microtemperatures. The constitutive equations are

$$\begin{aligned}
&\psi = \widehat{\psi}(A), \quad t_{ij} = \widehat{t}_{ij}(A), \quad m_{ij} = \widehat{m}_{ij}(A), \quad h_i = \widehat{h}_i(A),\\
&g = \widehat{g}(A), \quad \varepsilon_i = \widehat{\varepsilon}_i(A), \quad \eta = \widehat{\eta}(A), \quad q_i = \widehat{q}_i(A), \quad q_{ij} = \widehat{q}_{ij}(A),\\
&Q_i = \widehat{Q}_i(A),
\end{aligned} \tag{2.21}$$

where $A = (e_{ij}, \kappa_{ij}, \zeta_i, \varphi, \theta, \theta_{,i}, T_i, T_{i,j})$. We assume that the response functions $\widehat{\psi}$, $\widehat{t}_{ij}$, $\widehat{m}_{ij}$, $\widehat{h}_i$, $\widehat{g}$, $\widehat{\varepsilon}_i$, $\widehat{\eta}$, $\widehat{q}_i$, $\widehat{q}_{rs}$ and $\widehat{Q}_r$ are of class C^1 on their domain $\check{\mathcal{D}}$ which is the set of all $A = (e_{ij}, \kappa_{ij}, \zeta_i, \varphi, \theta, \theta_{,j}, T_i, T_{i,j})$.

We denote

$$\rho\psi = \sigma. \tag{2.22}$$

It follows from (2.20)–(2.22) that

$$\begin{aligned}
&\sigma = \widehat{\sigma}(e_{ij}, \kappa_{ij}, \zeta_i, \varphi, \theta, T_i),\\
&t_{ij} = \frac{\partial\sigma}{\partial e_{ij}}, \quad m_{ij} = \frac{\partial\sigma}{\partial\kappa_{ij}}, \quad h_i = \frac{\partial\sigma}{\partial\zeta_i}, \quad g = -\frac{\partial\sigma}{\partial\varphi},\\
&\rho\eta = -\frac{\partial\sigma}{\partial\theta}, \quad \rho\varepsilon_i = \frac{\partial\sigma}{\partial T_i}
\end{aligned} \tag{2.23}$$

and

$$q_i\theta_{,i} - T_0 q_{ij}T_{j,i} + T_0(q_i - Q_i)T_i \geqslant 0. \tag{2.24}$$

In the linear theory we can take σ in the form

$$\begin{aligned}
2\sigma = {} & A_{ijrs}e_{ij}e_{rs} + 2B_{ijrs}e_{ij}\kappa_{rs} + C_{ijrs}\kappa_{ij}\kappa_{rs} + 2D_{ij}e_{ij}\varphi + 2E_{ij}\kappa_{ij}\varphi + 2F_{ijk}e_{ij}\zeta_k - 2a_{ij}e_{ij}\theta + 2L_{ijk}e_{ij}T_k\\
& + 2G_{ijk}\kappa_{ij}\zeta_k + A_{ij}\zeta_i\zeta_j - 2b_{ij}\kappa_{ij}\theta + M_{ijk}\kappa_{ij}T_k + 2B_i\zeta_i\varphi - 2d_i\zeta_i\theta - 2N_{ij}\zeta_iT_j + \xi\varphi^2 - 2F\varphi\theta\\
& + 2R_i\varphi T_i - a\theta^2 - 2b_i\theta T_i - B_{ij}T_iT_j,
\end{aligned} \tag{2.25}$$

where the constitutive coefficients have the following symmetries:

$$A_{ijrs} = A_{rsij}, \quad C_{ijrs} = C_{rsij}, \quad A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}. \tag{2.26}$$

It follows from (2.23), (2.25) and (2.26) that

$$\begin{aligned}
t_{ij} &= A_{ijrs}e_{rs} + B_{ijrs}\kappa_{rs} + F_{ijk}\zeta_k + D_{ij}\varphi + L_{ijk}T_k - a_{ij}\theta,\\
m_{ij} &= B_{rsij}e_{rs} + C_{ijrs}\kappa_{rs} + G_{ijk}\zeta_k + E_{ij}\varphi + M_{ijk}T_k - b_{ij}\theta,\\
h_i &= F_{rsi}e_{rs} + G_{rsi}\kappa_{rs} + A_{ij}\zeta_j + B_i\varphi - N_{ij}T_j - d_i\theta,\\
g &= -D_{ij}e_{ij} - E_{ij}\kappa_{ij} - B_i\zeta_i - \xi\varphi - R_iT_i + F\theta,\\
\rho\eta &= a_{ij}e_{ij} + b_{ij}\kappa_{ij} + d_i\zeta_i + F\varphi + b_iT_i + a\theta,\\
\rho\varepsilon_i &= L_{rsi}e_{rs} + M_{rsi}\kappa_{rs} - N_{ji}\zeta_j + R_i\varphi - B_{ij}T_j - b_i\theta.
\end{aligned} \tag{2.27}$$

In view of (2.24), the linear approximations for q_i, Q_i and q_{ij} are given by

$$q_i = k_{ij}\theta_{,j} + H_{ij}T_j, \quad q_{ij} = -P_{ijrs}T_{s,r}, \quad Q_i = (k_{ij} - K_{ij})\theta_{,j} + (H_{ij} - \Lambda_{ij})T_j, \tag{2.28}$$

where the constitutive coefficients k_{ij}, H_{ij}, K_{ij}, Λ_{ij} and P_{ijrs} satisfy the inequality

$$k_{ij}\theta_{,i}\theta_{,j} + (H_{ji} + T_0K_{ij})\theta_{,j}T_i + T_0\Lambda_{ij}T_iT_j + T_0P_{ijrs}T_{j,i}T_{s,r} \geqslant 0. \tag{2.29}$$

By (2.17), (2.19), (2.22), (2.23) and (2.27) we find that, in the linear theory, the balance of energy (2.14) reduces to

$$\rho T_0\dot{\eta} = q_{j,j} + \rho S. \tag{2.30}$$

In the context of the linear theory the balance of the first moment of energy (2.15) becomes

$$\rho\dot{\varepsilon}_i = q_{ji,j} + q_i - Q_i + \rho G_i. \tag{2.31}$$

The basic equations of the theory are: the equations of motion (2.1), (2.11) and (2.12), the energy equations (2.30) and (2.31), the constitutive equations (2.27) and (2.28), and the geometrical equations (2.13). To these equations we must adjoin initial conditions and boundary conditions. The initial conditions are

$$\begin{aligned}
&u_i(x,0) = u_i^0(x), \quad \dot{u}_i(x,0) = v_i^0(x), \quad \varphi_i(x,0) = \varphi_i^0(x),\\
&\dot{\varphi}_i(x,0) = \nu_i^0(x), \quad \varphi(x,0) = \varphi^0(x), \quad \dot{\varphi}(x,0) = w^0(x),\\
&\theta(x,0) = \theta^0(x), \quad T_i(x,0) = T_i^0(x), \quad x \in \overline{B},
\end{aligned} \tag{2.32}$$

where $u_i^0, v_i^0, \varphi_i^0, \nu_i^0, \varphi^0, w^0, \theta^0$ and T_i^0 are given. Let S_r, $(r = 1, 2, \ldots, 10)$, be subsets of ∂B such that $\overline{S}_1 \cup S_2 = \overline{S}_3 \cup S_4 = \overline{S}_5 \cup S_6 = \overline{S}_7 \cup S_8 = \overline{S}_9 \cup S_{10} = \partial B$, $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = S_7 \cap S_8 = S_9 \cap S_{10} = \emptyset$.

From (2.9) and (2.10) we find that the components of surface moments have the form

$$\mu_{ij} = \frac{1}{3}h\delta_{ij} - \frac{1}{2}\varepsilon_{ijr}m_r,$$

where

$$h = h_kn_k, \quad m_i = m_{ji}n_j. \tag{2.33}$$

We consider the following boundary conditions:

$$\begin{aligned}
&u_i = \widetilde{u}_i \text{ on } \overline{S}_1 \times I, \quad \varphi_i = \widetilde{\varphi}_i \text{ on } \overline{S}_3 \times I, \quad \varphi = \widetilde{\varphi} \text{ on } \overline{S}_5 \times I,\\
&\theta = \widetilde{\theta} \text{ on } \overline{S}_7 \times I, \quad T_i = \widetilde{T}_i \text{ on } \overline{S}_9 \times I, \quad t_{ji}n_j = \widetilde{t}_i \text{ on } S_2 \times I,\\
&m_{ji}n_j = \widetilde{m}_i \text{ on } S_4 \times I, \quad h_kn_k = \widetilde{h} \text{ on } S_6 \times I, \quad q_jn_j = \widetilde{q} \text{ on } S_8 \times I,\\
&q_{ki}n_k = \widetilde{\Lambda}_i \text{ on } S_{10} \times I,
\end{aligned} \tag{2.34}$$

where $\widetilde{u}_i, \widetilde{\varphi}_i, \widetilde{\varphi}, \widetilde{\theta}, \widetilde{T}_i, \widetilde{t}_i, \widetilde{m}_i, \widetilde{h}, \widetilde{q}$ and $\widetilde{\Lambda}_i$ are prescribed functions, and $I = (0, \infty)$.

We assume that: (i) f_i, g_i, ℓ, S and G_i are continuous on $\overline{B} \times [0, \infty)$; (ii) $\rho, I_{ij}, J, u_i^0, v_i^0, \varphi_i^0, \nu_i^0, \varphi^0, w^0, \theta^0$ and T_i^0 are continuous on $\overline{B}$; (iii) $I_{rs} = I_{sr}$ and the constitutive coefficients satisfy the symmetry relations (2.26); (iv) the constitutive coefficients are continuous differentiable on $\overline{B}$; (v) $\widetilde{u}_i, \widetilde{\varphi}_i, \widetilde{\varphi}, \widetilde{\theta}$ and $\widetilde{T}_i$ are continuous on $S_1 \times I$, $S_3 \times I$, $S_5 \times I$, $S_7 \times I$ and $S_9 \times I$, respectively; (vi) $\widetilde{t}_i, \widetilde{m}_i, \widetilde{h}, \widetilde{q}$ and $\widetilde{\Lambda}_i$ are continuous in time and piecewise regular on $S_2 \times I$, $S_4 \times I$, $S_6 \times I$, $S_8 \times I$ and $S_{10} \times I$, respectively.

Let M and N be non-negative integers. We say that f is of class $C^{M,N}$ on $B \times I$ if f is continuous on $B \times I$ and the functions

$$\frac{\partial^m}{\partial x_i \partial x_j \dots \partial x_p}\left(\frac{\partial^n f}{\partial t^n}\right), \quad m \in \{0,1,2,\dots,M\},\ n \in \{0,1,2,\dots,N\},\ m+n \leqslant \max\{M,N\},$$

exist and are continuous on $B \times I$. We write C^M for $C^{M,M}$.

By an admissible process $p = \{u_i, \varphi_i, \varphi, \theta, T_i, e_{ij}, \kappa_{ij}, \zeta_i, t_{ij}, m_{ij}, h_i, g, \eta, \varepsilon_i, q_i, Q_i, q_{ji}\}$ we mean an ordered array of functions u_i, φ_i, φ, θ, T_i, e_{ij}, κ_{ij}, ζ_i, t_{ij}, m_{ij}, h_k, g, η, ε_i, q_j, Q_i and q_{ji} defined on $\overline{B} \times [0,\infty)$ with the following properties: (i) $u_i, \varphi_i, \varphi \in C^2$; $\theta, T_i \in C^{2,1}$; $e_{ij}, \kappa_{ij}, \zeta_k \in C^{1,0}$, $t_{ij}, m_{ij}, h_k, q_i, q_{ji} \in C^{1,0}$; $g, Q_i \in C^0$; $\eta, \varepsilon_i \in C^{0,1}$ on $B \times I$; (ii) $u_i, \dot{u}_i, \ddot{u}_i, u_{i,j}, \varphi_i, \dot{\varphi}_i, \ddot{\varphi}_i, \varphi_{i,j}, \varphi, \dot{\varphi}, \ddot{\varphi}, \varphi_{i,j}, \theta, \theta_{,j}, T_i, T_{i,j}, e_{ij}, \kappa_{ij}, \zeta_k, t_{ij}, t_{ji,j}, m_{ji}, m_{ji,j}, h_i, h_{j,j}, g, \eta, \dot{\eta}, \varepsilon_i, \dot{\varepsilon}_i, q_i, q_{ji}, Q_k, q_{j,j}$ and $q_{ji,j}$ are continuous on $\overline{B} \times [0,\infty)$.

By a solution of the mixed problem we mean an admissible process which satisfies the Eqs. (2.1), (2.11), (2.12), (2.30), (2.31), (2.27), (2.28) and (2.13) on $B \times I$, the boundary conditions (2.34) and the initial conditions (2.32).

We note that in the case of isotropic and homogeneous bodies, the constitutive equations become

$$\begin{aligned}
&t_{ij} = \lambda e_{rr}\delta_{ij} + (\mu+\kappa)e_{ij} + \mu e_{ji} + \mu_0\varphi\delta_{ij} - \beta_0\theta\delta_{ij},\\
&m_{ij} = \alpha\kappa_{rr}\delta_{ij} + \beta\kappa_{ji} + \gamma\kappa_{ij} + b_0\varepsilon_{ijk}\zeta_k + \mu_1\varepsilon_{ijk}T_k,\\
&h_i = a_0\zeta_i - \mu_2 T_i + b_0\varepsilon_{rsi}\kappa_{rs},\\
&g = -\mu_0 e_{rr} - \xi\varphi + \beta_1\theta,\\
&\rho\eta = \beta_0 e_{rr} + \beta_1\varphi + a\theta, \quad \rho\varepsilon_i = \mu_1\varepsilon_{rsi}\kappa_{rs} - \mu_2\zeta_i - bT_i,\\
&q_i = k\theta_{,i} + k_1 T_i, \quad Q_i = (k_1 - k_2)T_i + (k - k_3)\theta_{,i},\\
&q_{ij} = -k_4 T_{r,r}\delta_{ij} - k_5 T_{i,j} - k_6 T_{j,i},
\end{aligned} \tag{2.35}$$

where δ_{ij} is the Kronecker delta and λ, μ, κ, α, β, γ, ξ, b, a, k, β_0, β_1, a_0, b_0, μ_0, μ_1, μ_2, k and k_s $(s = 1,2,\dots,6)$ are prescribed constants. It follows from (2.1), (2.11), (2.12), (2.30), (2.31), (2.35), and (2.13) that the field equations of the theory of homogeneous and isotropic bodies, with $I_{ij} = I_1\delta_{ij}$, can be expressed as

$$\begin{aligned}
&(\mu+\kappa)\Delta u_i + (\lambda+\mu)u_{j,ji} + \kappa\varepsilon_{ijk}\varphi_{k,j} + \mu_0\varphi_{,i} - \beta_0\theta_{,i} + \rho f_i = \rho\ddot{u}_i,\\
&\gamma\Delta\varphi_i + (\alpha+\beta)\varphi_{j,ji} + \kappa\varepsilon_{ijk}u_{k,j} - 2\kappa\varphi_i + \mu_1\varepsilon_{jik}T_{k,j} + \rho g_i = I_1\ddot{\varphi}_i,\\
&(a_0\Delta - \xi)\varphi - \mu_0 u_{r,r} - \mu_2 T_{j,j} + \beta_1\theta + \rho\ell = J\ddot{\varphi},\\
&k\Delta\theta + k_1 T_{j,j} - \beta_0 T_0\dot{u}_{r,r} - \beta_1 T_0\dot{\varphi} - c\dot{\theta} = -\rho S,\\
&k_6\Delta T_i + (k_4+k_5)T_{j,ji} + \mu_1\varepsilon_{rsi}\dot{\varphi}_{s,r} - \mu_2\dot{\varphi}_{,i} - b\dot{T}_i - k_2 T_i - k_3\theta_{,i} = \rho G_i,
\end{aligned} \tag{2.36}$$

where Δ is the Laplacian and $c = aT_0$. Of interest here is the coupling of microrotation with the microtemperatures. We note that the inequality (2.29) implies that (see Grot, 1969)

$$k \geqslant 0, \quad 3k_4 + k_5 + k_6 \geqslant 0, \quad k_5 + k_6 \geqslant 0, \quad k_6 - k_5 \geqslant 0, (k_1 + T_0 k_3)^2 \leqslant 4T_0 k k_2. \tag{2.37}$$

3. Uniqueness

In this section we present a uniqueness theorem in the dynamic theory of microstretch thermoelasticity with microtemperatures.

We consider the admissible process $p = \{u_i, \varphi_i, \varphi,\ \theta, T_i, e_{ij}, \kappa_{ij}, \zeta_i, t_{ij}, m_{ij}, h_i, g, \eta, \varepsilon_i, q_i, Q_i, q_{ji}\}$ and introduce the functions W_p and $\mathcal{D}_p$ on $B \times I$, defined by

$$\begin{aligned}
2W_p = {} & A_{ijrs}e_{ij}e_{rs} + 2B_{ijrs}e_{ij}\kappa_{rs} + C_{ijrs}\kappa_{ij}\kappa_{rs} + 2D_{ij}e_{ij}\varphi + 2E_{ij}\kappa_{ij}\varphi + 2F_{ijk}e_{ij}\zeta_k + 2L_{ijk}e_{ij}T_k + 2G_{ijk}\kappa_{ij}\zeta_k\\
& + A_{ij}\zeta_i\zeta_j + 2B_i\zeta_i\varphi + \xi\varphi^2,\\
2\Gamma_p = {} & a\theta^2 + 2b_i\theta T_i + B_{ij}T_iT_j,\\
\mathcal{D}_p = {} & k_{ij}\theta_{,i}\theta_{,j} + (H_{ji} + T_0K_{ij})\theta_{,j}T_i + T_0\Lambda_{ij}T_iT_j + T_0P_{ijrs}T_{j,i}T_{s,r}.
\end{aligned} \tag{3.1}$$

Clearly, the entropy inequality (2.29) implies that $\mathcal{D}_p$ is positive semidefinite,

$$\mathcal{D}_p \geqslant 0, \tag{3.2}$$

for any admissible process p.

Theorem 3.1. *Assume that*

(i) ρ *and* J *are strictly positive*;
(ii) I_{ij} *is positive definite*;
(iii) W_p *is a positive semidefinite quadratic form*;
(iv) Γ_p *is positive definite for any admissible process* p;
(v) *the symmetry relations* (2.26) *hold.*

Then, the mixed problem of thermoelasticity with microtemperatures has at most one solution.

Proof. With the help of the constitutive equations (2.27) and (3.1) we find that

$$t_{ij}\dot{e}_{ij} + m_{ij}\dot{\kappa}_{ij} + h_i\dot{\zeta}_i - g\dot{\varphi} + \rho\dot{\eta}\theta - \rho\dot{\varepsilon}_i T_i = \dot{W}_p + \dot{\Gamma}_p. \tag{3.3}$$

By using (2.13), (2.1), (2.11), (2.12), (2.27), (2.28), and (3.1) we obtain

$$\begin{aligned} t_{ij}\dot{e}_{ij} + m_{ij}\dot{\kappa}_{ij} + h_i\dot{\zeta}_i - g\dot{\varphi} + \rho\dot{\eta}\theta - \rho\dot{\varepsilon}_i T_i = {} & \left(t_{ji}\dot{u}_i + m_{ji}\dot{\varphi}_i + h_j\dot{\varphi} + \frac{1}{T_0}q_j\theta - q_{ji}T_i\right)_{,j} \\ & + \rho\left(f_i\dot{u}_i + g_i\dot{\varphi}_i + \ell\dot{\varphi} + \frac{1}{T_0}S\theta - G_iT_i\right) \\ & - \rho\ddot{u}_i\dot{u}_i - I_{ij}\ddot{\varphi}_j\dot{\varphi}_i - J\ddot{\varphi}\dot{\varphi} - \frac{1}{T_0}\mathcal{D}_p. \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}(2W_p + 2\Gamma_p + \rho\dot{\mathbf{u}}^2 + I_{ij}\dot{\varphi}_i\dot{\varphi}_j + J\dot{\varphi}^2) = {} & \left(t_{ji}\dot{u}_i + m_{ji}\dot{\varphi}_i + h_j\dot{\varphi} + \frac{1}{T_0}q_j\theta - q_{ji}T_i\right)_{,j} \\ & + \rho\left(f_i\dot{u}_i + g_i\dot{\varphi}_i + \ell\dot{\varphi} + \frac{1}{T_0}S\theta - G_iT_i\right) - \frac{1}{T_0}\mathcal{D}_p. \end{aligned} \tag{3.5}$$

We introduce the function E_p, associated to the process p, defined on I by

$$E_p = \frac{1}{2}\int_B (\rho\dot{\mathbf{u}}^2 + I_{ij}\dot{\varphi}_i\dot{\varphi}_j + J\dot{\varphi}^2 + 2W_p + 2\Gamma_p)\,\mathrm{d}v. \tag{3.6}$$

If we integrate the relation (3.5) over B and use the divergence theorem, then we get

$$\begin{aligned} \dot{E}_p = {} & \int_B \rho\left(f_i\dot{u}_i + g_i\dot{\varphi}_i + \ell\dot{\varphi} + \frac{1}{T_0}S\theta - G_iT_i\right)\mathrm{d}v + \int_{\partial B}\left(t_{ji}\dot{u}_i + m_{ji}\dot{\varphi}_i + h_j\dot{\varphi} + \frac{1}{T_0}q_j\theta - q_{ji}T_i\right)n_j\,\mathrm{d}a \\ & - \frac{1}{T_0}\int_B \mathcal{D}_p\,\mathrm{d}v. \end{aligned} \tag{3.7}$$

Let us assume that there are two solutions of the mixed problem, $p_\alpha = \{u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \varphi^{(\alpha)}, \theta^{(\alpha)}, T_i^{(\alpha)}, t_{ij}^{(\alpha)}, m_{ij}^{(\alpha)}, h_i^{(\alpha)}, g^{(\alpha)}, \eta^{(\alpha)}, \varepsilon_i^{(\alpha)}, q_j^{(\alpha)}, Q_i^{(\alpha)}, q_{ji}^{(\alpha)}\}, (\alpha = 1, 2)$.We define the process $\pi = (u_i^*, \varphi_i^*, \varphi^*, \theta^*, T_i^*, t_{ij}^*, m_{ij}^*, h_i^*, g^*, \eta^*, \varepsilon_i^*, q_j^*, Q_i^*, q_{ji}^*)$ by $u_i^* = u_i^{(1)} - u_i^{(2)}$, $\varphi_i^* = \varphi_i^{(1)} - \varphi_i^{(2)}$, $\varphi^* = \varphi^{(1)} - \varphi^{(1)}$, $\theta^* = \theta^{(1)} - \theta^{(2)}$, $T_i^* = T_i^{(1)} - T_i^{(2)}$, $t_{ij}^* = t_{ij}^{(1)} - t_{ij}^{(2)}$, $m_{ij}^* = m_{ij}^{(1)} - m_{ij}^{(2)}$, $h_i^* = h_i^{(1)} - h_i^{(2)}$, $g^* = g^{(1)} - g^{(2)}$, $\eta^* = \eta^{(1)} - \eta^{(2)}$, $\varepsilon_i^* = \varepsilon_i^{(1)} - \varepsilon_i^{(2)}$, $q_i^* = q_i^{(1)} - q_i^{(2)}$, $Q_i^* = Q_i^{(1)} - Q_i^{(2)}$, $q_{ij}^* = q_{ij}^{(1)} - q_{ij}^{(2)}$. Clearly, the process π corresponds to null data. Then, from (3.2) and (3.7) we obtain

$$\dot{E}_\pi \leqslant 0 \quad \text{on } [0, \infty).$$

With the help of initial data we can see that $E_\pi \leqslant 0$ on I. On the basis of the hypotheses of the theorem and (3.6) we obtain $\dot{u}_i^* = 0, \dot{\varphi}_i^* = 0, \dot{\varphi}^* = 0, \theta^* = 0$ and $T_i^* = 0$ on I. By using the initial data we find that $u_i^* = 0$, $\varphi_i^* = 0$ and $\varphi^* = 0$ on I, and the proof is complete. □

We note that the hypothesis (iv) implies that the specific heat $c = aT_0$ is strictly positive. Uniqueness results in various theories of thermoelasticity have been presented by Ieşan (2004).

4. Continuous dependence of solutions upon initial data and body loads

In this section we consider the linear theory of homogeneous and isotropic bodies and establish a continuous dependence result. It is convenient to have Eqs. (2.36) rewritten in non-dimensional form. We introduce the dimensionless variables

$$
\begin{aligned}
&x_i' = \frac{1}{\ell_0}x_i, \quad t' = \frac{c_1}{\ell_0}t, \quad u_i' = \frac{1}{\ell_0}u_i, \quad \varphi_i' = \varphi_i, \quad \varphi' = \varphi,\\
&\theta' = \frac{1}{T_0}\theta, \quad T_i' = T_i\ell_0,
\end{aligned}
\tag{4.1}
$$

where ℓ_0 is a standard length and $c_1 = [(\lambda + 2\mu + \kappa)/\rho_0]^{1/2}$. Introducing (4.1) into (2.36) and suppressing primes we find the equations

$$
\begin{aligned}
&\alpha_1 \Delta u_i + (1-\alpha_1)u_{j,ji} + \gamma_1 \varepsilon_{ijk}\varphi_{k,j} + \gamma_2 \varphi_{,i} - \kappa_1 \theta_{,i} + F_i = \ddot{u}_i\\
&\alpha_2 \Delta \varphi_i + \alpha_3 \varphi_{j,ji} + \gamma_1 \varepsilon_{ijk} u_{k,j} - 2\gamma_1 \varphi_i + \kappa_2 \varepsilon_{jik} T_{k,j} + \Phi_i = I_0 \ddot{\varphi}_i,\\
&(\alpha_4 \Delta - \alpha_5)\varphi - \gamma_2 u_{j,j} - \kappa_3 T_{j,j} + \kappa_4 \theta + H = J_0 \ddot{\varphi},\\
&K\Delta\theta - \kappa_1 \dot{u}_{j,j} - \kappa_4 \dot{\varphi} + \kappa_5 T_{j,j} - \xi_1 \dot{\theta} = -Q_0,\\
&\kappa_6 \Delta T_i + \kappa_7 T_{j,j} + \kappa_2 \varepsilon_{rsi} \dot{\varphi}_{s,r} - \kappa_3 \dot{\varphi}_{,i} - \xi_2 T_i - \xi_3 \dot{T}_i - \xi_4 \theta_{,i} = R_i,
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
&\alpha_1 = \frac{\mu+\kappa}{\rho c_1^2}, \quad \alpha_2 = \frac{\gamma}{\ell_0^2 \rho c_1^2}, \quad \alpha_3 = \frac{\alpha+\beta}{\ell_0^2 \rho c_1^2}, \quad \alpha_4 = \frac{a_0}{\ell_0^2 \rho c_1^2}, \quad \alpha_5 = \frac{\xi}{\rho c_1^2},\\
&\gamma_1 = \frac{\kappa}{\rho c_1^2}, \quad \gamma_2 = \frac{\mu_0}{\rho c_1^2}, \quad \kappa_1 = \frac{\beta_0}{\rho c_1^2}T_0, \quad \kappa_2 = \frac{\mu_1}{\rho \ell_0^2 c_1^2},\\
&\kappa_3 = \frac{\mu_2}{\rho \ell_0^2 c_1^2}, \quad \kappa_4 = \frac{\beta_1}{\rho c_1^2}T_0, \quad \kappa_5 = \frac{k_1}{\rho \ell_0 c_1^3}, \quad \kappa_6 = \frac{k_6}{\rho \ell_0^3 c_1^3},\\
&\kappa_7 = \frac{1}{\rho \ell_0^3 c_1^3}(k_4 + k_5), \quad \kappa_8 = \frac{1}{\rho \ell_0^3 c_1^3}k_4, \quad I_0 = \frac{I_1}{\rho \ell_0^2}, \quad J_0 = \frac{J}{\ell_0^2 \rho},\\
&K = \frac{k}{\ell_0 \rho c_1^3}T_0, \quad \xi_1 = \frac{c}{\rho c_1^2}T_0, \quad \xi_2 = \frac{k_2}{\rho \ell_0 c_1^3},\\
&\xi_3 = \frac{b}{\rho \ell_0^2 c_1^2}, \quad \xi_4 = \frac{k_3}{\rho \ell_0 c_1^3}T_0, \quad F_i = \frac{\ell_0}{c_1^2}f_i,\\
&\Phi_i = \frac{1}{c_1^2}g_i, \quad H = \frac{1}{c_1^2}\ell_0, \quad Q_0 = \frac{\ell_0}{c_1^3}S, \quad R_i = \frac{1}{c_1^3}G_i.
\end{aligned}
\tag{4.3}
$$

To the equations (4.2) we add the initial conditions (2.32) and the boundary conditions

$$
u_i = \widetilde{u}_i, \quad \varphi_i = \widetilde{\varphi}_i, \quad \varphi = \widetilde{\varphi}, \quad \theta = \widetilde{\theta}, \quad T_i = \widetilde{T}_i \quad \text{on } \partial B \times [0, t_1],
\tag{4.4}
$$

where $\widetilde{u}_i, \widetilde{\varphi}_i, \widetilde{\varphi}, \widetilde{\theta}$ and $\widetilde{T}_i$ are prescribed functions, and t_1 is a given positive constant.

We consider two solutions $\{u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \varphi^{(\alpha)}, \theta^{(\alpha)}, T_i^{(\alpha)}\}$, $(\alpha = 1,2)$, of the equations (4.2) corresponding to the external data systems $\mathcal{I}^{(\alpha)} = \{F_i^{(\alpha)}, \Phi_i^{(\alpha)}, H^{(\alpha)}, Q_0^{(\alpha)}, R_i^{(\alpha)}, \widetilde{u}_i, \widetilde{\varphi}_i, \widetilde{\varphi}, \widetilde{\theta}, \widetilde{T}_i, u_i^{0(\alpha)}, v_i^{0(\alpha)}, \varphi_i^{0(\alpha)}, v_i^{0(\alpha)}, \varphi^{0(\alpha)}, w^{0(\alpha)}, \theta^{0(\alpha)}, T_i^{0(\alpha)}\}$, $(\alpha = 1,2)$, respectively. We introduce the functions $u_i = u_i^{(1)} - u_i^{(2)}, \varphi_i = \varphi_i^{(1)} - \varphi_i^{(2)}, \varphi = \varphi^{(1)} - \varphi^{(2)}$, $\theta = \theta^{(1)} - \theta^{(2)}, T_i = T_i^{(1)} - T_i^{(2)}$. Obviously, $\{u_i, \varphi_i, \varphi, \theta, T_i\}$ is a solution of the problem corresponding to the external data system $\Im=\{F_i,\Phi_i,H,Q_0,R_i,0,0,0,0,0,0,u_i^0,v_i^0,\varphi_i^0,v_i^0,\varphi^0,w^0,T_i^0\}$, where $F_i=F_i^{(1)}-F_i^{(2)}$, $\Phi_i=\Phi_i^{(1)}-\Phi_i^{(2)}$, $H=H^{(1)}-H^{(2)}, Q_0=Q_0^{(1)}-Q_0^{(2)}$, $R_i=R_i^{(1)}-R_i^{(2)}$, $u_i^0=u_i^{0(1)}-u_i^{0(2)}$, $v_i^0=v_i^{0(1)}-v_i^{0(2)}$, $v_i^0=v_i^{0(1)}-v_i^{0(2)}$, $\varphi^0=\varphi^{0(1)}-\varphi^{0(2)}$, $w^0=w^{0(1)}-w^{0(2)}$, $\theta^0=\theta^{0(1)}-\theta^{0(2)}$ and $T_i^0=T_i^{0(1)}-T_i^{0(2)}$. We denote this problem by $(\mathcal{P})$.

We define the function ψ on $[0,t_1]$ by

$$\psi=\frac{1}{2}\int_B\left(\dot u_i\dot u_i+I_0\dot\varphi_i\dot\varphi_i+J_0\dot\varphi^2+2U+\xi_1\theta^2+\xi_3T_iT_i+2\int_0^t\mathcal{D}\,\mathrm{d}t\right)\mathrm{d}v,\tag{4.5}$$

where

$$\begin{aligned}2U={}&(1-2\alpha_1+\gamma_1)e_{rr}e_{ss}+\alpha_1e_{ij}e_{ij}+(\alpha_1-\gamma_1)e_{ji}e_{ij}+2\gamma_2e_{rr}\varphi\\&+\tau\gamma_{rr}\gamma_{ss}+(\alpha_3-\tau)\gamma_{ji}\gamma_{ij}+\alpha_2\gamma_{ij}\gamma_{ij}+2b_1\varepsilon_{jik}\varphi_{,k}\gamma_{ji}+\alpha_4\varphi_{,i}\varphi_{,i}+\alpha_5\varphi^2,\\\mathcal{D}={}&K\theta_{,i}\theta_{,i}+(\kappa_5+\xi_4)\theta_{,i}T_i+\xi_2T_iT_i\\&+\kappa_8T_{r,r}T_{s,s}+(\kappa_7-\kappa_8)T_{j,i}T_{i,j}+\kappa_6T_{i,j}T_{i,j}.\end{aligned}\tag{4.6}$$

In (4.6) we have used the notations

$$e_{ij}=u_{j,i}+\varepsilon_{jik}\varphi_k,\quad \gamma_{ij}=\varphi_{j,i},\quad \tau=\frac{\alpha}{\rho\ell_0^2c_1^2}.\tag{4.7}$$

We note that (2.37) and (4.3) imply that

$$\mathcal{D}\geqslant 0,\tag{4.8}$$

for any T_i,$\theta_{,j}$ and $T_{i,j}$.

In what follows we assume that the elastic potential U and the function $\mathcal{D}$ are positive definite quadratic forms. Thus, there exist the positive constants λ_1, λ_2, K_1 and K_2 such that

$$\begin{aligned}&\lambda_1(e_{ij}e_{ij}+\gamma_{ij}\gamma_{ij}+\varphi_{,i}\varphi_{,i}+\varphi^2)\leqslant U\leqslant\lambda_2(e_{ij}e_{ij}+\gamma_{ij}\gamma_{ij}+\varphi_{,i}\varphi_{,i}+\varphi^2),\\&K_1(\theta_{,i}\theta_{,i}+T_iT_i+T_{i,j}T_{i,j})\leqslant\mathcal{D}\leqslant K_2(\theta_{,i}\theta_{,i}+T_iT_i+T_{i,j}T_{i,j}),\end{aligned}\tag{4.9}$$

for all the variables e_{ij}, γ_{ij}, $\varphi_{,k}$, φ, $\theta_{,i}$, T_j, $T_{r,s}$ and any $t\in[0,t_1]$.

Theorem 4.1. *Let $p=\{u_i,\varphi_i,\varphi,\theta,T_i\}$ be a solution of the problem $(\mathcal{P})$. Then*

$$\dot\psi=\int_B(F_i\dot u_i+\Phi_i\dot\varphi_i+H\dot\varphi+Q_0\theta-R_iT_i)\,\mathrm{d}v.\tag{4.10}$$

Proof. We introduce the notations

$$\begin{aligned}&\pi_{ji}=(1-2\alpha_1+\gamma_1)e_{rr}\delta_{ij}+\alpha_1e_{ji}+(\alpha_1-\gamma_1)e_{ij}+\gamma_2\varphi\delta_{ij}-\kappa_1\theta\delta_{ij},\\&\mu_{ji}=\tau\gamma_{rr}\delta_{ij}+(\alpha_3-\tau)\gamma_{ij}+\alpha_2\gamma_{ji}+b_1\varepsilon_{jik}\varphi_{,k}+\kappa_2\varepsilon_{jik}T_k,\\&s_i=\alpha_4\varphi_{,i}-\kappa_3T_i+b_1\varepsilon_{rsi}\gamma_{rs},\quad f=-\gamma_2e_{rr}-\alpha_5\varphi+\kappa_4\theta,\\&\Pi=\kappa_1e_{rr}+\kappa_4\varphi+\xi_1\theta,\quad \chi_i=\kappa_2\varepsilon_{rsi}\gamma_{rs}-\kappa_3\varphi_{,i}-\xi_3T_i,\\&\sigma_i=K\theta_{,i}+\kappa_5T_i,\quad \Gamma_i=(\kappa_5-\xi_2)T_i+(K-\xi_4)\theta_{,i},\\&\lambda_{ij}=-\kappa_8T_{r,r}\delta_{ij}-(\kappa_7-\kappa_8)T_{i,j}-\kappa_6T_{j,i},\end{aligned}\tag{4.11}$$

where e_{ij} and γ_{ij} are defined in (4.7) and $b_1=b_0/(\rho\ell_0^2c_1^2)$.

The equations (4.2) can be written in the form

$$\begin{aligned}&\pi_{ji,j}+F_i=\ddot u_i,\quad \mu_{ji,j}+\varepsilon_{ijk}s_{jk}+\Phi_i=I_0\ddot\varphi_i,\\&s_{j,j}+f+H=J_0\ddot\varphi,\quad \dot\Pi=\sigma_{j,j}+Q_0,\\&\dot\chi_i=\lambda_{ji,j}+\sigma_i-\Gamma_i+R_i.\end{aligned}\tag{4.12}$$

By (4.11) and (4.6) we get

$$\pi_{ji}\dot{e}_{ji} + \mu_{ij}\dot{\gamma}_{ij} + s_i\dot{\varphi}_{,i} - f\dot{\varphi} + \dot{\Pi}\theta - \dot{\chi}_i T_i = \frac{1}{2}\frac{\partial}{\partial t}(2U + \xi_1\theta^2 + \xi_3 T_i T_i). \tag{4.13}$$

On the other hand, with the aid of (4.7) and (4.12) we obtain

$$\begin{aligned}\pi_{ji}\dot{e}_{ji} + \mu_{ij}\dot{\gamma}_{ij} + s_i\dot{\varphi}_{,i} - f\dot{\varphi} + \dot{\Pi}\theta - \dot{\chi}_i T_i = {}& (\pi_{ji}\dot{u}_i + \mu_{ji}\dot{\varphi}_i + s_j\dot{\varphi} + \sigma_j\theta - \lambda_{ji}T_i)_{,j} + F_i\dot{u}_i + \Phi_i\dot{\varphi}_i + H\dot{\varphi} \\ & + Q_0\theta - R_iT_i - \ddot{u}_i\dot{u}_i - I_0\ddot{\varphi}_i\dot{\varphi}_i - J_0\ddot{\varphi}\dot{\varphi} - \mathcal{D},\end{aligned} \tag{4.14}$$

where $\mathcal{D}$ is given by (4.6). From (4.13) and (4.14) we find that

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\left(2U + \xi_1\theta^2 + \xi_3 T_iT_i + \dot{u}_i\dot{u}_i + I_0\dot{\varphi}_i\dot{\varphi}_i + J_0\dot{\varphi}^2 + 2\int_0^t \mathcal{D}\,\mathrm{d}t\right) \\ &\quad = (\pi_{ji}\dot{u}_i + \mu_{ji}\dot{\varphi}_i + s_j\dot{\varphi} + \sigma_j\theta - \lambda_{ji}T_i)_{,j} + F_i\dot{u}_i + \Phi_i\dot{\varphi}_i + H\dot{\varphi} + Q_0\theta - R_iT_i.\end{aligned} \tag{4.15}$$

If we integrate (4.15) over B and use the divergence theorem and the boundary conditions, then we obtain (4.10). □

Let us introduce the functions ζ and P on $[0, t_1]$ by

$$\begin{aligned}\zeta &= \left\{\int_B [\dot{u}_i\dot{u}_i + \dot{\varphi}_i\dot{\varphi}_i + \dot{\varphi}^2 + e_{ij}e_{ij} + \gamma_{ij}\gamma_{ij} + \varphi_{,i}\varphi_{,i} + \theta^2 + T_iT_i + \int_0^t (\theta_{,i}\theta_{,i} + T_iT_i + T_{j,k}T_{j,k})\,\mathrm{d}t]\,\mathrm{d}v\right\}^{1/2}, \\ P &= \left[\int_B (F_iF_i + \Phi_i\Phi_i + H^2 + Q_0^2 + R_iR_i)\,\mathrm{d}v\right]^{1/2}.\end{aligned} \tag{4.16}$$

Theorem 4.2. *Assume that*

(i) ρ, I_0, J_0, ξ_1 *and* ξ_3 *are strictly positive*;
(ii) U *and* $\mathcal{D}$ *are positive definite.*

Then there exist the positive constants ρ_1 *and* ρ_2 *such that*

$$\zeta(t) \leqslant \rho_1\zeta(0) + \rho_2\int_0^t P(s)\,\mathrm{d}s, \quad t \in [0, t_1). \tag{4.17}$$

Proof. With the help of the Schwartz inequality, from (4.10) and (4.16) we obtain

$$\dot{\psi} \leqslant P\left[\int_B (\dot{u}_i\dot{u}_i + \dot{\varphi}_i\dot{\varphi}_i + \dot{\varphi}^2 + \theta^2 + T_iT_i)\,\mathrm{d}v\right]^{1/2}. \tag{4.18}$$

By (4.16) and (4.18) we find

$$\dot{\psi} \leqslant P\zeta,$$

so that

$$\psi(t) \leqslant \psi(0) + \int_0^t P(s)\zeta(s)\,\mathrm{d}s, \quad t \in [0, t_1]. \tag{4.19}$$

It follows from (4.5) and (4.9) that

$$\psi(t) \geqslant \tau_1\zeta^2(t), \quad \psi(0) \leqslant \tau_2\zeta^2(0), \quad t \in [0, t_1], \tag{4.20}$$

where

$$\tau_1 = \frac{1}{2}\min(1, I_0, J_0, 2\lambda_1, \xi_1, \xi_3, 2K_1), \quad \tau_2 = \frac{1}{2}\max(1, I_0, J_0, 2\lambda_2, \xi_1, \xi_3, 2K_2).$$

From (4.19) and (4.20) we obtain

$$\zeta^2(t) \leqslant \rho_1^2\zeta^2(0) + 2\rho_2\int_0^t P(s)\zeta(s)\,\mathrm{d}s, \quad t \in [0, t_1], \tag{4.21}$$

where

$$\rho_1 = \left(\frac{\tau_2}{\tau_1}\right)^{1/2}, \quad \rho_2 = \frac{1}{2\tau_1}.$$

With the help of the Gronwall inequality, from (4.21) we obtain the desired result. □

We note that the conditions $\xi_1 > 0$ and $\xi_3 > 0$ imply that $c > 0$ and $b > 0$, respectively.

5. The equilibrium theory

In this section we study the equilibrium theory of thermoelastic bodies with microtemperatures. In the case of equilibrium the basic equations of the theory consist of the equations of equilibrium

$$\begin{aligned} &t_{ji,j} + \rho f_i = 0, \\ &m_{ji,j} + \varepsilon_{irs}t_{rs} + \rho\ell = 0, \\ &h_{j,j} + g + \rho\ell = 0, \end{aligned} \tag{5.1}$$

the balance of energy

$$q_{j,j} + \rho S = 0, \tag{5.2}$$

the balance of the first moment of energy

$$q_{ji,j} + q_i - Q_i + \rho G_i = 0, \tag{5.3}$$

the constitutive equations (2.27), (2.28), and the geometrical equations (2.13). In the case of the first boundary-value problem the boundary conditions are

$$u_i = \widetilde{u}_i, \quad \varphi_i = \widetilde{\varphi}_i, \quad \varphi = \widetilde{\varphi}, \quad \theta = \widetilde{\theta}, \quad T_i = \widetilde{T}_i \quad \text{on } \partial B, \tag{5.4}$$

where $\widetilde{u}_i, \widetilde{\varphi}_i, \widetilde{\varphi}, \widetilde{\theta}$ and $\widetilde{T}_i$ are given functions. In the second boundary-value problem the boundary conditions are

$$t_{ji}n_j = \widetilde{t}_i, \quad m_{ji}n_j = \widetilde{m}_i, \quad h_jn_j = \widetilde{h}, \quad q_jn_j = \widetilde{q}, \quad q_{ji}n_j = \widetilde{\Lambda}_i \quad \text{on } \partial B, \tag{5.5}$$

where $\widetilde{t}_i, \widetilde{m}_i, \widetilde{h}, \widetilde{q}$ and $\widetilde{\Lambda}_i$ are prescribed.

Remark. In the case of equilibrium the theory is not coupled, in the sense that we can first study the problem of finding the functions θ and T_i, and then the problem of finding the functions u_i, φ_i and φ.

By a rigid state we mean an ordered array of functions $(u_i^*, \varphi_i^*, \varphi^*, \theta^*, T_i^*)$ of the form

$$u_i^* = c_i^{(1)} + \varepsilon_{ijk}c_j^{(2)}x_k, \quad \varphi_i^* = c_j^{(2)}, \quad \varphi^* = 0, \quad \theta^* = c^{(3)}, \quad T_i^* = 0, \tag{5.6}$$

where $c_i^{(1)}$, $c_i^{(2)}$ and $c^{(3)}$ are arbitrary constants.

Now, we present a uniqueness result. Let W_p and $\mathcal{D}_p$ be defined by (3.1).

Theorem 5.1. *Assume that W_p and $\mathcal{D}_p$ be positive definite. Then,*

(i) *the first boundary-value problem has at most one solution*;
(ii) *any two solutions of the second boundary-value problem are equal modulo a rigid state.*

Proof. In view of (5.2) and (5.3) we obtain

$$\begin{aligned} \rho\left(\frac{1}{T_0}S\theta - G_iT_i\right) &= -\frac{1}{T_0}q_{j,j}\theta + (q_{ji,j} + q_i - Q_i)T_i \\ &= -\left(\frac{1}{T_0}q_j\theta - q_{ji}T_i\right)_{,j} + \frac{1}{T_0}q_j\theta_{,j} - q_{ji}T_{i,j} + (q_i - Q_i)T_i. \end{aligned} \tag{5.7}$$

With the aid of the divergence theorem, from (5.7) we find that

$$\int_B \mathcal{D}_p \,\mathrm{d}v = \int_B \rho(S\theta - T_0 G_i T_i)\,\mathrm{d}v + \int_{\partial B}(q_j n_j \theta - T_0 q_{ji} n_j T_i)\,\mathrm{d}a. \tag{5.8}$$

Suppose that there are two solutions. Then their difference $y = (\overline{u}_i, \overline{\varphi}_i, \overline{\varphi}, \overline{\theta}, \overline{T}_i)$ corresponds to null data. From (5.8) we obtain

$$\int_B \mathcal{D}_y \,\mathrm{d}v = 0. \tag{5.9}$$

Since $\mathcal{D}_p$ is positive definite for any process p, we conclude from (5.9) that

$$\overline{\theta} = C, \quad \overline{T}_i = 0,$$

where C is an arbitrary constant. In the case of the first boundary-value problem we obtain $C = 0$. The remaining part of the proof is a direct consequence of the uniqueness theorem from the isothermal theory of elastic bodies (cf. Eringen, 1999).

In the equilibrium theory of homogeneous and isotropic bodies, the basic equations (2.36) become

$$\begin{aligned}
&(\mu+\kappa)\Delta u_i + (\lambda+\mu)u_{j,ji} + \kappa\varepsilon_{ijk}\varphi_{k,j} + \mu_0\varphi_{,i} - \beta_0\theta_{,i} = -\rho f_i,\\
&\gamma\Delta\varphi_i + (\alpha+\beta)\varphi_{j,ji} + \kappa\varepsilon_{ijk}u_{k,j} - 2\kappa\varphi_i + \mu_1\varepsilon_{jik}T_{k,j} = -\rho g_i,\\
&(a_0\Delta - \xi)\varphi - \mu_0 u_{j,j} - \mu_2 T_{j,j} + \beta_1\theta = -\rho\ell,\\
&k\Delta\theta + k_1 T_{j,j} = -\rho S,\\
&k_6\Delta T_i + (k_4+k_5)T_{j,ji} - k_2 T_i - k_3\theta_{,i} = \rho G_i.
\end{aligned} \tag{5.10}$$

In this case, the first boundary-value problem consists in the finding of functions u_i, φ_i, φ, θ and T_i that satisfy the equations (5.10) on B and the conditions (5.4) on ∂B. Clearly, by (2.35) and (2.13) we can express the boundary conditions (5.5) in terms of the functions u_i, φ_i, φ, θ and T_i. □

6. Effects of a concentrated heat source

In this section we study the effect of a concentrated heat source in an isotropic and homogeneous body that occupies the entire three-dimensional euclidean space. We consider the theory of equilibrium and assume that

$$f_i = 0, \quad g_i = 0, \quad \ell = 0, \quad \rho S = M(r), \quad G_i = 0, \tag{6.1}$$

where M is a prescribed function, $r = [(x_i - y_i)(x_i - y_i)]^{1/2}$ and (y_1, y_2, y_2) is a fixed point.

The conditions at infinity are

$$\begin{aligned}
&u_i = \mathrm{O}(1), \quad \varphi_i = \mathrm{O}(1), \quad \varphi = \mathrm{O}(r^{-1}), \quad \theta = \mathrm{O}(r^{-1}), \quad T_i = \mathrm{O}(r^{-1}),\\
&u_{i,j} = \mathrm{O}(r^{-1}), \quad \varphi_{i,j} = \mathrm{O}(r^{-1}), \quad \varphi_{,i} = \mathrm{O}(r^{-2}), \quad \theta_{,j} = \mathrm{O}(r^{-2}), \quad T_{i,j} = \mathrm{O}(r^{-2}).
\end{aligned}$$

We note that the behaviour at infinity of displacements is the same as in classical thermoelasticity. We seek the solution in the form

$$u_i = \Psi_{,i}, \quad \varphi_i = 0, \quad \varphi = \Phi, \quad \theta = V, \quad T_i = W_{,i}, \tag{6.2}$$

where Ψ, Φ, V and W are unknown functions which depend only of the variable r. The field equations (5.10) are satisfied if Ψ, Φ, V and W satisfy the equations

$$\begin{aligned}
&d_1\Delta\Psi + \mu_0\Phi - \beta_0 V = 0,\\
&(a_0\Delta - \xi)\Phi - \mu_0\Delta\Psi - \mu_2\Delta W + \beta_1 V = 0,\\
&k\Delta V + k_1\Delta W = -M,\\
&(d_2\Delta - k_2)W - k_3 V = 0,
\end{aligned} \tag{6.3}$$

where

$$d_1 = \lambda + 2\mu + \kappa, \quad d_2 = k_4 + k_5 + k_6. \tag{6.4}$$

We assume that the functions W_p and $\mathcal{D}_p$ defined by (3.1) are positive definite. It is a simple matter to see that the positive definiteness of W_p and $\mathcal{D}_p$ implies that

$$a_0 > 0, \quad d_1 > 0, \quad k > 0, \quad d_2 > 0, \quad \xi d_1 - \mu_0^2 > 0, \quad kk_2 - k_1k_3 > 0.$$

We introduce the notations

$$D_1 = a_0 d_1 \Delta(\Delta - \ell_1^2), \quad D_2 = kd_2\Delta(\Delta - \ell_2^2), \tag{6.5}$$

where

$$\ell_1 = \left(\frac{\xi_1 d_1 - \mu_0^2}{a_0 d_1}\right)^{1/2}, \quad \ell_2 = \left(\frac{kk_2 - k_1k_3}{kd_2}\right)^{1/2}. \tag{6.6}$$

Theorem 6.1. *Let*

$$\begin{aligned}
&\Psi = \mu_0[(\beta_1 d_2 - \mu_2 k_3)\Delta - \beta_1 k_2]\chi + \beta_0(a_0\Delta - \xi)(d_2\Delta - k_2)\chi,\\
&\Phi = -d_1\Delta[(\beta_1 d_2 - \mu_2 k_3)\Delta - \beta_1 k_2]\chi + \mu_0\beta_0\Delta(d_2\Delta - k_2)\chi,\\
&V = (d_2\Delta - k_2)D_1\chi,\\
&W = k_3 D_1\chi,
\end{aligned} \tag{6.7}$$

where χ is a function of class C^8 which satisfies the equation

$$D_1D_2\chi = -M. \tag{6.8}$$

Then Ψ, Φ, V and W satisfy the equations (6.3).

Proof. Let us substitute the functions Ψ, Φ, V and W given by (6.7) into the left hand side of (6.3). For the first equation from (6.3) we get

$$\begin{aligned}
d_1\Delta\Psi + \mu_0\Phi - \beta_0 V &= \mu_0 d_1\Delta[(\beta_1 d_2 - \mu_2 k_3)\Delta - \beta_1 k_2]\chi + d_1\beta_0(a_0\Delta - \xi)(d_2\Delta - k_2)\Delta\chi - \mu_0 d_1\Delta[(\beta_1 d_2 - \mu_2 k_3)\\
&\quad \Delta - \beta_1 k_2]\chi + \mu_0^2\beta_0\Delta(d_2\Delta - k_2)\chi - \beta_0 D_1(d_2\Delta - k_2)\chi\\
&= \Delta(d_2\Delta - k_2)[\mu_0^2\beta_0 - \beta_0 d_1(a_0\Delta - \xi) - \beta_0\mu^2 + d_1\beta_0(a_0\Delta - \xi)]\chi = 0.
\end{aligned}$$

In view of (6.8), for the third equation of (6.3) we obtain

$$k\Delta V + k_1\Delta W = k\Delta D_1(kd_2\Delta - kk_2 + k_1k_3)\chi = D_1D_2\chi = -M.$$

Similarly, we can show that the other equations of (6.3) are satisfied. $\square$

Eq. (6.8) can be written in the form

$$(\Delta - \ell_1^2)(\Delta - \ell_2^2)\Delta\Delta\chi = -p_0 M, \tag{6.9}$$

where

$$p_0 = (a_0 k d_1 d_2)^{-1}. \tag{6.10}$$

Let us consider the functions ψ_j $(j = 1, 2, 3, 4)$, which satisfy the equations

$$(\Delta - \ell_1^2)\psi_1 = -p_0 M, \quad (\Delta - \ell_2^2)\psi_2 = -p_0 M, \quad \Delta\Delta\psi_3 = -p_0 M, \quad \Delta\psi_4 = -p_0 M. \tag{6.11}$$

It is a simple matter to see that the solution of the equation (6.9) can be expressed in the form

$$\chi = a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + a_4\psi_4, \tag{6.12}$$

where

$$a_1 = \frac{1}{\ell_1^4(\ell_1^2 - \ell_2^2)}, \quad a_2 = -\frac{1}{\ell_2^4(\ell_1^2 - \ell_2^2)}, \quad a_3 = \frac{1}{\ell_1^2\ell_2^2}, \quad a_4 = \frac{\ell_1^2 + \ell_2^2}{\ell_1^4\ell_2^4}. \tag{6.13}$$

Let us investigate the effect of a concentrated heat source. We assume that

$$M = \delta(\mathbf{x} - \mathbf{y}), \tag{6.14}$$

where $\delta(\cdot)$ is the Dirac delta and y is a fixed point. In this case, from (6.11) and (6.14) we obtain

$$\psi_1 = \frac{p_0}{4\pi r}\exp(-\ell_1 r), \quad \psi_2 = \frac{p_0}{4\pi r}\exp(-\ell_2 r), \quad \psi_3 = \frac{p_0}{8\pi}r, \quad \psi_4 = \frac{p_0}{4\pi r}, \tag{6.15}$$

It follows from (6.12), (6.13) and (6.15) that

$$\chi = \frac{p_0}{4\pi\ell_1^4\ell_2^4(\ell_1^2 - \ell_2^2)r}[\ell_2^4\exp(-\ell_1 r) - \ell_1^4\exp(-\ell_2 r)] + \frac{p_0}{8\pi\ell_1^4\ell_2^4 r}[\ell_1^2\ell_2^2 r^2 + 2(\ell_1^2 + \ell_2^2)]. \tag{6.16}$$

If we substitute the function χ given by (6.16) into (6.7) then we find that for $\mathbf{x} \neq \mathbf{y}$, we have

$$\begin{aligned}
\Psi &= \frac{p_0}{4\pi r}[c_{11}\exp(-\ell_1 r) + c_{12}\exp(-\ell_2 r) + a_3 b_{13} r^2 + c_{13}],\\
\Phi &= \frac{p_0}{4\pi r}[c_{21}\exp(-\ell_1 r) + c_{22}\exp(-\ell_2 r) + b_{22}a_3],\\
V &= \frac{p_0}{4\pi r}[c_{31}\exp(-\ell_2 r) + a_0 a_3 d_1 k_2 \ell_1^2],\\
W &= \frac{a_0 d_1 k_3}{4\pi r}[a_2(\ell_2^2 - \ell_1^2)\ell_2^2\exp(-\ell_2 r) - a_3\ell_1^2],
\end{aligned} \tag{6.17}$$

where

$$\begin{aligned}
&c_{11} = a_1\ell_1^2 b_{11} + a_1\ell_1^4 b_{12} + a_1 b_{13}, \quad c_{12} = a_2\ell_2^2 b_{11} + a_2\ell_2^4 b_{12} + a_2 b_{13},\\
&c_{13} = a_3 b_{11} + a_4 b_{13}, \quad c_{21} = a_1 b_{21}\ell_1^4 + a_1 b_{22}\ell_1^2,\\
&c_{22} = a_2 b_{21}\ell_2^4 + a_2 b_{22}\ell_2^2, \quad c_{31} = a_0 a_2\ell_2^2(\ell_2^2 - \ell_1^2)d_1(d_2\ell_2^2 - k_2),\\
&b_{11} = \mu_0(\beta_1 d_2 - \mu_2 k_3) - \beta_0(\xi d_2 + a_0 k_2), \quad b_{12} = a_0\beta_0 d_2,\\
&b_{13} = \beta_0\xi k_2 - \beta_1\mu_0 k_2, \quad b_{21} = \mu_0\beta_0 d_2 - d_1(\beta_0 d_2 - \mu_2 k_3),\\
&b_{22} = (\beta_1 d_1 - \mu_0\beta_0)k_2.
\end{aligned}$$

From (6.2) and (6.17) we obtain the components of the displacements u_i, the function φ, the temperature θ, and the microtemperatures T_i which correspond to a concentrated heat source.

7. Conclusions

The results established in this paper can be summarized as follows:

(a) We have used the theory of thermodynamics of a continuum with microstructure established by Grot (1969) to derive a linear theory of microstretch elastic bodies with microtemperatures. A microelement of the continuum is equipped with the mechanical degrees of freedom for rigid rotations and microdilatation, in addition to the classical translation degrees of freedom.
(b) We have established a uniqueness result in the dynamical theory of anisotropic bodies.
(c) The filed equations of the theory of homogeneous and isotropic solids are presented. The salient feature of these equations is the coupling of microrotation vector with the microtemperatures. It is known that in the classical theory of Coserat thermoelasticity for isotropic bodies, the microrotation vector is independent of the thermal field.
(d) We have established the continuous dependence of solutions upon initial data and body loads. Consequently, the mathematical model is well posed.
(e) We have studied the effect of a concentrated heat source in an isotropic and homogeneous body that occupies the three-dimensional euclidean space.

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References

Casas, P., Quintanilla, R., 2005. Exponential stability in thermoelasticity with microtemperatures. Int. J. Eng. Sci. 43, 33–47.

Ciarletta, M., Ieşan, D., 1993. Non-Classical Elastic Solids. Longman Scientific & Technical, London.

Ericksen, J., Truesdell, C., 1958. Exact theory of stress and strain in rods and shells. Arch. Ration. Mech. Anal. 1, 295–323.

Eringen, A.C., 1999. Microcontinuum Field Theories, I. Foundations and Solids. Springer-Verlag, New York, Berlin, Heidelberg.

Eringen, A.C., Suhubi, E.S., 1964. Nonlinear theory of simple microelastic solids. Int. J. Eng. Sci. 2, 389–403.

Green, A.E., 1965. Micro-materials and multipolar continuum mechanics. Int. J. Eng. Sci. 3, 533–537.

Green, A.E., Rivlin, R.S., 1964. Multipolar continuum mechanics. Arch. Ration. Mech. Anal. 17, 113–147.

Grot, R., 1969. Thermodynamics of a continuum with microstructure. Int. J. Eng. Sci. 7, 801–814.

Ieşan, D., 2001. On a theory of micromorphic elastic solids with microtemperatures. J. Thermal Stresses 24, 737–752.

Ieşan, D., 2004. Thermoelastic Models of Continua. Kluwer Academic Publishers, Dordrecht.

Koh, S.L., 1973. Theory of a second-order microfluid. Rheol. Acta 12, 418–430.

Kunin, I.A., 1983. Elastic Media with Microstructure. Springer-Verlag, Berlin.

Mindlin, R.D., 1964. Microstructure in linear elasticity. Arch. Ration. Mech. Anal. 16, 51–77.

Riha, P., 1975. On the theory of heat-conducting micropolar fluids with microtemperatures. Acta Mech. 23, 1–8.

Riha, P., 1976. On the microcontinuum model of heat conduction in materials with inner structure. Int. J. Eng. Sci. 14, 529–535.

Riha, P., 1977. Poiseuille flow of microthermopolar fluids. Acta Tech. CSAV 22, 602–614.

Scalia, A., Svanadze, M., 2006. On the representation of solutions of the theory of thermoelasticity with microtemperatures. J. Thermal Stresses 29, 849–863.

Svanadze, M., 2003. Boundary value problems of the theory of thermoelasticity with microtemperatures. Proc. GAMM, PAMM 3, 188–189.

Svanadze, M., 2004. Fundamental solutions of the equations of the theory of thermoelasticity with microtemperatures. J. Thermal Stresses 27, 151–170.

Verma, P.D.S., Singh, D.V., Singh, K., 1979. Hagen–Poiseuille flow of microthermopolar fluids in a circular pipe. Acta Tech. CSAV 24, 402–412.