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Which claw-free graphs are strongly perfect?

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Abstract

A set S of pairwise nonadjacent vertices in an undirected graph G is called a *stable transversal* of G if S meets every maximal (with respect to set-inclusion) clique of G. G is called *strongly perfect* if all its induced subgraphs (including G itself) have stable transversals. A *claw* is a graph consisting of vertices a, b, c, d and edges ab, ac, ad. We characterize claw-free strongly perfect graphs by five infinite families of forbidden induced subgraphs. This result—whose validity had been conjectured by Ravindra [Research problems, Discrete Math. 80 (1990) 105–107]—subsumes the characterization of strongly perfect line-graphs that was discovered earlier by Ravindra [Strongly perfect line graphs and total graphs, Finite and Infinite Sets. Colloq. Math. Soc. János Bolyai 37 (1981) 621–633].

Keywords: Stable transversal; Strongly perfect graph; Claw-free graph

1. Introduction

Our terminology is mostly standard (see, for instance, [3]). A *claw* is a graph with vertices x, a, b, c and edges xa, xb, xc. A *hole* is a chordless cycle of length at least four. An *antihole* is the complement of a hole. A hole or antihole is said to be *odd* if it has an odd number of vertices; it is said to be *even* if it has an even number of vertices. We say that a graph is an *odd refinement* of a graph F if it arises from F by repeated applications of the following operation: choose an edge that belongs to no triangle and replace this edge by a path with an odd number of edges. (In particular, every graph is an odd refinement of itself.)

A set S of pairwise nonadjacent vertices in a graph G is called a *stable transversal* of G if S meets every maximal (with respect to set-inclusion) clique of G. Berge and Duchet [2] defined G to be *strongly perfect* if all its induced subgraphs (including G itself) have stable transversals.

Theorem 1.1. *A claw-free graph G is strongly perfect if and only if it contains no induced subgraph that is an odd hole, an antihole with at least six vertices, or an odd refinement of one of the graphs F_1, F_2, F_3 shown in Fig.* 1.

Since line-graphs are claw-free, our theorem subsumes the characterization of strongly perfect line-graphs that was discovered earlier by Ravindra [11]: a line-graph G is strongly perfect if and only if it contains no induced subgraph that is an odd hole or an odd refinement of one of the graphs F_1, F_2, F_3. In fact, the validity of Theorem 1.1 had been conjectured by Ravindra [12].

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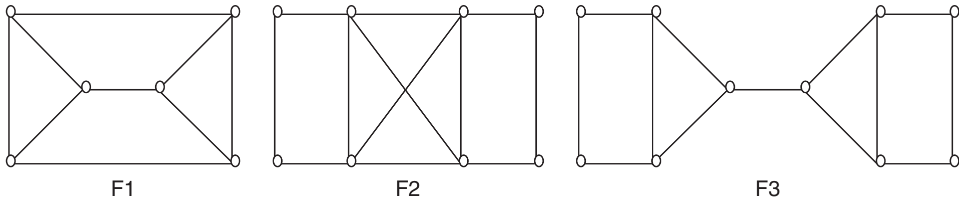

Fig. 1. Three claw-free graphs which are not strongly perfect.

Chvátal [5] proposed calling a linear order $<$ on the set of vertices of an undirected graph *perfect* if no chordless path with vertices a, b, c, d and edges ab, bc, cd has $a < b$ and $d < c$; a graph is called *perfectly orderable* if it admits a perfect order. He proved [5] that

every perfectly orderable graph is strongly perfect (1)

and, later on [7], characterized the class of claw-free perfectly orderable graphs as follows:

A claw-free graph is perfectly orderable if and only if
it contains no induced subgraph that is an odd hole,
an antihole with at least six vertices, or an odd refinement of one of certain
graphs $F_1, \ldots, F_7$
(such that F_1, F_2, F_3 are as in Fig. 1).

It was this theorem and its proof that motivated our work.

It is easy to check that no odd hole has a stable transversal, no antihole with at least six vertices has a stable transversal, and no odd refinement of one of F_1, F_2, F_3 has a stable transversal; these observations amount to the "only if" part of our theorem. The proof of the "if" part takes up most of the rest of this paper and begins with the following observations:

(i) If G has connected components $G_1, G_2, \ldots, G_k$, and if each G_i has a stable transversal S_i, then $S_1 \cup S_2 \cup \cdots \cup S_k$ is a stable transversal of G.
(ii) If x is a *simplicial* vertex (defined as a vertex whose neighbors form a clique) and if S is a stable transversal of $G - \{x\}$, then either S or $S \cup \{x\}$ is a stable transversal of G.
(iii) If two vertices u, v are *twins* (defined as two vertices such that no vertex distinct from both of them is adjacent to precisely one of them) and if S is a stable transversal of $G - \{v\}$, then either S or $S \cup \{v\}$ is a stable transversal of G.

Anstee and Farber [1], Lubiw [10], and Hoffman et al. [9] proved independently a theorem that was restated by Chvátal [7] as follows:

$$\left.\begin{array}{l}\text{If } G \text{ is the complement of a bipartite graph}\\ \text{and if } G \text{ contains no induced antihole with at least six vertices}\\ \text{then } G \text{ is perfectly orderable.}\end{array}\right\} \quad (2)$$

An instant corollary of (2) and (1) goes as follows:

(iv) If G is the complement of a bipartite graph and if G contains no induced antihole with at least six vertices, then G is strongly perfect.

In proving the “if” part of our theorem, we shall proceed by induction on the number of vertices of G. The induction hypothesis and (i)–(iv) allow us to assume that

G is connected,
G contains no simplicial vertices,
G contains no twins,
G is not the complement of a bipartite graph. (3)

Our analysis splits into two parts, depending on whether G has a clique-cutset or not; in each case, we determine the structure of G completely.

Throughout the remainder of this paper, we reserve the letter G for a graph that is connected, contains no simplicial vertex, contains no twins, is not the complement of a bipartite graph, and contains no induced subgraph that is an odd hole, an antihole with at least six vertices, or an odd refinement of one of F_1, F_2, F_3.

2. When G has no clique-cutset

In this section, we define “peculiar graphs” and “necklaces”. We prove that these graphs have stable transversals (Theorems 2.1 and 2.2) and that every G (satisfying our assumptions) that has no clique-cutset is either a peculiar graph or a necklace (Theorem 2.3).

Following Chvátal and Sbihi [8], we call a graph *peculiar* if it can be obtained as follows. Begin with a complete graph K whose set of vertices is split into pairwise disjoint nonempty sets $A_1, B_1, A_2, B_2, A_3, B_3$. Then, for each $i = 1, 2, 3$, remove at least one edge with one endpoint in A_i and the other endpoint in B_{i+1} (here, subscript 4 is interpreted as 1). Finally, add pairwise disjoint nonempty cliques K_1, K_2, K_3 and, for each $i = 1, 2, 3$, make each vertex in K_i adjacent to all the vertices in $K - (A_i \cup B_i)$.

Theorem 2.1. *If H is peculiar then H has a stable transversal.*

Proof. This is a straightforward corollary of (1) and Claim 1.1 of [7], which says “If H is peculiar then it is perfectly orderable”. □

A vertex x is said to *dominate* a vertex y if each neighbor of y except possibly x (which may or may not be a neighbor of y and is not adjacent to itself) is a neighbor of x.

The basic block in building necklaces is an *elementary bead*, which is a connected graph whose set of vertices can be partitioned into pairwise disjoint nonempty cliques $Q_1, \ldots, Q_n$ with $n \geqslant 3$ and n odd, so that

(i) if $x \in Q_i$, $y \in Q_j$ and $xy \in E$, then $j = i - 1$ or $j = i$ or $j = i + 1$;
(ii) for each i such that $2 \leqslant i \leqslant n - 1$, at least one of $Q_{i-1} \cup Q_i$ and $Q_i \cup Q_{i+1}$ is a clique;
(iii) every vertex in Q_1 has a neighbor in Q_2 and every vertex in Q_n has a neighbor in Q_{n-1};
(iv) if x, y are distinct vertices in the same Q_j, then one of x, y dominates the other.

We shall call Q_1 and Q_n the two *poles* of this bead; whenever convenient, we shall refer to the elementary bead simply as $Q_1, \ldots, Q_n$.

Lemma 2.1. *Every elementary bead $Q_1, \ldots, Q_n$ has*

- *a stable transversal that contains a vertex from Q_1 and a vertex from Q_n,*
- *a stable transversal that contains no vertex from Q_1 and no vertex from Q_n.*

Proof. By (iv) and since dominance is a transitive relation, each Q_i includes a vertex that dominates all the other vertices of Q_i; let x_i denote this vertex. Now (iii) guarantees that $Q_1 \cup \{x_2\}$ is a clique and that $Q_n \cup \{x_{n-1}\}$ is a clique; this observation and (ii) imply that

(v) no Q_i is a *maximal* clique.

Now consider an arbitrary maximal clique C of the bead. Since C is a clique, (i) guarantees that $C \subseteq Q_i \cup Q_{i+1}$ for some i; since C is a *maximal* clique, (v) guarantees that $C \not\subseteq Q_i$ and $C \not\subseteq Q_{i+1}$; in turn, maximality of C guarantees that C includes both x_i and x_{i+1}. Thus $\{x_1, x_3, \ldots, x_n\}$ is a stable transversal that contains a vertex from each of Q_1 and Q_n; the set $\{x_2, x_4, \ldots, x_{n-1}\}$ is a stable transversal that contains no vertex from Q_1 or Q_n. □

A *compound bead* is a graph constructed as follows:

- Begin with pairwise disjoint graphs $D_1, \ldots, D_n$ such that $n \geqslant 2$, all except possibly one of $D_1, \ldots, D_n$ are elementary beads, and the exceptional D_i (if any) is a single vertex.
- If D_i is an elementary bead then let A_i and C_i denote its two poles; if D_i is a single vertex, x, then set $A_i = C_i = \{x\}$.
- Add nonempty disjoint sets A, C of vertices disjoint from all of $D_1, \ldots, D_n$ and make each of $A \cup A_1 \cup \cdots \cup A_n$ and $C \cup C_1 \cup \cdots \cup C_n$ into a clique.

We shall call A and C the two *poles* of this bead.

Lemma 2.2. *If H is a compound bead, then H has a stable transversal that contains no vertex from either pole of H.*

Proof. Say H is made of $D_1, \ldots, D_n$ and two poles A, C. Without loss of generality, we may assume that all of D_i with $1 \leqslant i < n$ are elementary beads. For all $i = 1, \ldots, n-1$, let S_i be a stable transversal of D_i that contains no vertex from either pole of D_i. If D_n is a single vertex, x, then set $S_n = \{x\}$; if D_n is an elementary bead, then let S_n be a stable transversal of D_n that contains vertices from both poles of D_n. To see that $S_1 \cup \cdots \cup S_n$ is a stable transversal of H, note that each maximal clique of H is either a maximal clique of some D_i or else one of $A \cup A_1 \cup \cdots \cup A_n$, $C \cup C_1 \cup \cdots \cup C_n$. □

Let $\{B_1, \ldots, B_k\}$ be a set of at least two pairwise disjoint beads such that each B_i is either elementary or compound; let A_i and C_i denote the two poles of B_i. If, for every $i = 1, \ldots, k$, we have $|C_i| = |A_{i+1}|$, then the graph obtained by identifying pole C_i with pole A_{i+1} for every $i = 1, \ldots, k$ is called a *necklace*. Here, A_{k+1} is interpreted as A_1.

Theorem 2.2. *If H is a necklace, then H has a stable transversal.*

Proof. Say H is made of $B_1, \ldots, B_k$. Lemmas 2.1 and 2.2 guarantee that each B_i with $i = 1, \ldots, k$ has a stable transversal, S_i, that contains no vertex from either of its poles. To see that $S_1 \cup \cdots \cup S_k$ is a stable transversal of H, note that each maximal clique of H is a maximal clique of some B_i. □

Theorem 2.3. *If G has no clique-cutset, then G is either a peculiar graph or a necklace.*

Our proof of Theorem 2.3 takes up the remainder of the present section.
In their Section 5, Chvátal and Sbihi [8] proved (even though not quite stated) the following result:

$$\left.\begin{array}{l}\text{If a claw-free graph with no hole of length at least five}\\ \text{and no odd antihole}\\ \text{has no clique-cutset,}\\ \text{then it is either the complement of a bipartite graph}\\ \text{or a peculiar graph.}\end{array}\right\} \tag{4}$$

This result allows us to assume that G contains a hole of length at least six.
Throughout the remainder of this section, we reserve symbols

- H for an arbitrary but fixed hole in G such that H has at least six vertices;
- $w_1, \ldots, w_k$ (with subscript arithmetic modulo k) for the vertices of H in their natural cyclic order;
- N for the set of all the vertices in $G - H$ that have at least one neighbor in H.

We shall say that a vertex x is a

- 2-*vertex* if it has two neighbors in H and these neighbors are w_i, w_{i+1} for some i;
- 3-*vertex* if it has three neighbors in H and these neighbors are w_i, w_{i+1}, w_{i+2} for some i;
- 4-*vertex* if it has four neighbors in H and these neighbors are $w_i, w_{i+1}, w_j, w_{j+1}$ for some i and j of different parities.

Claim 2.1. *Every vertex in N is a* 2-*vertex or a* 3-*vertex or a* 4-*vertex.*

Justification. If $x \in N$ then x is adjacent to some w_i; note that x must be adjacent to at least one of w_{i-1} and w_{i+1} (else $w_i w_{i+1} w_{i-1} x$ would be a claw); and that x cannot have three pairwise nonadjacent neighbors (else x would be the center of a claw). It follows that the set of neighbors of x in H is $\{w_i, w_{i+1}\}$ or $\{w_i, w_{i+1}, w_{i+2}\}$ for some i, or $\{w_i, w_{i+1}, w_j, w_{j+1}\}$ for some i and j. In the last case, i and j must have different parities (else the graph induced by H and x would contain an odd hole).

Claim 2.2. *Let x be a* 2-*vertex adjacent to w_i and w_{i+1}. If x has a neighbor y in N such that $yw_i \notin E$, then y is a* 3-*vertex adjacent to $w_{i+1}, w_{i+2}, w_{i+3}$.*

Justification. By Claim 2.1, y is a 2-vertex or a 3-vertex or a 4-vertex. If y is a 2-vertex, then the graph induced by H along with x and y either contains an odd hole or is an odd refinement of F_1. If y is a 3-vertex, then y, x and the two nonadjacent neighbors of y in H form a claw unless y is adjacent to $w_{i+1}, w_{i+2}, w_{i+3}$. If y is a 4-vertex, then y, x and some two nonadjacent neighbors of y (both distinct from w_{i+1}) form a claw.

Claim 2.3. *For every* 2-*vertex x adjacent to w_i and w_{i+1}, precisely one of the following three statements holds true*:

(a) *x has a neighbor in N nonadjacent to w_i*,
(b) *x has a neighbor in N nonadjacent to w_{i+1}*,
(c) *x has a neighbor outside $H \cup N$.*

Justification. First, let us derive a contradiction from the assumption that all three of (a), (b), and (c) are false. Since x is not a simplicial vertex, it must have two nonadjacent neighbors, say y and z; since y and z are nonadjacent neighbors of a 2-vertex, at least one of them is outside H, say $y \notin H$; since (c) is false, $y \in N$; since (a) and (b) are both false, y is adjacent to both w_i and w_{i+1}. Now (since y and z are nonadjacent) $z \notin H$, and so (since all three of (a), (b), and (c) are false) z is a vertex in N adjacent to both w_i and w_{i+1}. To avoid a claw on $w_{i+1}w_{i+2}yz$, at least one of y and z must be adjacent to w_{i+2}, say $zw_{i+2} \in E$. Now, by Claim 2.1, z is a 3-vertex; in particular, $zw_{i-1} \notin E$. Next, to avoid a claw on $w_i w_{i-1} yz$, we must have $yw_{i-1} \in E$. Finally, by Claim 2.1, y is also a 3-vertex, and so

$$w_1 \dots w_{i-1} yxz w_{i+2} \dots w_k$$

is an odd hole, a contradiction.

Next, let us derive a contradiction from the assumption that (a) and (b) are both true: x has neighbors y and z in N such that $yw_i \notin E$ and $zw_{i+1} \notin E$. By Claim 2.2, y is a 3-vertex adjacent to $w_{i+1}, w_{i+2}, w_{i+3}$ and z is a 3-vertex adjacent to w_{i-2}, w_{i-1}, w_i. Since $yw_{i+1}w_{i+3}z$ is not a claw, y and z are nonadjacent. But then

$$w_1 \dots w_{i-2} zxy w_{i+3} \dots w_k$$

is an odd hole, a contradiction.

Finally, let us derive a contradiction from the assumption that (a) and (c) are both true: x has neighbors y and z such that $y \in N$, $yw_i \notin E$, and $z \notin H \cup N$. By Claim 2.2, y is a 3-vertex adjacent to $w_{i+1}, w_{i+2}, w_{i+3}$. But then one of $xw_i yz$ and $yzw_{i+1}w_{i+3}$ is a claw, a contradiction.

For each $j = 1, 2, \ldots, k$, let Q_j denote the set that consists of

- w_j;
- all the 2-vertices x adjacent to w_{j-1} and w_j such that some 3-vertex is adjacent to x, w_j, w_{j+1}, w_{j+2};
- all the 2-vertices x adjacent to w_j and w_{j+1} such that some 3-vertex is adjacent to x, w_{j-2}, w_{j-1}, w_j;
- all the 3-vertices adjacent to w_{j-1}, w_j, w_{j+1},

and let T_j denote the set that consists of

- all the 2-vertices adjacent to w_j and w_{j+1} that have a neighbor outside $H \cup N$.

In addition, let S denote the set of all the 4-vertices.

Claim 2.4. *Each vertex of $H \cup N$ belongs to precisely one of $Q_1, \ldots, Q_k, T_1, \ldots, T_k, S$.*

Justification. Straightforward from Claims 2.1, 2.3, and 2.2.

Our next aim is to describe the structure of the graph induced by $H \cup (N - S)$, which we denote by M.

Claim 2.5. *Let x and y be adjacent vertices in M.*
If $x \in Q_j$ then $y \in Q_{j-1} \cup T_{j-1} \cup Q_j \cup T_j \cup Q_{j+1}$.
If $x \in T_j$ then $y \in Q_j \cup T_j \cup Q_{j+1}$.

Justification. If at least one of x and y belongs to H, then Claim 2.5 follows trivially from Claims 2.1 and 2.4. If both x and y belong to $N - S$, then Claim 2.5 is implied by the following three claims:

(i) if x and y are adjacent 2-vertices then x and y have the same neighbors in H;
(ii) if x is a 2-vertex adjacent to w_i, w_{i+1} and if y is a 3-vertex adjacent to x then $y \in Q_{i-1} \cup Q_i \cup Q_{i+1}$;
(iii) if x is a 3-vertex in Q_j and if y is a 3-vertex adjacent to x then $y \in Q_{j-1} \cup Q_j \cup Q_{j+1}$.

Validity of (i) and (ii) follows directly from Claim 2.2. Let us justify (iii). Trivially, $y \in Q_i$ for some i. Since $xw_{j-1}w_{j+1}y$ is not a claw, i must be one of $j-2, j-1, j, j+1, j+2$; hence symmetry allows us to assume that i is one of $j, j+1, j+2$. If $i = j+2$ then

$$w_1 \ldots w_{j-1}xyw_{j+3} \ldots w_k$$

is an odd hole, a contradiction.

Claim 2.5 tells us that certain pairs of vertices in M must be nonadjacent. Now we shall prove that certain pairs of vertices in M must be adjacent:

- for each j, at least one of $Q_{j-1} \cup T_{j-1} \cup Q_j$ and $Q_j \cup T_j \cup Q_{j+1}$ is a clique (Claim 2.10);
- if $T_j \neq \emptyset$, then $Q_j \cup T_j \cup Q_{j+1}$ is a clique (a corollary of Claims 2.6 and 2.9).

For each $j = 1, 2, \ldots, k$, set
$j \in S_1$ if there are nonadjacent vertices x_j and y_j such that x_j is a 3-vertex adjacent to w_{j-1}, w_j, w_{j+1} and y_j is a 3-vertex adjacent to w_j, w_{j+1}, w_{j+2};
$j \in S_2$ if there are adjacent vertices x_j and y_j such that x_j is a 2-vertex adjacent to w_{j-1}, w_j and y is a 3-vertex adjacent to w_j, w_{j+1}, w_{j+2};
$j \in S_3$ if there are adjacent vertices x_j and y_j such that x_j is a 3-vertex adjacent to w_{j-1}, w_j, w_{j+1} and y_j is a 2-vertex adjacent to w_{j+1}, w_{j+2}.
We shall say that w_jw_{j+1} is *special* if $j \in S_1 \cup S_2 \cup S_3$; all the remaining edges w_jw_{j+1} will be called *normal*.

Claim 2.6. *If some vertex is adjacent to w_j, w_{j+1} and not adjacent to w_{j-1}, w_{j+2}, then $w_j w_{j+1}$ is a normal edge.*

Justification. Assume the contrary: some vertex z is adjacent to w_j, w_{j+1} and not adjacent to $w_{j-1}w_{j+2}$, and yet $w_j w_{j+1}$ is a special edge. We have three cases to consider.

Case 1: $j \in S_1$. We must have $zx_j \in E$ (to avoid a claw on $w_{j+1}w_{j+2}zx_j$) and $zy_j \in E$ (to avoid a claw on $w_j w_{j-1} z y_j$). Now z must be a 2-vertex (if $zw_t \in E$ and $t \neq j, j+1$ then $zx_j y_j w_t$ is a claw). But then

$$w_1 \ldots w_{j-1} x_j z y_j w_{j+2} \ldots w_k$$

is an odd hole, a contradiction.

Case 2: $j \in S_2$. We must have first $zy_j \in E$ (to avoid a claw on $w_j w_{j-1} z y_j$) and then $zx_j \in E$ (to avoid a claw on $y_j w_{j+2} z x_j$). Now z must be a 2-vertex (if $zw_t \in E$ and $t \neq j, j+1$ then $zx_j w_{j+1} w_t$ is a claw). But then

$$w_1 \ldots w_{j-1} x_j z w_{j+1} \ldots w_k$$

is an odd hole, a contradiction.

Case 3: $j \in S_3$. This is a mirror image of Case 2.

Claim 2.7. *No two special edges share a vertex.*

Justification. We only need derive a contradiction for the assumption that, for some j, both j and $j+1$ belong to $S_1 \cup S_2 \cup S_3$. By Claim 2.6, $j \notin S_3$ and $j+1 \notin S_2$. Four cases remain.

Case 1: $j \in S_1$, $j+1 \in S_1$. We must have $x_j y_{j+1} \notin E$, for otherwise

$$w_1 \ldots w_{j-1} x_j y_{j+1} w_{j+3} \ldots w_k$$

is an odd hole. Then we must have $x_j x_{j+1} \in E$ (to avoid a claw on $w_{j+1} x_j x_{j+1} y_{j+1}$) and $y_j y_{j+1} \in E$ (to avoid a claw on $w_{j+1} y_{j+1} x_j y_j$). In particular, $y_j \neq x_{j+1}$, and so $y_j x_{j+1} \in E$ to avoid a claw on $w_j w_{j-1} y_j x_{j+1}$. But then

$$w_1 \ldots w_{j-1} x_j x_{j+1} y_j y_{j+1} w_{j+3} \ldots w_k$$

is an odd hole, a contradiction.

Case 2: $j \in S_1$, $j+1 \in S_3$. We must have $x_j y_{j+1} \notin E$ (to avoid a claw on $x_j w_{j-1} w_{j+1} y_{j+1}$); now $x_j x_{j+1} \in E$, for otherwise

$$w_1 \ldots w_{j-1} x_j w_{j+1} x_{j+1} y_{j+1} w_{j+3} \ldots w_k$$

is an odd hole. In particular, $y_j \neq x_{j+1}$. Now we must have first $y_j x_{j+1} \in E$ (to avoid a claw on $w_{j+2} w_{j+3} y_j x_{j+1}$) and then $y_j y_{j+1} \in E$ (to avoid a claw on $x_{j+1} x_j y_j y_{j+1}$). But then

$$w_1 \ldots w_{j-1} x_j w_{j+1} y_j y_{j+1} w_{j+3} \ldots w_k$$

is an odd hole, a contradiction.

Case 3: $j \in S_2$, $j+1 \in S_1$. This is a mirror image of Case 2.

Case 4: $j \in S_2$, $j+1 \in S_3$. We must have $x_j y_{j+1} \notin E$ for otherwise

$$w_1 \ldots w_{j-1} x_j y_{j+1} w_{j+3} \ldots w_k$$

is an odd hole. Then we must have $x_j x_{j+1} \notin E$ (to avoid a claw on $x_{j+1} x_j w_{j+1} y_{j+1}$) and $y_j y_{j+1} \notin E$ (to avoid a claw on $y_j y_{j+1} w_{j+1} x_j$). In particular, $y_j \neq x_{j+1}$, and so $y_j x_{j+1} \in E$ to avoid a claw on $w_j w_{j-1} y_j x_{j+1}$. But then

$$w_1 \ldots w_{j-1} x_j y_j x_{j+1} y_{j+1} w_{j+3} \ldots w_k$$

is an odd hole, a contradiction.

Claim 2.8. *Every normal edge is contained in a unique maximal clique of G.*

Justification. Consider any edge $w_j w_{j+1}$ that extends to triangles $w_j w_{j+1} x$ and $w_j w_{j+1} y$ such that x and y are nonadjacent. To avoid a claw on $w_j w_{j-1} x y$, at least one of x and y must be adjacent to w_{j-1}, say $x w_{j-1} \in E$. Now

x is a 3-vertex; in particular, $xw_{j+2} \notin E$. To avoid a claw on $w_{j+1}w_{j+2}xy$, we must have $yw_{j+2} \in E$. Now y is also a 3-vertex, and so $j \in S_1$.

Claim 2.9. *If $w_j w_{j+1}$ is a normal edge, then $Q_j \cup T_j \cup Q_{j+1}$ is a clique.*

Justification. By virtue of Claim 2.8, we only need prove that each vertex x in $Q_j \cup T_j \cup Q_{j+1}$ other than w_j and w_{j+1} is adjacent to both w_j and w_{j+1}. For this purpose, assume the contrary, say $xw_{j+1} \notin E$. Since $x \in Q_j \cup T_j \cup Q_{j+1}$, it follows that x is a 2-vertex adjacent to w_{j-1}, w_j and that some 3-vertex y is adjacent to x, w_j, w_{j+1}, w_{j+2}. Thus $j \in S_2$, a contradiction.

Claim 2.10. *For each j, at least one of $Q_{j-1} \cup T_{j-1} \cup Q_j$ and $Q_j \cup T_j \cup Q_{j+1}$ is a clique.*

Justification. Straight from Claims 2.7 and 2.9.

Next, we turn our attention to 4-vertices.

Claim 2.11. *Let z be a 4-vertex and let C be the set of all the neighbors of z. Then there are subscripts r and s of different parities such that*

$$C = (Q_r \cup T_r \cup Q_{r+1}) \cup (Q_s \cup T_s \cup Q_{s+1})$$

and such that

both $w_r w_{r+1}$ and $w_s w_{s+1}$ are normal edges.

Justification. By definition, z has four neighbors in H and these neighbors are $w_r, w_{r+1}, w_s, w_{s+1}$ for some r and s of different parities. By Claim 2.6, each of $w_r w_{r+1}$ and $w_s w_{s+1}$ is a normal edge; hence Claims 2.8 and 2.9 guarantee that both $Q_r \cup T_r \cup Q_{r+1} \cup \{z\}$ and $Q_s \cup T_s \cup Q_{s+1} \cup \{z\}$ are cliques. To complete the proof, we will show that

(∗) z has no neighbors outside $(Q_r \cup T_r \cup Q_{r+1}) \cup (Q_s \cup T_s \cup Q_{s+1})$.

To justify claim (∗), assume the contrary: z has a neighbor y such that

$$y \notin (Q_r \cup T_r \cup Q_{r+1}) \cup (Q_s \cup T_s \cup Q_{s+1}).$$

Trivially, $y \notin H$. Since z, y, one of w_r, w_{r+1} and one of w_s, w_{s+1} do not form a claw, symmetry allows us to assume that y is adjacent to w_r and w_{r+1}. If y is a 2-vertex then Claim 2.4 implies that $y \in Q_r \cup T_r \cup Q_{r+1}$; if y is a 3-vertex then $y \in Q_r \cup Q_{r+1}$ by definition; hence Claim 2.1 guarantees that y is a 4-vertex. Now we distinguish between two cases.

Case 1: *z and y do not have the same set of neighbors in H.* In this case, the subgraph of G induced by H along with z and y contains an odd refinement of F_2, a contradiction.

Case 2: *z and y do have the same set of neighbors in H.* Since z and y are not twins, some vertex x is adjacent to precisely one of them, say $xz \in E$ and $xy \notin E$. By Claim 2.6, both $w_r w_{r+1}$ and $w_s w_{s+1}$ are normal edges; by Claim 2.8, each of them extends to a *unique* maximal clique; since $xy \notin E$ and since $yw_r w_{r+1}$, $yw_s w_{s+1}$ are triangles, x must be nonadjacent to at least one of w_r, w_{r+1} and to at least one of w_s, w_{s+1}. But then z, x, one of w_r, w_{r+1} and one of w_s, w_{s+1} induce a claw, a contradiction.

Claim 2.12. *If a vertex x of $H \cup N$ has a neighbor y outside $H \cup N$ then $x \in T_i$ for some i.*

Justification. Note that x must be a 2-vertex: else x, its two nonadjacent neighbors on H, and y would form a claw. By Claim 2.4, $x \in T_i$ or $x \in Q_i$ for some i; in the latter case, x is adjacent to a 3-vertex z such that z is nonadjacent to a neighbor w of x on H; but then $xyzw$ is a claw, a contradiction.

Let us enumerate all the connected components of $G-(H \cup N)$ as $R_1, \ldots, R_t$; for each $i=1, \ldots, t$, let C_i denote the set of vertices outside R_i that have at least one neighbor in R_i. By a *bridge of* H, we shall mean

- either the subgraph of G induced by some $R_i \cup C_i$
- or a 4-vertex.

Claim 2.13. *The vertex-sets of all the bridges of* H *are pairwise disjoint and their union is the vertex-set of* $G-(Q_1 \cup \cdots \cup Q_k)$.

Justification. We propose to show that

(i) $C_1 \cup \cdots \cup C_t \subseteq T_1 \cup \cdots \cup T_k$,
(ii) $C_1, \ldots, C_t$ are pairwise disjoint,
(iii) $T_1 \cup \cdots \cup T_k \subseteq C_1 \cup \cdots \cup C_t$;
the rest will follow from Claim 2.4.

Proof of (i): Let x be a vertex in some C_i. By definition, x has a neighbor, y, in R_i and $x \in H \cup N$. The conclusion follows from Claim 2.12.

Proof of (ii): If a vertex belonged to C_i and C_j with $i \neq j$ then this vertex, its neighbor in R_i, its neighbor in R_j, and its neighbor in H would form a claw, a contradiction.

Proof of (iii): Straight from the definition of T_j.

Claim 2.14. *For every* $i=1, \ldots, t$, *there are subscripts* r *and* s *of different parities such that*

$$C_i \cap T_r \neq \emptyset, \quad C_i \cap T_s \neq \emptyset, \quad C_i \subseteq T_r \cup T_s$$

and such that

both $w_r w_{r+1}$ *and* $w_s w_{s+1}$ *are normal edges.*

Justification. Claims 2.12 and 2.4 guarantee that $C_i \subseteq T_1 \cup \cdots \cup T_k$. Let I denote the set of all subscripts r such that $C_i \cap T_r \neq \emptyset$. Claim 2.6 guarantees that $w_r w_{r+1}$ is a normal edge whenever $r \in I$; in turn, Claim 2.9 guarantees that $Q_r \cup T_r \cup Q_{r+1}$ is a clique whenever $r \in I$. Since G has no clique-cutset, we have $|I|>1$; to complete the proof, we will show that

$(*)$ $\quad I$ cannot include two subscripts of the same parity.

To justify claim $(*)$, assume the contrary: two subscripts, r and s, in I have the same parity. Let F be the subgraph of G induced by H along with a chordless path from $C_i \cap T_r$ to $C_i \cap T_s$ with all interior vertices in R_i. Clearly, F consists of two triangles joined by three vertex-disjoint paths; two of these paths (segments of H) are odd; hence F contains an odd hole or an odd refinement of F_1, a contradiction.

Claim 2.15. *For every bridge of* H, *precisely four vertices in* H *have at least one neighbor in the bridge. These four vertices are* $w_r, w_{r+1}, w_s, w_{s+1}$ *with* r *and* s *of different parities*; *both* $w_r w_{r+1}, w_s, w_{s+1}$ *are normal edges.*

Justification. Straight from Claims 2.14 and 2.11.

We shall refer to the two edges $w_r w_{r+1}$ and $w_s w_{s+1}$ in Claim 2.15 as the *edges of attachment* of the bridge. By removing the edges of attachment of a bridge B, hole H is disconnected into two disjoint paths that we will call the *segments* of B; note that, by Claim 2.15, each of the two segments has an even number of edges. (In the special case when $r=s+1$ or $s=r+1$, one of the two segments consists of just a single vertex of H, and so it has no edges at all.)

By a *spine* of a bridge X that is induced by some $R_i \cup C_i$, we shall mean a chordless path from T_r to T_s with all internal nodes in $X-(T_r \cup T_s)$ and r, s as in Claim 2.14; by the *spine* of the bridge that is a 4-vertex x, we shall mean the degenerate path consisting of this single vertex x.

Claim 2.16. *Every spine of every bridge has an even number of edges.*

Justification. Assuming the contrary, the subgraph of G induced by the spine of the bridge and either of its two segments would contain an odd hole, a contradiction.

Claim 2.17. *No two bridges share precisely one edge of attachment.*

Justification. Assume the contrary: bridge X has edges of attachment $w_a w_{a+1}$, $w_b w_{b+1}$ and bridge Y has edges of attachment $w_b w_{b+1}$, $w_c w_{c+1}$. By symmetry, we may assume that a, b, c is the cyclic order of these three (distinct) subscripts. By Claim 2.15, edge $w_b w_{b+1}$ is normal; by Claim 2.8, it extends into a *unique* maximal clique; hence the subgraph of G induced by any spine of X, any spine of Y, and the path $w_{a+1} \dots w_c$ is an odd refinement of F_2, a contradiction.

We shall say that two bridges *cross* if the edges of attachment of one bridge lie in different segments of the other bridge.

Claim 2.18. *No two bridges cross.*

Justification. Assume the contrary: bridge X has edges of attachment $w_a w_{a+1}$, $w_b w_{b+1}$, bridge Y has edges of attachment $w_c w_{c+1}$, $w_d w_{d+1}$, and a, c, b, d is the cyclic order of these four (distinct) subscripts. By Claim 2.15,

(i) $a \not\equiv b \bmod 2$ and $c \not\equiv d \bmod 2$

symmetry allows us to assume that

(ii) $a \equiv c \bmod 2$.

Let P_X be a spine of X, let x be the endpoint of P_X such that $x w_a w_{a+1}$ is a triangle, and let us consider P_X oriented towards x; let P_Y be a spine of Y, let y be the endpoint of P_Y such that $y w_c w_{c+1}$ is a triangle, and let us consider P_Y oriented away from y. By (i), (ii), and Claim 2.16, the two triangles $x w_a w_{a+1}$ and $y w_c w_{c+1}$ along with the three paths

$$w_{a+1} \dots w_c,$$
$$w_{c+1} \dots w_b P_X,$$
$$P_Y w_{d+1} \dots w_a$$

induce an odd refinement of F_1, a contradiction.

Claim 2.19. *If bridge X has edges of attachment $w_a w_{a+1}$, $w_b w_{b+1}$, bridge Y has edges of attachment $w_c w_{c+1}$, $w_d w_{d+1}$, and a, b, c, d is the cyclic order of these four distinct subscripts, then b, c have different parities, and d, a have different parities.*

Justification. Assume the contrary: without loss of generality, b, c have the same parity. But then the subgraph of G induced by any spine of X, any spine of Y, and the path $w_{a+1} \dots w_d$ is an odd refinement of F_3, a contradiction.

Three bridges are *parallel* if, informally speaking, two of these bridges attach to different segments of the third bridge: more rigorously, bridges X, Y, Z are parallel if

- X has edges of attachment $w_a w_{a+1}$, $w_f w_{f+1}$;
- Y has edges of attachment $w_b w_{b+1}$, $w_e w_{e+1}$;
- Z has edges of attachment $w_c w_{c+1}$, $w_d w_{d+1}$, and a, b, c, d, e, f is the cyclic order of these six distinct subscripts.

Claim 2.20. *No three bridges are parallel.*

Justification. By Claim 2.19, a, b have different parities, b, c have different parities, and a, c have different parities, which is impossible.

Let $A(H)$ denote the set of all the edges of attachment of bridges of H. With this notation, Claims 2.17, 2.18, and 2.20 can be summarized as follows.

Claim 2.21. *The elements of $A(H)$ can be enumerated, in their natural cyclic order, as*

$$e_1,\ f_1,\ e_2,\ f_2, \ldots, e_d,\ f_d,$$

so that every bridge of H has some e_t and f_t (with the same t) for its two edges of attachment.

Claim 2.22. *Let r, s be subscripts of different parities such that $r \neq s+1$ and*

($\star$) *no edge on the path $w_r w_{r+1} \ldots w_{sn+1}$ belongs to $A(H)$*

and let B be the subgraph of G induced by $Q_r \cup Q_{r+1} \cup \cdots \cup Q_{s+1}$. Then

(a) $T_r = \cdots = T_s = \emptyset$;
(b) *B is an elementary bead with poles Q_r, Q_{s+1};*
(c) *no vertex in $B - (Q_r \cup Q_{s+1})$ has a neighbor outside B.*

Justification. (a) Note that by Claim 2.13 and by definition, any vertex of T_i belongs to some bridge whose edges of attachment include $w_i w_{i+1}$. The conclusion is guaranteed by assumption ($\star$).

(b) By definition, B is connected; by Claim 2.4, sets $Q_r, \ldots, Q_{s+1}$ are pairwise disjoint; by Claim 2.10, each Q_j is a clique. As for the four conditions in the definition of an elementary bead, (i) is guaranteed by Claim 2.5, (ii) is guaranteed by Claim 2.10, and (iii) follows from the definition of Q_j. To verify (iv), assume the contrary: there are vertices x, y in some Q_j and vertices u, v in B such that u is adjacent to x but not to y and such that v is adjacent to y but not to x. By Claim 2.5, both u and v belong to $Q_{j-1} \cup Q_j \cup Q_{j+1}$. By Claim 2.10, at least one of $Q_{j-1} \cup Q_j$ and $Q_j \cup Q_{j+1}$ is a clique; symmetry allows us to assume that $Q_{j-1} \cup Q_j$ is a clique. Now $u, v \in Q_{j+1}$; in turn, $Q_j \cup Q_{j+1}$ is not a clique, and so Claim 2.10 guarantees that $Q_{j+1} \cup Q_{j+2}$ is a clique. But then the hole $xyvu$ and the path $w_{j+2} \ldots w_k w_1 \ldots w_{j-1}$ induce an odd refinement of F_1, a contradiction.

(c) Consider an arbitrary vertex x in $B - (Q_r \cup Q_{s+1})$. By Claim 2.13, x belongs to no bridge of H, and so all its neighbors come from $H \cup N$; Claim 2.11 and assumption ($\star$) guarantee that x is adjacent to no 4-vertex. But then (a) and Claim 2.5 guarantee that all the neighbors of x come from B.

So far in our analysis, H has been an *arbitrary* hole with at least six vertices; from now on, we shall assume that

no hole H' with at least six vertices has $|A(H')| > |A(H)|$.

Claim 2.23. *Let X be a bridge of H with edges of attachment $w_r w_{r+1}, w_s w_{s+1}$. If X is not a 4-vertex, then X is an elementary bead with poles Q'_1, Q'_n such that*

- $Q'_1 \subseteq T_r$,
- $Q'_n \subseteq T_s$,
- *no vertex in $X - (Q'_1 \cup Q'_n)$ has a neighbor outside X.*

Justification. Claim 2.21allows us to assume (after switching subscripts if necessary) that

(i) no edge on the path $w_{r+1} \ldots w_s$ belongs to $A(H)$,
(ii) $s + 1 \neq r$.

Let P_X be a spine of X and let H' denote the hole induced by P_X and $w_{s+1} \ldots w_r$. From (ii), it follows that H' has at least six vertices. Enumerate the vertices of H' in their natural cyclic order as $w'_1, \ldots, w'_p$ so that $w'_1 \ldots w'_n$ is P_X and $w'_{n+1} = w_{s+1}$, $w'_p = w_r$. By Claim 2.6,

(iii) $w_s w_{s+1}$ and $w_r w_{r+1}$ are normal edges of H,
(iv) $w'_n w'_{n+1}$ and $w'_p w'_1$ are normal edges of H';

in turn, (iii), (iv), and Claim 2.8 imply that

(v) a vertex is adjacent to w_s, w_{s+1} if and only if it is adjacent to w'_n, w'_{n+1},
(vi) a vertex is adjacent to w_r, w_{r+1} if and only if it is adjacent to w'_p, w'_1.

We claim that

(vii) $A(H) - \{w_r w_{r+1}, w_s w_{s+1}\} \subseteq A(H')$.

To justify (vii), consider an arbitrary $w_j w_{j+1}$ in $A(H) - \{w_r w_{r+1}, w_s w_{s+1}\}$. By (i), $w_j w_{j+1}$ is on the path $w_{s+1} \dots w_r$; by definition, w_j and w_{j+1} have a common neighbor y in some bridge Y of H; by Claim 2.21, $Y \neq X$. If y is a 4-vertex of H, then (i) and Claim 2.21 guarantee that y is a 4-vertex of H', and so $w_j w_{j+1} \in A(H')$; else y has a neighbor z in Y such that $z \notin H \cup N$. By definition, every neighbor of z belongs to Y; in particular (since $Y \neq X$), z has no neighbor in P_X. Hence z has no neighbor in H', and so $w_j w_{j+1} \in A(H')$.

In addition, we claim that

(viii) $w'_n w'_{n+1}, w'_p w'_1 \in A(H')$.

To justify (viii), we propose to show that H' has a bridge with spine $w_{r+1} \dots w_s$. If $w_{r+1} = w_s$, then w_{r+1} is a 4-vertex of H' and we are done; hence we may assume that $w_{r+1} \neq w_s$. Now let w_j be an arbitrary interior vertex of $w_{r+1} w_{r+2} \dots w_s$. Since H is a hole, w_j neither belongs to nor has a neighbor in $w_{s+1} \dots w_r$. Since X is not a 4-vertex, $n \geqslant 3$; (since X is a bridge of H) w_j neither belongs to nor has a neighbor in the interior of $w'_1 \dots w'_n$ and (by Claim 2.14) w'_1 and w'_n are distinct 2-vertices of H nonadjacent to w_j. To summarize, w_j neither belongs to nor has a neighbor in H', and so we are done.

Define $Q'_1, \dots, Q'_p$ in the same way we defined $Q_1, \dots, Q_k$, except that H' is now used in place of H; then let B stand for the subgraph of G induced by $Q'_1 \cup \dots \cup Q'_n$. Claim 2.16 guarantees that n is odd; since X is not a 4-vertex, $n \geqslant 3$; (vii), (viii), and the assumption that $|A(H')| \leqslant |A(H)|$ guarantee that no edge on the path $w'_1 \dots w'_n$ belongs to $A(H')$. Hence H', 1, $n-1$ satisfy assumptions of Claim 2.22 in place of H, r, s; in turn, this claim guarantees that

(ix) B is an elementary bead with poles Q'_1, Q'_n,
(x) no vertex in $B - (Q'_1 \cup Q'_n)$ has a neighbor outside B.

By definition, X is induced by some $R_i \cup C_i$; Claim 2.14 guarantees that

(xi) $C_i \subseteq T_r \cup T_s$.

We propose to show that

(xii) $Q'_2 \cup \dots \cup Q'_{n-1} \subseteq R_i$,
(xiii) $Q'_1 \subseteq C_i \cap T_r$ and $Q'_n \subseteq C_i \cap T_s$.

Proof of (xii): Consider an arbitrary vertex v in $Q'_2 \cup \dots \cup Q'_{n-1}$. By definition, v equals or is adjacent to one of $w'_2, \dots, w'_{n-1}$; by definition, all of $w'_2, \dots, w'_{n-1}$ belong to R_i; hence $v \in X$. By definition, v is adjacent to neither of w'_{n+1} nor of w'_p; hence $v \notin T_r \cup T_s$; but then (xi) guarantees that $v \notin C_i$.

Proof of (xiii): Consider an arbitrary vertex v in Q'_n. By definition, v has a neighbor in Q'_{n-1}; hence (xii) guarantees that $v \in X$. By (v), every vertex in Q'_n is adjacent to w'_{n+1}; in particular, v is adjacent to w_{s+1}, and so $v \in C_i$. Now (xi) guarantees that $v \in T_r \cup T_s$; since v is adjacent to w_{s+1}, we have $v \notin T_r$. Hence $Q'_n \subseteq C_i \cap T_s$; a mirror image of this argument shows that $Q'_1 \subseteq C_i \cap T_r$.

From (xii), (x), and (xiii), it follows that $Q'_2 \cup \dots \cup Q'_{n-1} = R_i$; in turn, this identity, (x), and (xiii) imply that $Q'_1 \cup Q'_n = C_i$.

Claim 2.24. *If H has no bridge, then G is a necklace.*

Justification. Let B_1 be the subgraph of G induced by $Q_1 \cup Q_2 \cup Q_3$ and let B_2 be the subgraph of G induced by $Q_3 \cup \dots \cup Q_k \cup Q_1$. By of Claim 2.22(b), B_1 and B_2 are elementary beads with poles Q_1 and Q_3; by assumption and by Claim 2.13, every vertex of G belongs to one of B_1, B_2. In turn, Claim 2.22(c) guarantees that every edge of G belongs to one of B_1, B_2.

Claim 2.24 allows us to assume that

H has at least one bridge.

With $e_1, f_1, \ldots, e_d, f_d$ as in Claim 2.21, let J denote the set of all subscripts j such that some e_t is $w_j w_{j+1}$ or some f_t is $w_{j-1}w_j$. (Note that there may be i and j such that $f_t = w_{j-1}w_j$ and $e_{t+1} = w_j w_{j+1}$.)

Claim 2.25. *Every two subscripts in J have the same parity.*

Justification. Straight from Claims 2.19 and 2.15.

Enumerate all the sets Q_j with $j \in J$ in their cyclic order as $A_1, \ldots, A_m$. We are going to prove that G is a necklace made out of certain beads $B_1, \ldots, B_m$ such that the two poles of each B_i are A_i and A_{i+1}; these beads are defined as follows.

Each A_i is some Q_r; its successor A_{i+1} (with A_{m+1} interpreted as A_1) is some Q_{s+1}; let B_i be the subgraph of G induced by the union of $Q_r \cup Q_{r+1} \cup \cdots \cup Q_{s+1}$ and all the bridges of H with edges of attachment $w_r w_{r+1}$, $w_s w_{s+1}$.

Claim 2.26. *Each B_i is a bead with poles A_i, A_{i+1}; no vertex in $B_i - (A_i \cup A_{i+1})$ has a neighbor outside B_i.*

Justification. With r, s as in the definition of B_i, Claim 2.25 guarantees that

(a) r and s have different parities.

With $e_1, f_1, \ldots, e_d, f_d$ as in Claim 2.21, we shall distinguish between two cases.

Case 1: $f_t = w_{r-1}w_r$ and $e_{t+1} = w_{s+1}w_{s+2}$ for some t. In this case, no edge on the path $w_r w_{r+1} \ldots w_{s+1}$ is an edge of attachment of a bridge of H, and so the desired conclusion follows from Claim 2.22.

Case 2: $e_t = w_r w_{r+1}$ and $f_t = w_s w_{s+1}$ for some t. Let D_1 be the subgraph of G induced by $Q_{r+1} \cup \cdots \cup Q_s$ and enumerate all the bridges of H with edges of attachment $w_r w_{r+1}$, $w_s w_{s+1}$ as $D_2, \ldots, D_n$. By Claims 2.15 and 2.9,

(b) $Q_r \cup T_r \cup Q_{r+1}$ and $Q_s \cup T_s \cup Q_{s+1}$ are cliques.

We propose to show that

(c) if $r + 1 = s$, then D_1 is a single vertex
and the set of its neighbors is $Q_r \cup T_r \cup T_s \cup Q_{s+1}$.

For this purpose, note that $D_1 = Q_{r+1} = Q_s$ and consider an arbitrary vertex x in D_1. By assumption of Case 2, if one of the two edges $w_r w_{r+1}$, $w_s w_{s+1}$ is an edge of attachment of a bridge X then the other edge is also an edge of attachment of X; since $r + 1 = s$, Claim 2.21 guarantees that no 4-vertex is adjacent to both w_r, w_{r+1} and that no 4-vertex is adjacent to both w_s, w_{s+1}; now Claim 2.11 guarantees that no 4-vertex is adjacent to x. By Claim 2.12, no vertex outside $H \cup N$ is adjacent to x; in turn, Claim 2.5 guarantees that all the neighbors of x come from $Q_r \cup T_r \cup D_1 \cup T_s \cup Q_{s+1}$. Combining this observation with (b), we conclude that the set of neighbors of x is $Q_r \cup T_r \cup (D_1 - \{x\}) \cup T_s \cup Q_{s+1}$. But then we must have $|D_1| = 1$: else any two vertices in D_1 would be twins.

By Claim 2.22 (with $r + 1$, $s - 1$ in place of r, s),

(d) if $r + 1 \neq s$, then D_1 is an elementary bead with poles Q_{r+1}, Q_s and no vertex in $D_1 - (Q_{r+1} \cup Q_s)$ has a neighbor outside D_1.

By Claim 2.11,

(e) if some D_j with $1 < j \leqslant n$ is a single vertex, then the set of its neighbors is $Q_r \cup T_r \cup Q_{r+1} \cup Q_s \cup T_s \cup Q_{s+1}$.

By Claim 2.23,

(f) if some D_j with $1 < j \leqslant n$ is not a single vertex,
then it is an elementary bead with poles U_j, V_j so that
$U_j \subseteq T_r$, $V_j \subseteq T_s$,
and no vertex in $D_j - (U_j \cup V_j)$ has a neighbor outside D_j.

Claims 2.15 and 2.8 guarantee that $w_r w_{r+1}$ is contained in a unique maximal clique of G. Let C denote this clique. Since (by Claim 2.11) no two 4-vertices are adjacent, C includes at most one 4-vertex; (e) guarantees that C includes every 4-vertex that equals some D_j with $1 < j \leqslant n$; if D_1 is a single vertex, then (c) guarantees that $C \subseteq Q_r \cup T_r \cup D_1$, and so C includes no 4-vertices. We conclude that

(g) at most one D_j (with $1 \leqslant j \leqslant n$) is a single vertex.

By Claims 2.13 and 2.21, every vertex of $T_r \cup T_s$ belongs to a bridge with edges of attachment $w_r w_{r+1}$, $w_s w_{s+1}$. It follows that

(h) $T_r \cup T_s \subseteq B_i$.

By Claims 2.13 and 2.4, $D_1, D_2, \ldots, D_n, Q_r, Q_s$ are pairwise vertex-disjoint; now (b)–(g) imply that B_i is a compound bead with poles Q_r, Q_{s+1}. Finally, consider an arbitrary vertex x in B_i that has a neighbor outside B_i. From (c)–(f) and (h) we conclude that $x \in Q_r \cup T_r \cup Q_{r+1} \cup Q_s \cup T_s \cup Q_{s+1}$; by Claims 2.5 and 2.22(a), $x \notin T_r \cup Q_{r+1} \cup Q_s \cup T_s$.

By Claims 2.13, 2.4, and 2.21, the m sets $B_i - (A_i \cup A_{i+1})$ are pairwise disjoint and each vertex of G belongs to one of $B_1, B_2, \ldots, B_m$. By Claims 2.5 and 2.25, no edge of G joins an A_i to an A_j with $i \neq j$; now Claim 2.26 guarantees that G is a necklace.

3. When G has a clique-cutset

In this section, we define "strings". We prove that these graphs have stable transversals (Theorem 3.1) and that every G (satisfying our assumptions) that has a clique-cutset is a string (Theorem 3.2).

A *large appendix* is a graph whose set of vertices can be partitioned into sets N, P, R so that

(i) $P \cup N$ is a clique,
(ii) no vertex in N has a neighbor in R,
(iii) the graph induced by $P \cup R$ is a necklace,
(iv) P is a maximal clique in the necklace induced by $P \cup R$.

We shall call N the *pole* of the large appendix.

Lemma 3.1. *If H is a large appendix, the H has a stable transversal that contains no vertex from the pole of H.*

Proof. By (iii) and Theorem 2.2, the graph induced by $P \cup R$ has a stable transversal, S; by (i) and (ii), each maximal clique in H is either a subset of $P \cup R$ or the clique $P \cup N$; property (iv) guarantees that S meets P. Hence S is a stable transversal of H. □

A *small appendix* is a graph whose set of vertices can be partitioned into sets N, P, Q, R so that

(i) the graph induced by $P \cup R$ is the complement of a bipartite graph,
(ii) the graph induced by $P \cup R$ contains no induced antihole with at least six vertices,
(iii) no vertex in P has two nonadjacent neighbors in R,
(iv) every vertex of R has a neighbor in P,
(v) P is nonempty,

(vi) $P \cup N$ is a clique,
(vii) every vertex in Q is adjacent to all the vertices in $P \cup Q \cup R$ except itself,
(viii) no vertex in N has a neighbor in R.

An arbitrary subset of N that includes all the vertices of N with no neighbors in Q may be called the *pole* of the small appendix.

Lemma 3.2. *If H is a small appendix, then H has a stable transversal which contains no vertex from the pole of H.*

Proof. The proof of Claim 2.30 in [6] shows that

(a) the subgraph of H induced by $P \cup R$ admits a perfect order in which all the vertices in P precede all the vertices in R.

In [5], Chvátal proved (even though not quite stated) the following result:

(b) Given a graph H with vertices perfectly ordered in a sequence $v_1, \ldots, v_n$, scan the sequence from v_1 to v_n and place each v_j in S if and only if none of its neighbors v_i (with $i < j$) has been placed in S. Then S is a stable transversal of H.

By (a), (b), and (v), the subgraph of H induced by $P \cup R$ has a stable transversal, S, that includes a vertex of P. By (vi)–(viii),

- each clique of H is contained in $P \cup Q \cup R$ or in $N \cup P \cup Q$,
- each maximal clique of H that is contained in $P \cup Q \cup R$ contains a maximal clique of the subgraph of H induced by $P \cup R$,
- each maximal clique of H that is contained in $N \cup P \cup Q$ contains P.

Hence S is a stable transversal of H. □

Let $\{B_0, \ldots, B_{k+1}\}$ be a set of pairwise disjoint graphs such that

- each of B_0, B_{k+1} is a small appendix or a large appendix and such that (if $k > 0$),
- each of $B_1, \ldots, B_k$ is either an elementary bead or a compound bead.

For each i with $1 \leqslant i \leqslant k$, let A_i and C_i denote the two poles of B_i; let C_0 denote the pole of B_0 and let A_{k+1} denote the pole of B_{k+1}. If, for every $i = 0, \ldots, k$, we have $|C_i| = |A_{i+1}|$, then the graph obtained by identifying pole C_i with pole A_{i+1} for every $i = 0, \ldots, k$ is called a *string*.

Theorem 3.1. *If H is a string, then H has a stable transversal.*

Proof. Say H is made of $B_0, \ldots, B_{k+1}$. Lemmas 3.1 and 3.2 guarantee that each of B_i with $i = 0$ and $i = k + 1$ has a stable transversal, S_i, that contains no vertex from its pole; Lemmas 2.1 and 2.2 guarantee that each B_i with $1 \leqslant i \leqslant k$ has a stable transversal, S_i, that contains no vertex from either of its poles. To see that $S_1 \cup \cdots \cup S_k$ is a stable transversal of H, note that each maximal clique of H is a maximal clique of some B_i. □

Theorem 3.2. *If G has a clique-cutset, then G is a string.*

Our proof of Theorem 3.2 takes up the remainder of the present section.

If a graph H has a clique-cutset C, then it is the union of graphs H_1, H_2 whose intersection is C and such that each H_i has at least one vertex outside C. If, in addition, the clique-cutset C is minimal and H_1 has no clique-cutset, then we shall call H_1 a *tip* of H and we shall call C a *hinge* of H_1. These terms come from [6]. We are going to prove that

- G contains two vertex-disjoint tips, H_1 and H_2 (Claim 3.5), and that, with N_i standing for the set of all vertices outside H_i that have at least one neighbor in H_i;
- the subgraph of G induced by $H_i \cup N_i$ is a large appendix with pole N_i or a small appendix whose pole is a subset of N_i (Claim 3.2).

Our first objective is to prove Claim 3.2; for this purpose, we begin with a claim concerning a few properties of tips in connected claw-free graphs.

If a vertex x is adjacent to all the vertices in a set S except possibly itself (x may or may not be in S and is not adjacent to itself), then we say that x is *S-universal*.

Claim 3.1. *Let $H^\star$ be a connected claw-free graph, let H be a tip of $H^\star$, let N be the set of all vertices outside H that have at least one neighbor in H, let C be the hinge of H, and let Q be the set of H-universal vertices of C. Then*

(i) *no vertex in C has two nonadjacent neighbors in $H - C$;*
(ii) *H is not a peculiar graph;*
(iii) *if $H^\star$ contains no simplicial vertex, then $C \neq Q$ and $(C - Q) \cup N$ is a clique;*
(iv) *if some vertex in $H - C$ has no neighbor in C, then H contains a hole of length at least five with two vertices in C;*
(v) *if every vertex in $H - C$ has a neighbor in C, then H contains a hole of length at most five with two vertices in C.*

Justification.

(i) If some vertex in C had two nonadjacent neighbors in $H - C$, then this vertex, its two nonadjacent neighbors in $H - C$, and one of its neighbors outside H would form a claw, a contradiction.
(ii) It is a routine matter to verify that each nonempty clique K in a peculiar graph includes a vertex that has two nonadjacent neighbors outside K; by (i), clique C in H lacks this property.
(iii) Since $H^\star$ contains no simplicial vertex, H is not a clique; (i) guarantees that $H - (C - Q)$ is a clique; hence $C \neq Q$. The proof of Claim 2.9 in [6] shows that (even if the assumption that $H^\star$ contains no simplicial vertex is dropped) all the vertices in $C - Q$ are N-universal. Now if $(C - Q) \cup N$ were not a clique, then N would not be a clique; but then an arbitrary vertex in $C - Q$, its neighbor in $H - C$, and two nonadjacent vertices in N would form a claw, a contradiction.
(iv) Let A be a connected component of the subgraph induced in H by all the vertices with no neighbors in C; let D be the set of all the vertices outside A with at least one neighbor in A. Since D is a cutset of H, it must contain two nonadjacent vertices, u_1 and u_2. Since each u_i is outside A, it has a neighbor in C; let v_i denote this neighbor. By (i), u_1 and u_2 have no common neighbor in C; trivially, there is a chordless path P from u_1 to u_2 with all internal vertices in A. The desired hole is Pv_1v_2.
(v) Let x be a vertex in $H - C$ with the smallest number of neighbors in C. Since x isnot simplicial, it has two nonadjacent neighbors, y and z.
Case 1: $y \in C, z \notin C$. Since z has at least as many neighbors in C as x, some vertex w in C is adjacent to z and not to x. The desired hole is $wyxz$.
Case 2: $y \notin C, z \notin C$. By assumption, y has a neighbor v in C; note that v is not adjacent to z (to avoid a claw on v, its neighbor outside H and y, z). By symmetry, some w in C is adjacent to z and not to y. If x is adjacent to at least one of v and w, then we are back in Case 1; else the desired hole is $vyxzw$.

Claim 3.2. *Let H be a tip of G and let N be the set of all vertices outside H that have at least one neighbor in H. Then the subgraph of G induced by $H \cup N$ is either a large appendix whose pole is N or small appendix whose pole is a subset of N.*

Justification. Let us partition the set of vertices in H as follows:

- P is the set of all the vertices in the hinge of H that are not H-universal,
- Q is the set of all the vertices in the hinge of H that are H-universal,
- $R = H - (P \cup Q)$.

We shall distinguish between two cases.

Case 1: *H contains a hole of length at least five*. By assumption of this case, H is not the complement of a bipartite graph; in particular, $Q \cup R$ is not a clique; hence $Q = \emptyset$ by Claim 3.1(i). We propose to show that N, P, R satisfy the four conditions in the definition of a large appendix.

(i) Guaranteed by Claim 3.1(iii).
(ii) Holds by definition.
(iii) Let $N_G(x)$ denote the set of neighbors of x in G and let $N_H(x)$ denote the set of neighbors of x in H. If $x \in R$, then $N_H(x) = N_G(x)$ and $N_H(x) \cap N = \emptyset$; if $x \in P$, then $N_G(x) = N_H(x) \cup N$ and (since P is a minimal cutset of G) $N_H(x) \cap R \neq \emptyset$. It follows that (since G contains no simplicial vertex) H contains no simplicial vertex and (since G contains no twins) H contains no twins; by assumption of this case, H is not the complement of a bipartite graph. Now Theorem 2.3 with H in place of G guarantees that H is either a peculiar graph or a necklace; by Claim 3.1(ii), H is not peculiar.
(iv) First, let us show that

$$(\star) \quad H \text{ contains a hole of length at least six with two vertices in } P.$$

By assumption of this case, H contains a hole H_0 of length at least five; since G contains no odd hole, the length of H_0 is at least six. If H_0 and P share two vertices, then $(\star)$ follows; hence we may assume that H_0 and P share at most one vertex. Under this assumption, Claim 3.1(i) guarantees that H_0 and P are vertex-disjoint. Let $P^\star$ denote the set of all vertices in P that have at least one neighbor in H_0. No vertex in $P^\star$ has precisely one neighbor in H_0 (else this vertex in $P^\star$ and three consecutive vertices of H_0 would form a claw); this observation and Claim 3.1(i) guarantee that each vertex in $P^\star$ is adjacent to precisely two vertices of H_0 and that these two vertices are adjacent. Now every two vertices in $P^\star$ must have the same set of neighbors in H_0 (else the subgraph of G induced by H_0 and these two vertices would either be an odd refinement of F_1 or contain an odd hole), and so some vertex of H_0 has no neighbor in P. But then Claim 3.1(iv) guarantees that H contains a hole of length at least five with two vertices in P; since G contains no odd hole, $(\star)$ follows.
Next, let $H^\star$ denote the hole featured in $(\star)$ and enumerate the vertices of $H^\star$ in their natural cyclic order as $w_1, \dots, w_k$ in such a way that $w_2, w_3 \in P$. If (iv) failed, then some vertex x in $H - P$ would be adjacent to both w_2 and w_3; by Claim 3.1(i), x would be also adjacent to both w_1 and w_4; but then the subgraph of G induced by $H^\star$ and x would contain an odd hole or a claw, a contradiction.

Case 2: *H contains no hole of length at least five*. We propose to show that N, P, Q, R satisfy the eight conditions in the definition of a small appendix.

(i) By the result of Chvátal and Sbihi (1988) referred to as (4) in the previous section, H is either the complement of a bipartite graph or a peculiar graph; by Claim 3.1(ii), H is not peculiar.
(ii) Guaranteed by our assumptions on G.
(iii) Guaranteed by Claim 3.1(i).
(iv) Guaranteed by Claim 3.1(iv) and the assumption that H contains no hole of length at least five.
(v), (vi) Guaranteed by Claim 3.1(iii).
(vii), (viii) Hold by definition.

Our next objective is to prove that G contains at least two vertex-disjoint tips, which is a generalization of Claim 2.7 in [6]. Actually, we shall prove an even more general statement, Claim 3.5. Our argument relies on the following auxiliary result, which is Claim 2.6 of [6].

Claim 3.3. *Let H be a claw-free graph whose set of vertices is partitioned into pairwise disjoint cliques R_1, R_2, C in such a way that no edge of H has one endpoint in R_1 and the other endpoint in R_2. Let H_i denote the subgraph of H induced by $R_i \cup C$. If neither H_1 nor H_2 has a clique-cutset, then H is the complement of a bipartite graph.*

Claim 3.4. *Let H be a connected claw-free graph such that H is not the complement of a bipartite graph and H contains no simplicial vertex. Then any two distinct tips of H are vertex-disjoint.*

Justification. Consider any two distinct tips, H_1 and H_2, of H. Let C_i denote the hinge of H_i; write $R_i = H_i - C_i$ and $C = C_1 \cap C_2$. We propose to show that

$$(\star) \quad H_1 \cap H_2 = C.$$

Since H_1 and H_2 are distinct, we may assume (switching subscripts if necessary) that $H_1 - H_2 \neq \emptyset$. Now (since $C_2 \cap H_1$ is not a cutset of H_1) $R_2 \cap H_1 = \emptyset$; next (since $R_2 \neq \emptyset$) $R_2 - H_1 \neq \emptyset$, and so (since $C_1 \cap H_2$ is not a cutset of H_2) $R_1 \cap H_2 = \emptyset$.

By ($\star$), our task reduces to proving that $C = \emptyset$. For this purpose, we shall distinguish between two cases.

Case 1: $C_1 = C_2$. Here, $C = C_1 = C_2$ and, since H is connected, $C \neq \emptyset$; we are going to show that this case cannot occur.

Note that $H - C$ has precisely two components: else there would be a claw on any vertex in C and its three neighbors in three distinct components of $H - C$. Let D_i denote the set of vertices in R_i that have at least one neighbor in C. If $D_1 = R_1$ and $D_2 = R_2$, then Claim 3.3 guarantees that at least one of D_1 and D_2 is not a clique; if $D_i \neq R_i$ for at least one i, then D_i is a cutset of H_i, and so D_i is not a clique; in either case, at least one of D_1 and D_2 is not a clique.

Symmetry allows us to assume that D_1 contains two nonadjacent vertices, u_1 and u_2. Let N_i denote the set of neighbors of u_i in C. By Claim 3.1(i), we have $N_1 \cap N_2 = \emptyset$; in particular, no vertex in $N_1 \cup N_2$ is H_1-universal; now Claim 3.1(iii) guarantees that $N_1 \cup N_2 \cup D_2$ is a clique. In fact, every vertex w in D_2 must be C-universal (else there would be a claw on a neighbor of w in N_1, w, u_1, and some vertex in $C - N_1 - N_2$ that is not adjacent to w), and so $C \cup D_2$ is a clique.

Since H_2 has no clique-cutset, we must have $R_2 = D_2$. Hence $C \cup R_2$ is a clique. But then all the vertices of R_2 are simplicial, a contradiction.

Case 2: $C_1 \neq C_2$. Here, we are going to derive a contradiction from the assumption that $C \neq \emptyset$.

Since C_1 and C_2 are minimal clique-cutsets, each $C_i - C$ is nonempty. Note that no vertex in R_2 has a neighbor in $C_1 - C$ and that some vertex in R_2 has a neighbor in C. Hence Claim 3.1(iii) guarantees that all vertices in $C_1 - C$ are H_1-universal; in turn, Claim 3.1(i) guarantees that $R_1 \cup (C_1 - C)$ is a clique. By symmetry, $R_2 \cup (C_2 - C)$ is also a clique.

Let H^0 denote the subgraph of H induced by $R_1 \cup R_2 \cup C$ and let H_i^0 denote the subgraph of H induced by $R_i \cup C$. Now the hypothesis of Claim 3.3 is satisfied with H^0, H_1^0, H_2^0 in place of H, H_1, H_2 (in particular, if H_i^0 had a clique-cutset D, then $D \cup (C_i - C)$ would be a clique-cutset in H_i, a contradiction); by this claim, the set of vertices of H^0 splits into two disjoint cliques; we may label these cliques A_1 and A_2 in such a way that $R_i \subseteq A_i$. Thus the set of vertices of $H_1 \cup H_2$ splits into two cliques, $A_1 \cup (C_1 - C)$ and $A_2 \cup (C_2 - C)$.

Since H is not the complement of a bipartite graph, it has a vertex outside $H_1 \cup H_2$; in turn, since H is connected, some vertex w outside $H_1 \cup H_2$ must have a neighbor in $C_1 \cup C_2$. Note that w has no neighbor in C (otherwise this neighbor y, a neighbor of y in R_1, a neighbor of y in R_2, and w would form a claw); hence symmetry allows us to assume that w has a neighbor in $C_1 - C$. By Claim 3.1(iii), all vertices in C are H_1-universal. Recalling that $R_1 \cup (C_1 - C)$ is a clique, we conclude that H_1 is a clique, and so all the vertices of R_1 are simplicial, a contradiction.

Claim 3.5. *Let H be a connected claw-free graph such that H contains no simplicial vertex and H is not the complement of a bipartite graph. If H contains a clique-cutset, then it contains at least two vertex-disjoint tips.*

Justification. A corollary of Claim 2.2 in [6] asserts that every connected claw-free graph with a clique-cutset contains at least two tips; in particular, H contains at least two tips; by Claim 3.4, these two tips are vertex-disjoint.

In particular, Claim 3.5 guarantees that G contains at least two vertex-disjoint tips; Claim 3.1 (iv) and (v) guarantee that each tip of G contains a hole with two vertices in the hinge of the tip. Throughout the remainder of this section, we reserve symbols

- H_1 and H_2 for arbitrary but fixed vertex-disjoint tips of H,
- C_i for the hinge of H_i,
- H_i^0 for an arbitrary but fixed hole in H_i that has two vertices in C_i, and
- N_i for the set of vertices outside H_i that have at least one neighbor in H_i,
- $w_1 \dots w_k$ for an arbitrary but fixed chordless path such that
  w_1 is the only w_j in N_1 and w_k is the only w_j in N_2,
- w_0 for an arbitrary but fixed vertex in $C_1 \cap H_1^0$,
- w_{k+1} for an arbitrary but fixed vertex in $C_2 \cap H_2^0$,
- P for the path $w_0 w_1 \dots w_k w_{k+1}$,
- N for the set of all the vertices in $G - (H_1 \cup H_2 \cup P)$
  that have at least one neighbor in P.

The following claim guarantees that $H_1 \cap N_2 = \emptyset$ and $H_2 \cap N_1 = \emptyset$; in particular, P is a *chordless* path.

Claim 3.6. *No edge of G joins a vertex of H_1 to a vertex of H_2.*

Justification. Let u_1, u_2 denote the two vertices in $H_1^0 \cap C_1$ and let v_1, v_2 denote the two vertices in $H_2^0 \cap C_2$. If G has an edge xy with $x \in H_1$ and $y \in H_2$, then trivially $x \in C_1$ and $y \in C_2$. Now Claim 3.1(iii) guarantees first that $\{u_1, u_2, y\}$ and $\{v_1, v_2, x\}$ are cliques and then that $\{u_1, u_2, v_1, v_2\}$ is a clique. But then the subgraph of G induced by $H_1^0 \cup H_2^0$ is an odd refinement of F_2, a contradiction.

Claim 3.7. *k is odd.*

Justification. If k were even, then Claim 3.1(iii) would guarantee that the subgraph of G induced by $H_1^0 \cup H_2^0 \cup P$ is an odd refinement of F_3, a contradiction.

The subsequent analysis of G resembles the analysis in the preceding section; the role played by H there is played by P here. Claims 3.8, 3.9, 3.10, and 3.12 are like Claims 2.1, 2.2, 2.3, and 2.4, respectively. Claim 3.14 is like Claim 2.5, Claim 3.15 is like Claim 2.6, and so on until Claim 3.27, which is like Claim 2.18. In addition, Claim 3.31 is like Claim 2.23.

We shall say that a vertex x is a

- *2-vertex* if it has two neighbors in P
- and these neighbors are w_i, w_{i+1} for some i;
- *3-vertex* if it has three neighbors in P
- and these neighbors are w_i, w_{i+1}, w_{i+2} for some i;
- *4-vertex* if it has four neighbors in P
- and these neighbors are $w_i, w_{i+1}, w_j, w_{j+1}$ for some i and j
  such that i is odd, j is even, and $i < j$.

Claim 3.8. *Every vertex in N is a 2-vertex or a 3-vertex or a 4-vertex.*

Justification. If x is a vertex of N adjacent to some w_i with $1 \leqslant i \leqslant k$, then it must be adjacent to at least one of w_{i-1} and w_{i+1} (else $w_i w_{i+1} w_{i-1} x$ would be a claw); furthermore, x cannot have three pairwise nonadjacent neighbors (else x would be the center of a claw); by Claim 3.1(iii), if x is adjacent to w_0 then it is adjacent to w_1; if it is adjacent to w_{k+1} then it is adjacent to w_k. From these observations, it follows that the set of neighbors of x in P is $\{w_i, w_{i+1}\}$ for some i or $\{w_i, w_{i+1}, w_{i+2}\}$ for some i, or $\{w_i, w_{i+1}, w_j, w_{j+1}\}$ for some i, j such that $i + 2 \leqslant j \leqslant k$. In the last case, i and j must have different parities (else H_1^0, H_2^0 and $w_1, \dots, w_i, x, w_{j+1}, \dots, w_k$ would induce in G an odd refinement of F_3). Now i must be odd, for otherwise H_1^0 and $x w_{i+1} \dots w_j$ along with $w_1 w_2 \dots w_i$ would induce in G an odd refinement of F_2 (when $i = 0$) or F_3 (when $i \geqslant 2$), a contradiction.

Claim 3.9. *Let x be a 2-vertex adjacent to w_i and w_{i+1}. If x has a neighbor y in N such that $yw_i \notin E$, then y is a 3-vertex adjacent to $w_{i+1}, w_{i+2}, w_{i+3}$.*

Justification. By Claim 3.8, y is a 2-vertex or a 3-vertex or a 4-vertex. If y is a 2-vertex, then the graph induced by $H_1^0 \cup H_2^0 \cup P$ along with x and y contains an odd hole of length at least five, or an odd refinement of F_2 or an odd refinement of F_3. If y is a 3-vertex or a 4-vertex, we argue as in the justification of Claim 2.2.

Claim 3.10. *For every 2-vertex x adjacent to w_i and w_{i+1}, with $1 \leqslant i \leqslant k-1$, precisely one of the following three statements holds true:*

(a) *x has a neighbor in N nonadjacent to w_i,*
(b) *x has a neighbor in N nonadjacent to w_{i+1},*
(c) *x has a neighbor in $G - (P \cup N \cup H_1 \cup H_2)$.*

Justification. First, let us derive a contradiction from the assumption that all three of (a), (b), and (c) are false. Since x is not a simplicial vertex, it must have two nonadjacent neighbors, say y and z; since y and z are nonadjacent neighbors of a 2-vertex, at least one of them is outside P, say $y \notin P$; since (c) is false, $y \in N \cup H_1 \cup H_2$. Since $xw_0 \notin E$ and $xw_{k+1} \notin E$, Claim 3.1(iii) guarantees that $y \notin H_1 \cup H_2$; since (a) and (b) are both false, y is adjacent to both w_i and w_{i+1}. Now (since y and z are nonadjacent) $z \notin P$, and so (since all three of (a), (b), and (c) are false and by Claim 3.1(iii)) z is a vertex in N adjacent to both w_i and w_{i+1}. As in the justification of Claim 2.3 (with H replaced by P) we may assume that z is a 3-vertex adjacent to w_i, w_{i+1}, w_{i+2} and that y is a 3-vertex adjacent to w_{i-1}, w_i, w_{i+1}. Hence

$$w_1 \ldots w_{i-1} yxz w_{i+2} \ldots w_k$$

is a chordless odd path; this path along with H_1^0 *and* H_2^0 induces an odd refinement of F_3, a contradiction.

Next, let us derive a contradiction from the assumption that (a) and (b) are both true: x has neighbors y and z in N such that $yw_i \notin E$ and $zw_{i+1} \notin E$. By Claim 3.9, y is a 3-vertex adjacent to $w_{i+1}, w_{i+2}, w_{i+3}$ and z is a 3-vertex adjacent to w_{i-2}, w_{i-1}, w_i. Since $yw_{i+1}w_{i+3}z$ is not a claw, y and z are nonadjacent. But then

$$w_1 \ldots w_{i-2} zxy w_{i+3} \ldots w_k$$

is a chordless odd path; this path together with H_1^0 and H_2^0 induces an odd refinement of F_3, a contradiction.

Finally, let us derive a contradiction from the assumption that (a) and (c) are both true: x has neighbors y and z such that $y \in N$, $yw_i \notin E$, and $z \notin P \cup N \cup H_1 \cup H_2$. By Claim 3.9, y is a 3-vertex adjacent to $w_{i+1}, w_{i+2}, w_{i+3}$. But then one of xw_iyz and $yzw_{i+1}w_{i+3}$ is a claw, a contradiction.

For each $j = 1, 2, \ldots, k$, let Q_j denote the set that consists of

- w_j;
- all the 2-vertices x adjacent to w_{j-1} and w_j such that some 3-vertex is adjacent to x, w_j, w_{j+1}, w_{j+2};
- all the 2-vertices x adjacent to w_j and w_{j+1} such that some 3-vertex is adjacent to x, w_{j-2}, w_{j-1}, w_j;
- all the 3-vertices adjacent to w_{j-1}, w_j, w_{j+1}, and for each $j = 0, 2, \ldots, k$, let T_j denote the set that consists of
- all the 2-vertices adjacent to w_j and w_{j+1} that have a neighbor outside $P \cup N \cup H_1 \cup H_2$.

In addition, let S denote the set of all the 4-vertices; set $J_1 = N_1 - Q_1$, $J_2 = N_2 - Q_k$.

Claim 3.11. *Both T_0 and T_k are empty.*

Justification. By symmetry, we only need to prove the claim for T_0. Assume the contrary, let x be a vertex of T_0. Let j be the smallest index such that there is a chordless path P' connecting x and w_j with all internal vertices outside $P \cup N \cup H_1 \cup H_2$. If such an index j exists, then $H_1^0 \cup P \cup P'$ either contains an odd hole or an odd refinement of F_2. Else, T_0 is a clique (by Claim 3.1(iii)) cutset (by definition) of G. Let G be the union of graphs G_1, G_2 so that $G_1 \cap G_2 \subseteq T_0$ is a minimal cutset of G, and both H_1 and H_2 are subgraphs of G_2. A corollary of Claim 2.2 in [6] asserts that G_1 contains a tip of G; Claim 3.1(iv) and (v) guarantee that G_1 contains a hole H' with at least four vertices. Let P' be a shortest chordless path in G_1 connecting a vertex of H' and a vertex of $G_1 \cap G_2$ so that no internal vertex of P' belongs to $G_1 \cap G_2$. But either $H' \cup P' \cup H_1^0$ or $H' \cup P' \cup \{w_1, \ldots, w_k\} \cup H_2^0$induces an odd refinement of F_3.

Claim 3.12. *Each vertex of $P \cup N$ belongs to precisely one of*

$$Q_1, \ldots, Q_k, T_1, \ldots, T_{k-1}, S, J_1, J_2.$$

Justification. Straightforward from Claims 3.8, 3.10, 3.9 and 3.11.

Our next aim is to describe the structure of the graph induced by $P \cup (N - S)$, which we denote by M.

Claim 3.13. (a) *If x is a vertex in J_i, then x has no neighbor outside $H_i \cup N_i$.*
(b) *Every vertex of J_i has a neighbor in C_i that is H_i-universal.*
(c) *If $H_i \cup N_i$ is a large appendix, then J_i is empty.*

Justification. (a) Symmetry allows us to set $i = 1$. Let x be an arbitrary vertex of J_1. Note that $xw_0 \in E$ (by Claim 3.1(iii)) and that $x \neq w_1$ (since $x \notin Q_1$). Hence, $x \notin P$. Claim 3.12 guarantees that x is not a 4-vertex; since x does not belong to Q_1, it is not a 3-vertex. Thus x is a 2-vertex adjacent to w_0 and w_1.

Now assume that x has a neighbor y outside $H_1 \cup N_1$. Since $x \notin T_i$, it follows that $y \in N \cup P \cup H_2$; since x is a 2-vertex, $y \notin P$ and $y \notin H_2$ (by Claim 3.1). We conclude that $y \in N$. Since $y \notin N_1$, we have $yw_0 \notin E$; now Claim 3.9 guarantees that $x \in Q_1$, a contradiction.

(b) Straight from (a), Claim 3.1(iii), and the assumption that G has no simplicial vertex.

(c) By Claim 3.2 and the definition of a large appendix, H_i contains a hole with at least six vertices. Since G has no claw, no vertex of C_i is H_i-universal. Hence, (b) implies that J_i is empty.

Claim 3.14. *Let x and y be adjacent vertices in M.*
If $x \in T_j$ then $y \in Q_j \cup T_j \cup Q_{j+1}$.
If $x \in Q_j$ then $y \in Q_{j-1} \cup T_{j-1} \cup Q_j \cup T_j \cup Q_{j+1}$ $(1 \leqslant j \leqslant k)$ where $Q_0 = J_1$ and $Q_{k+1} = J_2$.
If $x \in J_1$ then $y \in J_1 \cup Q_1$; if $x \in J_2$ then $y \in J_2 \cup Q_k$.

Justification. If at least one of x and y belongs to P, then Claim 3.14 follows trivially from Claims 3.1(iii) 3.8, 3.11, 3.12. If both x and y belong to $N - S$, then Claim 3.14 is implied by Claims 3.1(iii) 3.13, 3.11, and the following three claims:

(i) if x and y are adjacent 2-vertices then x and y have the same neighbors in P;
(ii) if x is a 2-vertex adjacent to w_i, w_{i+1} and if y is a 3-vertex adjacent to x then $y \in Q_{i-1} \cup Q_i \cup Q_{i+1}$;
(iii) if x is a 3-vertex in Q_j and if y is a 3-vertex adjacent to x then $y \in Q_{j-1} \cup Q_j \cup Q_{j+1}$.

Validity of (i) and (ii) follows directly from Claim 3.9. Let us justify (iii). Trivially, $y \in Q_i$ for some i. Since $xw_{j-1}w_{j+1}y$ is not a claw, i must be one of $j-2, j-1, j, j+1, j+2$; hence symmetry allows us to assume that i is one of $j, j+1, j+2$. If $i = j+2$ then

$$w_1 \ldots w_{j-1}xyw_{j+3} \ldots w_k$$

along with H_1^0 and H_2^0 induces an odd refinement of F_3, a contradiction.

Claim 3.14 tells us that certain pairs of vertices in M must be nonadjacent. Now we shall prove that certain pairs of vertices in M must be adjacent:

- for each $j = 1, \ldots, k$, at least one of $Q_{j-1} \cup T_{j-1} \cup Q_j$ and $Q_j \cup T_j \cup Q_{j+1}$ is a clique, with $Q_0 = J_1$ and $Q_{k+1} = J_2$ (Claim 3.19);
- *if* $T_j \neq \emptyset$, then $Q_j \cup T_j \cup Q_{j+1}$ is a clique (a corollary of Claims 3.15, 3.18).

For each $j = 1, 2, \ldots, k-1$, set

$j \in S_1$ if there are nonadjacent vertices x_j and y_j such that x_j is a 3-vertex adjacent to w_{j-1}, w_j, w_{j+1} and y_j is a 3-vertex adjacent to w_j, w_{j+1}, w_{j+2};

$j \in S_2$ if there are adjacent vertices x_j and y_j such that x_j is a 2-vertex adjacent to w_{j-1}, w_j and y is a 3-vertex adjacent to w_j, w_{j+1}, w_{j+2};

$j \in S_3$ if there are adjacent vertices x_j and y_j such that x_j is a 3-vertex adjacent to w_{j-1}, w_j, w_{j+1} and y_j is a 2-vertex adjacent to w_{j+1}, w_{j+2}.

We shall say that $w_j w_{j+1}$ is *special* if $j \in S_1 \cup S_2 \cup S_3$; all the remaining edges $w_j w_{j+1}$ will be called *normal*. (Note that $w_0 w_1$ and $w_k w_{k+1}$ are neither special nor normal.)

Claim 3.15. *If some vertex is adjacent to* w_j, w_{j+1} *and not adjacent to* w_{j-1}, w_{j+2} *with* $1 \leqslant j \leqslant k-1$, *then* $w_j w_{j+1}$ *is a normal edge.*

Justification. Assume the contrary: some vertex z is adjacent to w_j, w_{j+1} and not adjacent to $w_{j-1} w_{j+2}$ with $1 \leqslant j \leqslant k-1$, and yet $w_j w_{j+1}$ is a special edge. We have three cases to consider.

Case 1: $j \in S_1$. We must have $zx_j \in E$ (to avoid a claw on $w_{j+1} w_{j+2} z x_j$) and $zy_j \in E$ (to avoid a claw on $w_j w_{j-1} z y_j$). Now z must be a 2-vertex (if $zw_t \in E$ and $t \neq j, j+1$ then $zx_j y_j w_t$ is a claw). But then

$$w_1 \dots w_{j-1} x_j z y_j w_{j+2} \dots w_k$$

along with H_1^0 and H_2^0 induces an odd refinement of F_3, a contradiction.

Case 2: $j \in S_2$. We must have first $zy_j \in E$ (to avoid a claw on $w_j w_{j-1} z y_j$) and then $zx_j \in E$ (to avoid a claw on $y_j w_{j+2} z x_j$). Now z must be a 2-vertex (if $zw_t \in E$ and $t \neq j, j+1$ then $zx_j w_{j+1} w_t$ is a claw). But then

$$w_1 \dots w_{j-1} x_j z w_{j+1} \dots w_k$$

along with H_1^0 and H_2^0 induces an odd refinement of F_3, a contradiction.

Case 3: $j \in S_3$. This is a mirror image of Case 2.

Claim 3.16. *No two special edges share a vertex.*

Justification. We only need to derive a contradiction for the assumption that, for some j, both j and $j+1$ belong to $S_1 \cup S_2 \cup S_3$. By Claim 3.15, $j \notin S_3$ and $j+1 \notin S_2$. Four cases remain.

Case 1: $j \in S_1$, $j+1 \in S_1$. We must have $x_j y_{j+1} \notin E$, for otherwise

$$w_1 \dots w_{j-1} x_j y_{j+1} w_{j+3} \dots w_k$$

along with H_1^0 and H_2^0 induces an odd refinement of F_3, a contradiction.

Then we must have $x_j x_{j+1} \in E$ (to avoid a claw on $w_{j+1} x_j x_{j+1} y_{j+1}$) and $y_j y_{j+1} \in E$ (to avoid a claw on $w_{j+1} y_{j+1} x_j y_j$). In particular, $y_j \neq x_{j+1}$, and so $y_j x_{j+1} \in E$ to avoid a claw on $w_j w_{j-1} y_j x_{j+1}$. But then

$$w_1 \dots w_{j-1} x_j x_{j+1} y_j y_{j+1} w_{j+3} \dots w_k$$

along with H_1^0 and H_2^0 induces an odd refinement of F_3, a contradiction.

Case 2: $j \in S_1$, $j+1 \in S_3$. We must have $x_j y_{j+1} \notin E$ (to avoid a claw on $x_j w_{j-1} w_{j+1} y_{j+1}$); now $x_j x_{j+1} \in E$, for otherwise

$$w_1 \dots w_{j-1} x_j w_{j+1} x_{j+1} y_{j+1} w_{j+3} \dots w_k$$

along with H_1^0 and H_2^0 induces an odd refinement of F_3, a contradiction.

In particular, $y_j \neq x_{j+1}$. Now we must have first $y_j x_{j+1} \in E$ (to avoid a claw on $w_{j+2} w_{j+3} y_j x_{j+1}$) and then $y_j y_{j+1} \in E$ (to avoid a claw on $x_{j+1} x_j y_j y_{j+1}$). But then

$$w_1 \dots w_{j-1} x_j w_{j+1} y_j y_{j+1} w_{j+3} \dots w_k$$

along with H_1^0 and H_2^0 induces an odd refinement of F_3, a contradiction.

Case 3: $j \in S_2$, $j+1 \in S_1$. This is a mirror image of Case 2.

Case 4: $j \in S_2$, $j+1 \in S_3$. We must have $x_j y_{j+1} \notin E$ for otherwise

$$w_1 \dots w_{j-1} x_j y_{j+1} w_{j+3} \dots w_k$$

along with H_1^0 and H_2^0 induces an odd refinement of F_3, a contradiction.

Then we must have $x_j x_{j+1} \notin E$ (to avoid a claw on $x_{j+1} x_j w_{j+1} y_{j+1}$) and $y_j y_{j+1} \notin E$ (to avoid a claw on $y_j y_{j+1} w_{j+1} x_j$). In particular, $y_j \neq x_{j+1}$, and so $y_j x_{j+1} \in E$ to avoid a claw on $w_j w_{j-1} y_j x_{j+1}$. But then

$$w_1 \dots w_{j-1} x_j y_j x_{j+1} y_{j+1} w_{j+3} \dots w_k$$

along with H_1^0 and H_2^0 induces an odd refinement of F_3, a contradiction.

Claim 3.17. *Every normal edge is contained in a unique maximal clique of G.*

Justification. Same as that of Claim 2.9 (except that now $1 \leqslant j \leqslant k-1$).

Claim 3.18. *If $w_j w_{j+1}$ is a normal edge then $Q_j \cup T_j \cup Q_{j+1}$ is a clique.*

Justification. Same as that of Claim 2.9 (except that now we rely on Claim 3.17 rather than Claim 2.8 and $1 \leqslant j \leqslant k-1$).

Claim 3.19. *Set $Q_0 = J_1$ and $Q_{k+1} = J_2$.*
For each $j = 1, \dots, k$, at least one of $Q_{j-1} \cup T_{j-1} \cup Q_j$ and $Q_j \cup T_j \cup Q_{j+1}$ is a clique.

Justification. Straight from Claims 3.1(iii), 3.11, 3.16, and 3.18.

Next, we turn our attention to 4-vertices.

Claim 3.20. *Let z be a 4-vertex and let C be the set of all the neighbors of z. Then there is an odd subscript r and an even subscript s with $1 \leqslant r < s \leqslant k-1$ such that*

$$C = (Q_r \cup T_r \cup Q_{r+1}) \cup (Q_s \cup T_s \cup Q_{s+1})$$

and such that

both $w_r w_{r+1}$ and $w_s w_{s+1}$ are normal edges.

Justification. Same as that of Claim 2.11 with three modifications: first, we depend on Claims 3.15, 3.17, 3.18, 3.8, and 3.12 rather than Claims 2.6, 2.8, 2.9, 2.1, and 2.4; second, the role of H is replaced by P. Finally, indices r and s are such that $1 \leqslant r < s \leqslant k-1$, r odd and s even.

Claim 3.21. *If a vertex x of $P \cup N$ has a neighbor outside $P \cup N \cup H_1 \cup H_2$, then $x \in T_i$ for some $i = 1, \dots, k-1$.*

Justification. Let y denote a neighbor of x outside $P \cup N \cup H_1 \cup H_2$. Trivially, x must be a 2-vertex (else x, its two nonadjacent neighbors in P, and y form a claw). By Claims 3.12 and 3.13(a), $x \in T_i$ or $x \in Q_i$ for some i. If $x \in Q_i$, then x is adjacent to a 3-vertex z such that z is nonadjacent to a neighbor w of x on P; but then $xyzw$ is a claw, a contradiction.

Let us enumerate all the connected components of $G - (P \cup N \cup H_1 \cup H_2)$ as $R_1, \dots, R_t$; for each $i = 1, \dots, t$, let C_i denote the set of vertices outside R_i that have at least one neighbor in R_i. By a *bridge of P*, we shall mean

- either the subgraph of G induced by some $R_i \cup C_i$
- or a 4-vertex.

Claim 3.22. *The vertex-sets of all the bridges of P are pairwise disjoint and their union is the vertex-set of $G - (H_1 \cup J_1 \cup Q_1 \cup \dots \cup Q_k \cup J_2 \cup H_2)$.*

Justification. Same as that of Claim 2.13 except that the role of H is played by P and that we rely on Claims 3.12 and 3.21 instead of Claims 2.4 and 2.12.

Claim 3.23. *For every $i = 1, \ldots, t$, there is an odd subscript r and an even subscript s with $1 \leqslant r < s \leqslant k - 1$ such that*

$$C_i \cap T_r \neq \emptyset, \quad C_i \cap T_s \neq \emptyset, \quad C_i \subseteq T_r \cup T_s$$

and such that

both $w_r w_{r+1}$ and $w_s w_{s+1}$ are normal edges.

Justification. Claim 3.21 guarantees that $C_i \subseteq T_1 \cup \cdots \cup T_k$. Let I denote the set of all subscripts r such that $C_i \cap T_r \neq \emptyset$. Claims 3.15, 3.18 guarantee that $Q_r \cup T_r \cup Q_{r+1}$ is a clique whenever $r \in I$.

First, let us show that $|I| > 1$. If $I = \{r\}$, then T_r is a clique (by Claim 3.1(iii)) cutset (by definition) of G. Let G be the union of graphs G_1, G_2 so that $G_1 \cap G_2 \subseteq T_r$ is a minimal cutset of G, and both of H_1 and H_2 are subgraphs of G_2. A corollary of Claim 2.2 in [6] asserts that G_1 contains a tip of G. By Claim 3.1(iii), G_1 contains a hole H' with at least four vertices. Let P' be a shortest chordless path in G_1 connecting a vertex of H' and a vertex of $G_1 \cap G_2$ so that no internal vertex of P' belongs to $G_1 \cap G_2$. But either $H' \cup P' \cup H_1^0$ or $H' \cup P' \cup \{w_1, \ldots, w_k\} \cup H_2^0$ induces an odd refinement of F_3.

Next, let us show that I cannot include two subscripts of the same parity. Assume the contrary: two subscripts, r and s, in I have the same parity. Let F be the subgraph of G induced by $H_1^0 \cup P \cup H_2^0$ along with a chordless path P' from $C_i \cap T_r$ to $C_i \cap T_s$ with all interior vertices in R_i. F contains an odd hole (when P' has an even number of edges) or an odd refinement of F_3 (when P' has an odd number of edges), a contradiction.

Hence, $I = \{r, s\}$ such that r and s are of different parities. Let P' be a chordless path from $C_i \cap T_r$ to $C_i \cap T_s$ with all interior vertices in R_i. If r were even, then the subgraph induced by $H_1^0 \cup \{w_1, \ldots, w_r\} \cup P'$ is either an odd refinement of F_2 or an odd refinement of F_3.

Claim 3.24. *For every bridge of P, precisely four vertices in P have at least one neighbor in the bridge. These four vertices are $w_r, w_{r+1}, w_s, w_{s+1}$ with $1 \leqslant r < s \leqslant k - 1$, r odd, s even; both $w_r w_{r+1}, w_s, w_{s+1}$ are normal edges.*

Justification. Straight from Claims 3.23 and 3.20.

We shall refer to the two edges $w_r w_{r+1}$ and $w_s w_{s+1}$ in Claim 3.24 as the *edges of attachment* of the bridge. By removing the edges of attachment of a bridge, P is disconnected into three disjoint even paths; two such subpaths that contain either w_0 or w_{k+1} are of odd lengths (by Claim 3.24); we will call them the *external segments* of the bridge; we will call the third path $(w_{r+1} \ldots w_s)$ the *internal segment* segment of the bridge; an internal segment has an even length (by Claim 3.24). (In the special case when $s = r + 1$, the internal segment consists of just w_s.)

By a *spine* of a bridge X that is induced by some $R_i \cup C_i$, we shall mean a chordless path from T_r to T_s with all internal nodes in $X - (T_r \cup T_s)$ and r, s as in Claim 3.23 (with $R = R_i$, $C = C_i$); by the *spine* of the bridge that is a 4-vertex x, we shall mean the degenerate path consisting of this single vertex x.

Claim 3.25. *Every spine of every bridge has an even number of edges.*

Justification. Assuming the contrary, the subgraph of G induced by the odd spine and the internal segment of the bridge is an odd hole, a contradiction.

Claim 3.26. *No two bridges share precisely one edge of attachment.*

Justification. Assume the contrary, two bridges X and Y share one edge of attachment. By Claim 3.24 and by symmetry, we may assume that there are mutually distinct subscripts a, b, c with $1 \leqslant a < c < b \leqslant k - 1$, a, c odd, b even, so that bridge X has edges of attachment $w_a w_{a+1}$, $w_b w_{b+1}$ and bridge Y has edges of attachment $w_c w_{c+1}$, $w_b w_{b+1}$. Let $x_1 \ldots x_s$ be a spine of X so that $x_1 w_a w_{a+1}$ form a triangle and $x_s w_b w_{b+1}$ form a triangle; let $y_1, \ldots, y_t$ be a spine of Y oriented towards the neighbor of w_c in this spine. Then, $H_1^0, w_1, \ldots, w_a, x_1, \ldots, x_s$ and hole $w_{c+1}, \ldots, w_b, y_1, \ldots, y_t$ induce in G an odd refinement of F_3, a contradiction.

We shall say that two bridges *cross* if exactly one of the edges of attachment of one bridge lies in the internal segment of the other bridge.

Claim 3.27. *No two bridges cross.*

Justification. Assume the contrary: bridge X has edges of attachment $w_a w_{a+1}$, $w_b w_{b+1}$, bridge Y has edges of attachment $w_c w_{c+1}$, $w_d w_{d+1}$ such that a, c are odd, b, d are even, and $1 \leqslant a < c < b < d \leqslant k-1$. By Claim 3.26 and the assumption that X and Y cross, we must have $a \neq c$ and $b \neq d$. Let $x_1 \dots x_s$ be a spine of X such that $x_0 w_a w_{a+1}$ form a triangle and $x_s w_b w_{b+1}$ form a triangle; let $y_1, \dots, y_t$ be a spine of Y oriented towards the neighbor of w_c in this spine. Then, $H_1^0, w_1, \dots, w_a, x_1, \dots, x_s$ and hole $w_{c+1}, \dots, w_d, y_1, \dots, y_t$ induce in G an odd refinement of F_3, a contradiction.

Two bridges are *nested* if both of the two edges of attachment of one bridge lie in the internal segment of the other bridge.

Claim 3.28. *No two bridges are nested.*

Justification. Assume the contrary: bridge X has edges of attachment $w_a w_{a+1}$, $w_b w_{b+1}$, bridge Y has edges of attachment $w_c w_{c+1}$, $w_d w_{d+1}$ such that a, c are odd, b, d are even, and $1 \leqslant a < c < d < b \leqslant k-1$. Let $x_1 \dots x_s$ be a spine of X such that $x_1 w_a w_{a+1}$ is a triangle and $x_s w_b w_{b+1}$ is a triangle; let $y_1, \dots, y_t$ be a spine of Y oriented towards the neighbor of w_c in this spine. Then, $H_1^0, w_1, \dots, w_a, x_1, \dots, x_s, w_b, w_{b-1}, \dots, w_{d+1}$ and hole $w_{c+1}, \dots, w_d, y_1, \dots, y_t$ induce in G an odd refinement of F_3, a contradiction.

Let $A(P)$ denote the set of all the edges of attachment of bridges of P. With this notation, Claims 3.26–3.28 can be summarized as follows.

Claim 3.29. *The elements of $A(P)$ can be enumerated, in their natural order along P from w_1 to w_k, as*

$$e_1, \; f_1, \; e_2, \; f_2, \dots, e_d, f_d,$$

so that every bridge of P has some e_t and f_t (with the same t) for its two edges of attachment.

Claim 3.30. *Let r, s be subscripts of different parities such that $1 \leqslant r < s \leqslant k-1$ and*

($\star$) *no edge on the path $w_r \dots w_{s+1}$ belongs to $A(P)$*

and let B be the subgraph of G induced by $Q_r \cup Q_{r+1} \cup \dots \cup Q_{s+1}$. Then

(a) $T_r = \cdots T_s = \emptyset$;
(b) *B is an elementary bead with poles Q_r, Q_{s+1};*
(c) *no vertex in $B - (Q_r \cup Q_{s+1})$ has a neighbor outside B.*

Justification. (a) Same as that of Claim 2.22(a) except that we rely on Claim 3.22 instead of Claim 2.13.

(b) Same as that of Claim 2.22(b) with the following modifications. We rely on Claims 3.12,, 3.19, 3.14 instead of Claims 2.4, 2.10, 2.5 in proving conditions (i), (ii) and (iii) of an elementary bead. To show that condition (iv) holds for B, assume the contrary: if there are vertices x, y in Q_1 and vertices u, v in Q_2 such that u is adjacent to x but not to y and such that v is adjacent to y but not to x, then the subgraph induced by u, v, x, y along with H_1 contains an odd refinement of F_2. Similarly, if there are vertices x, y in some Q_k and vertices u, v in Q_{k-1} such that u is adjacent to x but not to y and such that v is adjacent to y but not to x, then the subgraph induced by u, v, x, y along with H_2 contains an odd refinement of F_2. For all other cases, we argue as in Claim 2.22(b) using appropriate claims listed above; the last sentence should read " But then the hole $xyvu$ and paths $w_1 \dots w_{j-1}$, $w_{j+2} \dots w_k$ induce an odd refinement of F_3, a contradiction."

(c) Consider an arbitrary vertex x in $B - (Q_r \cup Q_{s+1})$. By Claim 3.22, x belongs to no bridge of P, and so all its neighbors come from $P \cup N \cup H_1 \cup H_2$; by Claim 3.12 and by definition, x has no neighbor in $H_1 \cup H_2$; Claim 3.20 and assumption ($\star$) guarantee that x is adjacent to no 4-vertex. But then (a) and Claim 3.14 guarantee that all the neighbors of x come from B.

Claim 3.31. *Let X be a bridge of P with edges of attachment $w_r w_{r+1}$, $w_s w_{s+1}$ with $1 \leqslant r < s \leqslant k-1$, r odd, s even. If X is not a 4-vertex, then X is an elementary bead with poles Q'_r, Q'_{r+n} such that*

- $Q'_r \subseteq T_r$,
- $Q'_{r+n} \subseteq T_s$,
- *no vertex in $X - (Q'_r \cup Q'_{r+n})$ has a neighbor outside X.*

Justification. By Claim 3.29,

(i) no edge on the path $w_{r+1} \dots w_s$ belongs to $A(P)$.

Let P_X be a spine of X and let P' denote the chordless path induced by P_X and the two external segments $w_0 \dots w_r$ and $w_{s+1} \dots w_{k+1}$. Enumerate the vertices of P' as $w'_0, \dots, w'_{p+1}$ so that $w'_i = w_i$ for $i=0, \dots, r$, $P_X = w'_{r+1}, \dots, w'_{r+n}$, and $w'_{r+n+i} = w_{s+i}$ for $i = 1, \dots, k+1-s$.

By Claim 3.15,

(ii) $w_r w_{r+1}$ and $w_s w_{s+1}$ are normal edges of P,
(iii) $w'_r w'_{r+1}$ and $w'_{r+n} w'_{r+n+1}$ are normal edges of P';

in turn, (ii), (iii), and Claim 3.17 imply that

(iv) a vertex is adjacent to w_r, w_{r+1} if and only if it is adjacent to w'_r, w'_{r+1};
(v) a vertex is adjacent to w_s, w_{s+1} if and only if it is adjacent to w'_{r+n}, w'_{r+n+1}.

We claim that

(vi) $w'_r w'_{r+1}, w'_{r+n} w'_{r+n+1} \in A(P')$.

To justify (vi), we propose to show that P' has a bridge with spine $w_{r+1} \dots w_s$. If $w_{r+1} = w_s$, then w_{r+1} is a 4-vertex of P' and we are done; hence we may assume that $w_{r+1} \neq w_s$. Now let w_j be an arbitrary interior vertex of $w_{r+1} w_{r+2} \dots w_s$. By definition, w_j has no neighbor in any external segment of X. Since X is not a 4-vertex, $n \geqslant 3$; (since X is a bridge of H) w_j neither belongs to nor has a neighbor in the interior of $w'_{r+1} \dots w'_{r+n}$ and (by Claim 3.23) w'_{r+1} and w'_{r+n} are distinct 2-vertices of P nonadjacent to w_j. To summarize, w_j neither belongs to nor has a neighbor in P', and so we are done.

Define $Q'_1, \dots, Q'_p$ in the same way we defined $Q_1, \dots, Q_k$, except that P' is now used in place of P; then let B stand for the subgraph of G induced by

$Q'_{r+1} \cup \dots \cup Q'_{r+n}$. Claim 3.25 guarantees that n is odd; since X is not a 4-vertex, $n \geqslant 3$; Claim 3.29 and (vi) guarantee that no edge on the path $w'_{r+1} \dots w'_{r+n}$ belongs to $A(P')$. Hence $P', r+1, r+n-1$ satisfy assumptions of Claim 3.30 in place of P, r, s; in turn, this claim guarantees that

(vii) B is an elementary bead with poles Q'_{r+1}, Q'_{r+n};
(viii) no vertex in $B - (Q'_{r+1} \cup Q'_{r+n})$ has a neighbor outside B.

By definition, X is induced by some $R_i \cup C_i$; Claim 3.23 guarantees that

(ix) $C_i \subseteq T_r \cup T_s$.

Replacing Q'_i with Q'_{r+i} and w'_i with w'_{r+i} for $i = 1, \dots, n$ and using the arguments for proving (xii), (xiii) of Claim 2.23, we conclude that

(x) $Q'_{r+2} \cup \dots \cup Q'_{r+n-1} \subseteq R_i$,
(xi) $Q'_{r+1} \subseteq C_i \cap T_r$ and $Q'_{r+n} \subseteq C_i \cap T_s$.

From (x), (viii), and (xi), it follows that $Q'_{r+2} \cup \dots \cup Q'_{r+n-1} = R_i$; in turn, this identity, (viii), and (xi) imply that $Q'_{r+1} \cup Q'_{r+n} = C_i$.

Claim 3.32. *If P has no bridge, then G is a string.*

Justification. Let B_0 be the subgraph of G induced by $H_1 \cup J_1 \cup Q_1$; let B_1 be the subgraph of G induced by $Q_1 \cup Q_2 \cup Q_k$ and let B_2 be the subgraph of G induced by $H_2 \cup J_2 \cup Q_k$. Since P has no bridge, Claims 3.22 and 3.12 guarantee that every vertex of G belongs to precisely one of B_0, B_1, B_2. By Claim 3.2, B_0 is either a large appendix with pole N_1 or a small appendix whose pole is a subset of N_1. In the former case, Claim 3.13(c) guarantees that Q_1 is the pole of the large appendix; in the latter case, Claim 3.13(b) guarantees that Q_1 is a pole of the small appendix. Similarly, B_2 is either a large appendix or a small appendix with pole Q_k.

If $k=1$, then Claim 3.6 guarantees that G is a string. If $k>1$, then Claim 3.30(b) guarantees that B_1 is an elementary bead with poles Q_1 and Q_k. In turn, Claims 3.6 and 3.30(c) guarantee that every edge of G belongs to one of B_0, B_1, B_2.

Claim 3.32 allows us to assume that

P has at least one bridge.

With $e_1, f_1, \ldots, e_d, f_d$ as in Claim 3.29, let J denote the set of all subscripts j such that some e_t is $w_j w_{j+1}$ or some f_t is $w_{j-1}w_j$. (Note that there may be i and j such that $f_t = w_{j-1}w_j$ and $e_{t+1} = w_j w_{j+1}$.)

Claim 3.33. *If subscript $j \in J$, then $1 \leqslant j \leqslant k$ and j is odd.*

Justification. Straight from Claim 3.24.

Enumerate all the sets Q_j with $j \in J \cup \{1, k\}$ as $A_1, \ldots, A_{m+1}$ according to their order along P from w_1 to w_k. Note that $A_1 = Q_1$ and $A_{m+1} = Q_k$. We are going to prove that G is a string made out of two appendices B_0, B_{m+1} and certain beads $B_1, \ldots, B_m$ such that the pole of B_0 is A_1, the pole of B_{m+1} is A_{m+1}, and the two poles of each B_i with $1 \leqslant i \leqslant m$ are A_i and A_{i+1}. These appendices and beads are defined as follows.

Let B_0 be the subgraph induced by $H_1 \cup J_1 \cup Q_1$.

Let B_{m+1} be the subgraph induced by $H_2 \cup J_2 \cup Q_k$.

For $i = 1, \ldots, m$, A_i is some Q_r; its successor A_{i+1} is some Q_{s+1}; let B_i be the subgraph of G induced by the union of $Q_r \cup Q_{r+1} \cup \cdots \cup Q_{s+1}$ and all the bridges of P with edges of attachment $w_r w_{r+1}$, $w_s w_{s+1}$.

Claim 3.34.

(i) *B_0 is an appendix with pole A_1; B_{m+1} is an appendix with pole A_{m+1}; for $i = 1, \ldots, m$, B_i is a bead with poles A_i, A_{i+1}.*
(ii) *No vertex in $B_i - (A_i \cup A_{i+1})$ has a neighbor outside B_i for $i = 0, \ldots, m+1$, where $A_0 = A_{m+2} = \emptyset$.*

Justification. (i) By Claim 3.2, B_0 is either a large appendix with pole N_1 or a small appendix whose pole is a subset of N_1. In the former case, Claim 3.13(c) guarantees that Q_1 is the pole of the large appendix; in the latter case, Claim 3.13(b) guarantees that Q_1 is a pole of the small appendix. Similarly, B_{m+1} is either a large appendix or a small appendix with pole Q_k.

For $i = 1, \ldots, m$ and with r, s as in the definition of B_i, Claim 3.33 guarantees that

(a) $1 \leqslant r < s \leqslant k-1$, r is odd and s is even.

With $e_1, f_1, \ldots, e_d, f_d$ as in Claim 3.29, we shall distinguish between two cases.

Case 1: *$f_t = w_{r-1}w_r$ and $e_{t+1} = w_{s+1}w_{s+2}$ for some t.* In this case, no edge on the path $w_r w_{r+1} \ldots w_{s+1}$ is an edge of attachment of a bridge of P, and so the desired conclusion follows from Claim 3.30.

Case 2: *$e_t = w_r w_{r+1}$ and $f_t = w_s w_{s+1}$ for some t.* Let D_1 be the subgraph of G induced by $Q_{r+1} \cup \cdots \cup Q_s$ and enumerate all the bridges of P with edges of attachment $w_r w_{r+1}$, $w_s w_{s+1}$ as $D_2, \ldots, D_n$. By Claims 3.24 and 3.18,

(b) $Q_r \cup T_r \cup Q_{r+1}$ and $Q_s \cup T_s \cup Q_{s+1}$ are cliques.

Replacing Claims 2.11, 2.12, 2.5 with Claims 3.20, 3.21, 3.14 in the argument for proving (c) of Claim 2.26, we conclude that

(c) if $r+1=s$, then D_1 is a single vertex and the set of its neighbors is $Q_r \cup T_r \cup T_s \cup Q_{s+1}$.

By Claim 3.30 (with $r+1$, $s-1$ in place of r, s),

(d) if $r+1 \neq s$, then D_1 is an elementary bead with poles Q_{r+1}, Q_s

and no vertex in $D_1 - (Q_{r+1} \cup Q_s)$ has a neighbor outside D_1.

By Claim 3.20,

(e) if some D_j with $1 < j \leqslant n$ is a single vertex, then the set of its neighbors is $Q_r \cup T_r \cup Q_{r+1} \cup Q_s \cup T_s \cup Q_{s+1}$.

By Claim 3.31,

(f) if some D_j with $1 < j \leqslant n$ is not a single vertex, then it is an elementary bead with poles U_j, V_j so that $U_j \subseteq T_r$, $V_j \subseteq T_s$, and no vertex in $D_j - (U_j \cup V_j)$ has a neighbor outside D_j.

Using the same argument as in the proof of Claim 2.26 with Claims 2.15, 2.8, 2.11, 2.13, 2.21, in place of Claims 3.24, 3.17, 3.20, 3.22, 3.29, we conclude that

(g) at most one D_j (with $1 \leqslant j \leqslant n$) is a single vertex;

(h) $T_r \cup T_s \subseteq B_i$.

By Claims 3.22 and 3.12, $D_1, D_2, \ldots, D_n, Q_r, Q_s$ are pairwise vertex-disjoint; now (b)–(g) imply that B_i is a compound bead with poles Q_r, Q_{s+1}.

(ii) By Claim 3.6, by definition of a tip, and by Claim 3.13(a), it follows that no vertex of $(B_0 - Q_1)$ has a neighbor outside B_0; no vertex of $(B_{m+1} - Q_k)$ has a neighbor outside B_{m+1}. Finally, consider an arbitrary vertex x in B_i ($i \in \{1, \ldots, m\}$) that has a neighbor outside B_i. From (c)–(h) we conclude that $x \in Q_r \cup T_r \cup Q_{r+1} \cup Q_s \cup T_s \cup Q_{s+1}$; by Claims 3.14 and 3.30(a), $x \notin T_r \cup Q_{r+1} \cup Q_s \cup T_s$.

Write $A_0 = A_{m+2} = \emptyset$. By Claims 3.22, 3.12, and 3.29, the $(m+2)$ sets $B_i - (A_i \cup A_{i+1})$ are pairwise disjoint and each vertex of G belongs to one of $B_0, B_1, \ldots, B_{m+1}$. By Claims 3.14 and 3.33, no edge of G joins an A_i to an A_j with $i \neq j$; now Claim 3.34 guarantees that G is a string.

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