Elasticity solutions for plane anisotropic functionally graded beams

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Abstract

This paper considers the plane stress problem of generally anisotropic beams with elastic compliance parameters being arbitrary functions of the thickness coordinate. Firstly, the partial differential equation, which is satisfied by the Airy stress function for the plane problem of anisotropic functionally graded materials and involves the effect of body force, is derived. Secondly, a unified method is developed to obtain the stress function. The analytical expressions of axial force, bending moment, shear force and displacements are then deduced through integration. Thirdly, the stress function is employed to solve problems of anisotropic functionally graded plane beams, with the integral constants completely determined from boundary conditions. A series of elasticity solutions are thus obtained, including the solution for beams under tension and pure bending, the solution for cantilever beams subjected to shear force applied at the free end, the solution for cantilever beams or simply supported beams subjected to uniform load, the solution for fixed–fixed beams subjected to uniform load, and the one for beams subjected to body force, etc. These solutions can be easily degenerated into the elasticity solutions for homogeneous beams. Some of them are absolutely new to literature, and some coincide with the available solutions. It is also found that there are certain errors in several available solutions. A numerical example is finally presented to show the effect of material inhomogeneity on the elastic field in a functionally graded anisotropic cantilever beam.

Keywords: Anisotropic; Functionally graded material; Plane beam; Elasticity solution; Stress function

1. Introduction

Attentions have always been paid to the elasticity solutions for plane beams by scientists and engineers. Exact and analytical elasticity solutions for homogeneous isotropic beams can be obtained via Airy stress function, as shown in Timoshenko and Goodier (1970). These analytical solutions satisfy the exact force boundary conditions at the two longitudinal sides, but satisfy the simplified boundary conditions at the two beam ends. Three boundary conditions are usually prescribed at each end, for instance, axial force $T$,
bending moment $M$, and shear force $Q$ are usually given at a free end, while $u = v = 0$ and $\partial v/\partial x = 0$ or $\partial u/\partial y = 0$ on the neutral axes are prescribed at a fixed end, etc. Lekhnitskii (1968), using the stress function method, further obtained a series of solutions for plane anisotropic beams, including the one for beams subjected to simple tension, pure shear, and pure bending, the one for cantilever beams acted by a concentrated shear force at the tip, the one for uniformly loaded cantilever beams and simply supported beams, and the one for linearly loaded cantilever beams and simply supported beams. Silverman (1964) presented a general method to obtain stress function for orthotropic beams; the bending problems of cantilever beams subjected to a terminal shear force and cantilever beams subjected to uniform load and linearly distributed load were studied. For the purpose of analysis of stresses and displacements of anisotropic beams, Hashin (1967) expressed the stress function in terms of polynomials of the two coordinate variables; a cantilever beam subjected to shear force at the free end and a simply supported beam subjected to a uniform load were considered as examples to demonstrate his procedure. Despite of the above analytical solutions that have been obtained for many years, the analytical solution for beams with two ends fixed has not been reported yet. Ahmed et al. (1996) introduced a displacement function and presented a finite difference solution for a fixed–fixed isotropic beam subjected to uniform load. Ahmed et al. (1998) further investigated numerically a cantilever beam subjected to a distributed shear force at the free end, and comparison with the elasticity solution was made. Recently, Ding et al. (2005) derived an elasticity solution for a fixed–fixed plane isotropic beam subjected to uniform load with the aid of Airy stress function; the correctness of the solution was confirmed through comparison with the numerical solution of Ahmed et al. (1996). An elasticity solution for a fixed–simply supported plane isotropic beam subjected to uniform load was also presented in Ding et al. (2005). It is noted that the boundary conditions at the fixed end used in Ding et al. (2005) are the same as that employed by Timoshenko and Goodier (1970). Jiang and Ding (2005) employed displacement method to obtain an analytical solution for orthotropic cantilever beam subjected to a body force proportional to the density, which varies with one coordinate.

With regard to functionally graded beams, Sankar (2001) investigated simply supported orthotropic beams subjected to arbitrary normal stresses. He assumed that all the elastic compliance parameters are proportional to $e^{kz}$, where $k$ is a constant and $z$ is the thickness coordinate. Sankar and Tzeng (2002) considered the thermal stress problem of orthotropic beams, of which the elastic compliance parameters are proportional to $e^{kz}$, the thermo-mechanical coupling parameters are proportional to $e^{kz}$, and the temperature increment is proportional to $e^{kz}\sin k\zeta$. Under these restrictions, exact solutions have been found (Sankar, 2001; Sankar and Tzeng, 2002). For a simply supported orthotropic beam subjected to arbitrary normal stresses, Zhu and Sankar (2004) assumed that the elastic compliance parameters are proportional to a polynomial of $z$, for which exact solution can not be obtained by Fourier series expansion method. Thus, they sought for an approximate solution using Galerkin method. If the simply supported beam is anisotropic, Sankar’s method can not be used to obtain any exact solution, even for a homogeneous beam. Using the trial-and-error method, Lekhnitskii (1968) investigated nonhomogeneous orthotropic cantilever beams subjected to a transverse force and a bending moment at the free end. He assumed that the elastic compliance parameters are functions of the thickness coordinate, and did not impose any restriction on the form of these functions. It is remarkable that the stress expressions are still very simple and usable.

Hitherto, no general method has been developed for obtaining elasticity solutions of plane anisotropic functionally graded beams. Here, Silverman’s method (Silverman, 1964) will be generalized to establish a general way to obtain the stress function for anisotropic FG beams. No assumption will be imposed on the variation of the elastic compliance parameters along the beam thickness. In addition, body force varying with the coordinates will be considered. Totally six examples are presented to illustrate the application of the method, and hence the work of Lekhnitskii (1968) was extended in the round. When all the elastic compliance parameters are constant, the present solutions degenerate to those for homogeneous beams, among which the solutions for fixed–fixed homogeneous anisotropic beams subjected to uniform load are absolutely new to literature. The others are compared with the available elasticity solutions mentioned above, and a good agreement is obtained except for few mistakes found in several earlier solutions. Numerical results of a particular functionally graded anisotropic beam, of which only one elastic compliance coefficient varies with the thickness coordinate, are given in figure form to clearly show the effect of material inhomogeneity parameter on the displacement and stress field in the beam.
2. Basic formulations

The basic equations for plane stress static problems include the equations of equilibrium, strain-displacement relations and stress-strain relations as follows

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0,
\]

(1)

\[
e_x = \frac{\partial u}{\partial x}, \quad e_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},
\]

(2)

\[
e_x = s_{11} \sigma_x + s_{12} \sigma_y + s_{16} \tau_{xy}, \quad e_y = s_{12} \sigma_x + s_{22} \sigma_y + s_{26} \tau_{xy}, \quad \gamma_{xy} = s_{16} \sigma_x + s_{26} \sigma_y + s_{66} \tau_{xy},
\]

(3)

where \(\sigma_x, \sigma_y\) and \(\tau_{xy}\) denote the stress components, \(e_x, e_y\) and \(\gamma_{xy}\) are the strain components, \(u\) and \(v\) denote the displacement components, and \(F_x\) and \(F_y\) denote the body force components. In this paper, we consider functional graded materials (FGMs), whose elastic compliance parameters are functions of \(y\), i.e. \(s_{ij} = s_{ij}(y)\), \((i, j = 1, 2, 6)\).

In order to satisfy the equations of equilibrium, Eq. (1), we introduce stress function \(\phi\) as follows,

\[
\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - X, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} - Y, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y},
\]

(4)

where \(X\) and \(Y\) are called body force functions. They are special solutions of the following two equations, respectively

\[
F_x = \frac{\partial X}{\partial x}, \quad F_y = \frac{\partial Y}{\partial y}.
\]

(5)

Substituting Eq. (4) into Eq. (3), yields

\[
e_x = s_{11} \frac{\partial^2 \phi}{\partial y^2} + s_{12} \frac{\partial^2 \phi}{\partial x^2} - s_{16} \frac{\partial^2 \phi}{\partial x \partial y} - s_{11} X - s_{12} Y,
\]

(6)

\[
e_y = s_{12} \frac{\partial^2 \phi}{\partial y^2} + s_{22} \frac{\partial^2 \phi}{\partial x^2} - s_{26} \frac{\partial^2 \phi}{\partial x \partial y} - s_{12} X - s_{22} Y,
\]

\[
\gamma_{xy} = s_{16} \frac{\partial^2 \phi}{\partial y^2} + s_{26} \frac{\partial^2 \phi}{\partial x^2} - s_{66} \frac{\partial^2 \phi}{\partial x \partial y} - s_{16} X - s_{26} Y.
\]

The strain compatibility equations can be derived from Eq. (2)

\[
\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} - s_{ij} \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0.
\]

(7)

Substituting Eq. (6) into Eq. (7) gives

\[
\frac{\partial^2}{\partial y^2} \left( s_{11} \frac{\partial^2 \phi}{\partial y^2} + s_{12} \frac{\partial^2 \phi}{\partial x^2} - s_{16} \frac{\partial^2 \phi}{\partial x \partial y} \right) + \frac{\partial^2}{\partial x^2} \left( s_{12} \frac{\partial^2 \phi}{\partial y^2} + s_{22} \frac{\partial^2 \phi}{\partial x^2} - s_{26} \frac{\partial^2 \phi}{\partial x \partial y} \right) - \frac{\partial^2}{\partial x \partial y} \left( s_{16} \frac{\partial^2 \phi}{\partial y^2} + s_{26} \frac{\partial^2 \phi}{\partial x^2} - s_{66} \frac{\partial^2 \phi}{\partial x \partial y} \right)
\]

\[
= \frac{\partial^2}{\partial y^2} (s_{11} X + s_{12} Y) + \frac{\partial^2}{\partial x^2} (s_{12} X + s_{22} Y) - \frac{\partial^2}{\partial x \partial y} (s_{16} X + s_{26} Y).
\]

(8)

In the following, we assume that the body force functions take the form of

\[
X = \sum_{k=0}^{N_1} x^k X_k(y), \quad Y = \sum_{k=0}^{N_2} x^k Y_k(y),
\]

(9)
where \( X_k(y) \) and \( Y_k(y) \) are known functions of \( y \). By substituting the expression for \( X \) into Eq. (5), we find that \( X_0(y) \) contributes nothing to \( F_x \). For simplicity, we set \( X_0(y) = 0 \). Obviously, when all \( X_k(y) = 0 \), we have \( F_x = 0 \), and when all \( Y_k(y) = \text{const.} \), we have \( F_y = 0 \).

Consider the beam as shown in Fig. 1, which is under gravity and rotates about the axes \( y \) at an angular velocity \( \omega_0 \). The body force components are

\[
F_x = \rho x o_0^2, \quad F_y = \rho g,
\]

where \( \rho \) is the material density, and \( g \) the gravitational acceleration. If the density \( \rho \) takes the following polynomial form

\[
\rho = \sum_{j=0}^{m} x^j \rho_j(y),
\]

then the two body force functions \( X \) and \( Y \), which correspond to the inhomogeneous body force in Eq. (10), will have the same form as Eq. (9). In particular, if \( m = 0 \) or \( \rho_j(y) = \text{const.} \) \((j = 1, 2, \ldots, m)\) in Eq. (11), the material is called as density functionally graded material (Jiang and Ding, 2005).

3. Stress function

For beams subjected to distributed polynomial load on their edges as well as body force represented by Eq. (9), we assume that the stress function \( \phi \) to be

\[
\phi = \sum_{k=0}^{N} x^k \phi_k(y).
\]

Substituting Eqs. (9) and (12) into Eq. (8), and assuming \( N_1 = N_2 = N \), we obtain

\[
\sum_{k=0}^{N} x^k \{ (s_{11} \phi_k^" - s_{11} X_k - s_{12} Y_k)" - (k + 1)[(s_{16} \phi_k')'] + (s_{16} \phi_k')' - (s_{16} X_k + s_{26} Y_k') \} H(N - k) + (k + 1)(k + 2)[(s_{12} \phi_{k+2})' + s_{12} \phi_{k+2}' + (s_{66} \phi_{k+2}') - s_{12} X_{k+2} - s_{22} Y_{k+2}] H(N - k - 1) - (k + 1)(k + 2)(k + 3)[(s_{26} \phi_{k+3})' + s_{26} \phi_{k+3}'] H(N - k - 2) + (k + 1)(k + 2)(k + 3)(k + 4)s_{22} \phi_{k+4} H(N - k - 3) \} = 0,
\]

where \((\cdot)'\) and \((\cdot)''\) denote the first and the second derivatives, respectively, and

\[
H(x) = \begin{cases} 
0, & x \leq 0, \\
1, & x > 0.
\end{cases}
\]

From Eq. (13), we obtain the following differential equations which \( \phi_k \) satisfy

\[
(s_{11} \phi_k^" - s_{11} X_k - s_{12} Y_k)" - (k + 1)[(s_{16} \phi_k')'] + (s_{16} \phi_k')' - (s_{16} X_k + s_{26} Y_k') \} H(N - k) + (k + 1)(k + 2)[(s_{12} \phi_{k+2})' + s_{12} \phi_{k+2}' + (s_{66} \phi_{k+2}') - s_{12} X_{k+2} - s_{22} Y_{k+2}] H(N - k - 1) - (k + 1)(k + 2)(k + 3)[(s_{26} \phi_{k+3})' + s_{26} \phi_{k+3}'] H(N - k - 2) + (k + 1)(k + 2)(k + 3)(k + 4)s_{22} \phi_{k+4} H(N - k - 3) \} = 0,
\]

where \( k = N, N - 1, \ldots, 1, 0 \).
Integrating Eq. (15) once from the lower limit \(-h/2\), we obtain

\[
(s_{11}'\phi'_{N} - s_{11}X_{k} - s_{12}Y_{k})' - (k + 1)[(s_{16}'\phi'_{N+1})' + s_{16}\phi''_{N+1} - s_{16}X_{k+1} - s_{26}Y_{k+1}]H(N - k)
\]
\[
+ (k + 1)(k + 2)[(s_{12}'\phi'_{k+2})' + s_{66}\phi''_{k+2} + \int_{-\frac{h}{2}}^{y} s_{12}\phi''_{k+2} d\xi] - \int_{-\frac{h}{2}}^{y} (s_{12}X_{k+2} + s_{22}Y_{k+2})d\xi]H(N - k - 1)
\]
\[
- (k + 1)(k + 2)(k + 3)[s_{26}\phi'_{k+3} + \int_{-\frac{h}{2}}^{y} s_{56}\phi''_{k+3} d\xi]H(N - k - 2)
\]
\[
+ (k + 1)(k + 2)(k + 3)(k + 4) \int_{-\frac{h}{2}}^{y} s_{22}\phi''_{k+4} d\xi]H(N - k - 3) = a_{k},
\]

where \(k = N, N - 1, \ldots, 1, 0\), and \(a_{k}\) are integral constants.

The step to solve the set of differential equations in Eq. (16) is as follows. Firstly, set \(k = N\) to obtain \(\phi_{N}\). Secondly, set \(k = N - 1\) to obtain \(\phi_{N-1}\). Thirdly, set \(k = N - 2\) to obtain \(\phi_{N-2}\), and so on. Finally we can set \(k = 0\) to obtain \(\phi_{0}\).

When \(k = N\), we have from Eq. (16)

\[
(s_{11}'\phi''_{N} - s_{11}X_{N} - s_{12}Y_{N})' = a_{N}.
\]

Integrating Eq. (17) with respect to \(y\) once, twice and three times, yields, respectively

\[
\phi'_{N} = a_{N}y/s_{11} + b_{N}/s_{11} + X_{N} + s_{12}Y_{N}/s_{11},
\]

\[
\phi''_{N} = a_{N}f_{10}(y) + b_{N}f_{00}(y) + c_{N} + z_{00}^{N}(y),
\]

\[
\phi'''_{N} = a_{N}f_{11}(y) + b_{N}f_{01}(y) + c_{N}y + d_{N} + z_{01}^{N}(y),
\]

where \(b_{N}, c_{N}, d_{N}\) are integral constants, and

\[
f_{m0}(y) = \frac{1}{n!} \int_{-\frac{h}{2}}^{y} \frac{\zeta^{m}(y - \zeta)^{n}}{s_{11}(\zeta)} d\xi \quad (m, n = 0, 1, 2, \ldots),
\]

\[
\zeta_{00}^{N}(y) = \frac{1}{n!} \int_{-\frac{h}{2}}^{y} [X_{k}(\zeta) + s_{12}(\zeta)Y_{k}(\zeta)/s_{11}(\zeta)](y - \zeta)^{n} d\xi \quad (n = 0, 1, 2, \ldots, k = N, N - 1, \ldots, 1, 0).
\]

When \(k = N - 1\), we obtain from Eq. (16)

\[
(s_{11}'\phi''_{N-1} - s_{11}X_{N-1} - s_{12}Y_{N-1})' = a_{N-1} + N[(s_{16}'\phi'_{N})' + s_{16}\phi''_{N} - s_{16}X_{N} - s_{26}Y_{N}]
\]
\[
= a_{N-1} + N[(s_{16}'\phi'_{N})' + a_{N}s_{16}/s_{11} + b_{N}s_{16}/s_{11} + s_{16}s_{12}/s_{11} - s_{26}Y_{N}].
\]

Integration of Eq. (23) yields

\[
\phi''_{N-1} = a_{N-1}y/s_{11} + b_{N-1}/s_{11} + N\{a_{N}g_{10}'(y) + b_{N}g_{00}'(y) + c_{N}s_{16}/s_{11} + [z_{01}^{N}(y)]'\} + [z_{00}^{N-1}(y)]',
\]

where

\[
g_{m0}(y) = \frac{1}{n!} \int_{-\frac{h}{2}}^{y} \frac{s_{16}(\zeta)f_{m0}(\zeta) + f_{m0}(\zeta)(y - \zeta)^{n}}{s_{11}(\zeta)} d\xi \quad (m, n = 0, 1, 2, \ldots),
\]

\[
\zeta_{10}^{N}(y) = \frac{1}{n!} \int_{-\frac{h}{2}}^{y} \frac{s_{16}(\zeta)z_{00}(\zeta) + y_{0}^{k}(\zeta)(y - \zeta)^{n}}{s_{11}(\zeta)} d\xi \quad (n = 0, 1, 2, \ldots, k = N, N - 1, \ldots, 0),
\]

\[
f_{m0}^{6}(y) = \frac{1}{n!} \int_{-\frac{h}{2}}^{y} \frac{s_{16}(\zeta)\zeta^{m}(y - \zeta)^{n}}{s_{11}(\zeta)} d\xi \quad (m, n = 0, 1, 2, \ldots),
\]

\[
y_{0}^{k}(y) = \int_{-\frac{h}{2}}^{y} \frac{s_{16}(\zeta)s_{12}(\zeta)/s_{11}(\zeta) - s_{26}(\zeta)Y_{k}(\zeta)}{d\xi} \quad (k = N, N - 1, \ldots, 1).
\]
From Eq. (24), it is easy to obtain by integration
\[
\phi_N^{-1} = a_{N-1} f_{10}(y) + b_{N-1} f_{00}(y) + c_{N-1} + N[a_N g_{10}(y) + b_N g_{00}(y) + c_N f_{00}^g(y) + z_{10}^N(y)] + z_{00}^{N-1}(y),
\]
(29)
\[
\phi_N^{-1} = a_{N-1} f_{11}(y) + b_{N-1} f_{01}(y) + c_{N-1} y + d_{N-1} + N[a_N g_{11}(y) + b_N g_{01}(y) + c_N f_{01}^g(y) + z_{11}^N(y)]
+ z_{01}^{N-1}(y).
\]
(30)
When \( k = N - 2 \), we obtain from Eq. (16)
\[
(s_{11} \phi_N^m - s_{11} X_{N-2} - s_{12} Y_{N-2}) = a_{N-2} + (N - 1)(s_{16} \phi_N^m - s_{16} X_{N-1} - s_{26} Y_{N-1})
- N(N - 1) \left( s_{12} \phi_N^m + s_{66} \phi_N^m + \int_0^\frac{y}{z} (s_{12} \phi_N^m - s_{12} X_N - s_{22} Y_N) d\xi \right)
= a_{N-2} + (N - 1)(s_{16} \phi_N^m - a_{N-1} s_{16} y/s_{11} + b_{N-1} s_{16}/s_{11} + (Y_{N-1}^m)(y))
+ N(N - 1) \left( a_N [s_{16} g_{10}^g(y) - s_{66} f_{10}(y) - f_{10}^g(y)] + b_N [s_{16} g_{00}^g(y) - s_{66} f_{01}(y) - f_{01}^g(y)]
+ c_N (s_{16}^2/s_{11} - s_{66}) - s_{66} z_{10}^{N-1}(y) - Y_N^2(y) + s_{16} [-z_{11}^N(y)] - (s_{12} \phi_N^m),
\]
(31)
where
\[
f_{mn}^g(y) = \frac{1}{m!} \int_{-\frac{y}{z}}^y \frac{s_{12}(\xi)}{s_{11}(\xi)} \frac{z^m(y - \xi)^n}{z_{11}(\xi)} d\xi \quad (m, n = 0, 1, 2, \ldots),
\]
(32)
\[
Y_k^2(y) = \int_{-\frac{y}{z}}^y \left[ \frac{s_{12}(\xi)}{s_{11}(\xi)} - s_{22}(\xi) \right] Y_k(\xi) d\xi \quad (k = N, N - 1, \ldots, 1, 0).
\]
(33)
Integration of Eq. (31) yields
\[
\phi_N^m = a_{N-2} y/s_{11} + b_{N-2}/s_{11} + (N - 1)(a_{N-1} g_{10}^g(y) + b_{N-1} g_{00}^g(y) + c_{N-1} /s_{11} + (Y_{N-1}^m)(y))
+ N(N - 1) \left( a_N [g_{10}^g(y)] + b_N [g_{00}^g(y)] + c_N h_0(y) - d_N s_{12}/s_{11} + [z_{20}^g(y)] \right) + X_{N-2} + s_{12} Y_{N-2}/s_{11},
\]
(34)
where
\[
g_{mn}^g(y) = \frac{1}{m!} \int_{-\frac{y}{z}}^y \frac{s_{12}(\xi) g_{mn}^g(\xi) - s_{12}(\xi) f_{mn}(\xi) + B_m(\xi)](y - \xi)^n}{s_{11}(\xi)} d\xi,
\]
(35)
\[
h_0(y) = \frac{1}{m!} \int_{-\frac{y}{z}}^y \frac{s_{12}(\xi) f_{00}^g(\xi) - s_{12}(\xi) g_{00}(\xi) + B_0(\xi)](y - \xi)^n}{s_{11}(\xi)} d\xi,
\]
(36)
\[
z_{20}^g(y) = \frac{1}{m!} \int_{-\frac{y}{z}}^y \frac{s_{12}(\xi) g_{20}^g(\xi) - s_{12}(\xi) z_{20}(\xi) - z_2(\xi)(y - \xi)^n}{s_{11}(\xi)} d\xi,
\]
(37)
\[
B_m(y) = \int_{-\frac{y}{z}}^y \frac{s_{16}(\xi) g_{10}^g(\xi) - s_{66}(\xi) f_{10}(\xi) - f_{10}^g(y)}{s_{11}(\xi)} d\xi,
\]
(38)
\[
B_0(y) = \int_{-\frac{y}{z}}^y \frac{s_{16}(\xi) g_{00}^g(\xi) - s_{66}(\xi) f_{01}(\xi) - f_{01}^g(y)}{s_{11}(\xi)} d\xi,
\]
(39)
\[
z_N(y) = \int_{-\frac{y}{z}}^y \left\{ \left[ s_{66}(\xi) - s_{11}^N(\xi) \right] z_{00}^N(\xi) + Y_N^2(\xi) - Y_N^2(\xi) Y_N^2(\xi) \right\} d\xi,
\]
(40)
where \( m, n = 0, 1, 2, \ldots \) and \( k = N, N - 1, \ldots, 1, 0 \).
Integration of Eq. (34) yields
\[
\phi_N^{-2} = a_{N-2} f_{10}(y) + b_{N-2} f_{00}(y) + c_{N-2} + (N - 1)[a_{N-1} g_{10}(y) + b_{N-1} g_{00}(y) + c_{N-1} f_{00}^g(y) + z_{10}^N(y)]
+ N(N - 1) \left( a_N [g_{10}^g(y) + b_N g_{00}^g(y) + c_N h_0(y) - d_N f_{10}^g(y) + z_{10}^N(y)] \right) + z_{00}^{N-2}(y),
\]
(41)
\[
\phi_N^{-2} = a_{N-2} f_{11}(y) + b_{N-2} f_{01}(y) + c_{N-2} y + d_{N-2} + (N - 1)[a_{N-1} g_{11}(y) + b_{N-1} g_{01}(y) + c_{N-1} f_{01}^g(y) + z_{11}^{N-1}(y)]
+ N(N - 1)[a_N g_{11}(y) + b_N g_{01}^g(y) + c_N h_1(y) - d_N f_{01}^g(y) + z_{11}^N(y)] + z_{01}^{N-2}(y).
\]
(42)
With the above expressions we have obtained, we can readily obtain \( \phi_0 \) by virtue of Eq. (20) when \( N = 0 \). When \( N = 1 \), we can readily obtain \( \phi_1 \) from Eq. (20), and \( \phi_0 \) from Eq. (30). When \( N = 2 \), we can easily determine \( \phi_2 \) from Eq. (20), \( \phi_1 \) from Eq. (30), and \( \phi_0 \) from Eq. (42) . Thus, for an arbitrary \( N \), the expressions for all \( \phi_k \) \( (k = N, N - 1, \ldots, 1, 0) \) can be determined.

4. Axial force, bending moment, shear force and displacements

It is easy to obtain expressions of the axial force \( T \), the bending moment \( M \) and the shear force \( Q \) from Eqs. (4), (9) and (12)

\[
T = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dy = \sum_{k=0}^{N} \frac{x^k}{k+1} \left[ \phi_k \left( \frac{h}{2} \right) - \phi_k' \left( \frac{h}{2} \right) \right] - \int_{-\frac{h}{2}}^{\frac{h}{2}} X_k(\xi) \, d\xi,
\]

\[
M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \, dy = \sum_{k=0}^{N} \frac{x^k}{k+1} \left[ \phi_k \left( \frac{h}{2} \right) + \phi_k' \left( \frac{h}{2} \right) \right] - \int_{-\frac{h}{2}}^{\frac{h}{2}} X_k(\xi) \, d\xi,
\]

\[
Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \, dy = \sum_{k=1}^{N} k x^{k-1} \left[ \phi_k \left( \frac{h}{2} \right) - \phi_k' \left( \frac{h}{2} \right) \right].
\]

Substituting Eqs. (2), (9) and (12) into Eq. (6), assuming \( N_1 = N_2 = N \) as before, through integration and by virtue of Eq. (16), we obtain displacement expressions

\[
u = \sum_{k=0}^{N} \frac{x^k}{k+1} \left[ s_{11} \phi''_k - s_{11} X_k - s_{12} Y_k \right] + \sum_{k=2}^{N} k x^{k-1} s_{12} \phi_k - \sum_{k=1}^{N} x^k s_{16} \phi'_k - \int_{-\frac{h}{2}}^{\frac{h}{2}} F(\xi) \, d\xi + \omega y + u_0,
\]

\[
v = \sum_{k=0}^{N} \frac{x^k}{k+1} \left[ s_{12} \phi''_k - s_{12} X_k - s_{22} Y_k \right] \frac{x^2 a_k}{(k+1)(k+2)} + \sum_{k=2}^{N} k(k-1)x^{k-2} \int_{-\frac{h}{2}}^{\frac{h}{2}} s_{22} \phi_k \, d\xi
\]

From Eq. (46), we find that the integral constants \( u_0 \), \( v_0 \) and \( \omega \) represent rigid body displacements, which can be determined from the boundary conditions of a specific problem.

5. Boundary conditions

For the equilibrium problem corresponding to the stress function in Eq. (12), the boundary conditions at \( y = \pm h/2 \) are

\[
\sigma_y = p_1(x) = \sum_{k=0}^{N_1} p_{1k} x^k \quad \text{and} \quad \tau_{xy} = \tau_1(x) = \sum_{k=0}^{N_1} \tau_{1k} x^k \quad \text{at} \quad y = h/2,
\]

\[
\sigma_y = p_2(x) = \sum_{k=0}^{N_2} p_{2k} x^k \quad \text{and} \quad \tau_{xy} = \tau_2(x) = \sum_{k=0}^{N_2} \tau_{2k} x^k \quad \text{at} \quad y = -h/2,
\]

where \( p_{1k}, p_{2k}, \tau_{1k} \) and \( \tau_{2k} \) are all known constants.

We further consider the boundary conditions at two ends in the following:

1. Free end

The boundary conditions are

\[
Q = \overline{Q}, \quad T = \overline{T}, \quad M = \overline{M} \quad \text{at} \quad x = 0 \quad \text{or} \quad x = l,
\]

where \( \overline{Q}, \overline{T} \) and \( \overline{M} \) are prescribed resultant forces.
2. Fixed end
The boundary conditions are

\[ u = 0, \quad v = 0, \quad \frac{\partial v}{\partial x} = 0 \quad \text{or} \quad \frac{\partial u}{\partial y} = 0 \quad \text{at point} \ (0, 0) \quad \text{or} \ (l, 0), \]  

(51)

where and hereafter, \((a, b)\) indicates, as usual, a point with \(x\)-coordinate \(a\) and \(y\)-coordinate \(b\).

3. Hinged end
The boundary conditions are

\[ T = 0, \quad M = 0 \quad \text{at} \ x = 0 \quad \text{or} \quad x = l, \quad v = 0 \quad \text{at point} \ (0, 0), \quad v = 0 \quad \text{at point} \ (l, 0). \]  

(52)

For simply supported beams with two ends hinged, considering the equilibrium of the whole beam (see Fig. 1), can give the equivalent expressions of boundary conditions that are more convenient for application. Assuming that the following equation is valid

\[ \int_0^l \tau_1(x)dx - \int_0^l \tau_2(x)dx + \sum_{k=0}^N \frac{k+1}{k+1} \int_{-\frac{l}{2}}^{\frac{l}{2}} X_k(y)dy = 0, \]  

(53)

we can give the boundary conditions for simply supported beams at the two ends as

\[ T_0 = 0, \]  

(54a)

\[ M_0 = 0 \]  

(54b)

and

\[ Q_0 = \frac{1}{7} \int_0^l (l-x)[p_1(x) - p_2(x)]dx + \frac{h}{27} \int_0^l [\tau_1(x) + \tau_2(x)]dx + \sum_{k=0}^N \frac{k+1}{(k+1)(k+2)} \int_{-\frac{l}{2}}^{\frac{l}{2}} Y_1(y)dy + \sum_{k=1}^N p^{k-1} \int_{-\frac{l}{2}}^{\frac{l}{2}} X_k(y)dy, \]  

(54c)

\[ u = v = 0 \quad \text{at point} \ (0, 0), \]  

(55)

\[ v = 0 \quad \text{at point} \ (l, 0). \]  

(56)

Obviously, if there is no body force and tangential forces applied at the boundaries, Eq. (53) always holds. \(Q_0, T_0\) and \(M_0\) in Eq. (54), which denote the shear force, axial load and bending moment at \(x = 0\), can be expressed with stress function by virtue of Eqs. (43)–(45)

\[ Q_0 = \phi_1 \left( -\frac{h}{2} \right) - \phi_1 \left( \frac{h}{2} \right), \]  

(57)

\[ T_0 = \phi_0 \left( \frac{h}{2} \right) - \phi_0 \left( -\frac{h}{2} \right), \]  

(58)

\[ M_0 = h \left[ \phi_0 \left( \frac{h}{2} \right) + \phi_0 \left( -\frac{h}{2} \right) \right] - \phi_0 \left( \frac{h}{2} \right) + \phi_0 \left( -\frac{h}{2} \right). \]  

(59)

6. Applications

With the formulations presented above, the procedure for solving specific boundary value problem is as follows. Firstly, determine \(N\) in Eq. (12) according to the load condition. In fact, from Eqs. (4), (9), (12), (48) and (49), we know that \(N = \max[\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6]\), where \(\Omega_1 = N_1\) if \(X \neq 0\) and \(\Omega_1 = 0\) if \(X = 0, \Omega_2 = N_2 + 2\) if \(Y \neq 0\) and \(\Omega_2 = 0\) if \(Y = 0, \Omega_3 = N_3 + 2\) if \(p_1 \neq 0\) and \(\Omega_3 = 0\) if \(p_1 = 0, \Omega_4 = N_4 + 1\) if \(\tau_1 \neq 0\) and \(\Omega_4 = 0\) if \(\tau_1 = 0, \Omega_5 = N_5 + 2\) if \(p_2 \neq 0\) and \(\Omega_5 = 0\) if \(p_2 = 0, \Omega_6 = N_6 + 1\) if \(\tau_2 \neq 0\) and \(\Omega_6 = 0\) if \(\tau_2 = 0\). Whereas in theoretical study, \(N\) must be given or assumed a priori, and consequently, it is demanded that \(N_1 \leq N, N_2 \leq N - 2, N_3 \leq N - 2, N_4 \leq N - 1, N_5 \leq N - 2\) and \(N_6 \leq N - 1\), which implies that, in application of Eq. (16) or Eq. (15), we must set \(Y_N(y) = 0\) and \(Y_{N-1}(y) = 0\). Secondly, it can be found
that every \( \phi_k(y) \) contains four integral constants, i.e. \( a_k, b_k, c_k \) and \( d_k \), thus the stress function represented by Eq. (12) contains a total of \( 4(N + 1) \) integral constants. Since the linear terms involving \( d_1, c_0 \) and \( d_0 \) make no contribution to the stress field, we shall omit \( c_0 y \) and \( d_0 \) in the expression of \( \phi_0 \), and omit \( d_1 \) in the expression of \( \phi_1 \), and totally \( 4N + 1 \) constants will appear in the expressions of stresses. The boundary conditions at \( y = \pm h/2 \) as in Eqs. (48) and (49) will give \( 2(N - 1) + 2N = 4N - 2 \) algebraic equations, and the boundary conditions at the two ends of the beam will provide another six algebraic equations. Therefore, we have \( 4N + 4 \) algebraic equations all together, which can be used to determine \( 4N + 4 \) arbitrary constants, i.e. \( a_N, b_N, c_N, d_N, a_{N-1}, b_{N-1}, c_{N-1}, d_{N-1}, \ldots, a_2, b_2, c_2, d_2, a_1, b_1, c_1, a_0, b_0, a_0, b_0, a_0, b_0, c_0 \) and \( d_0 \).

Firstly, we investigate the application of the stress function represented by Eq. (12) when \( N = 0 \). No body force will be considered in this case, because when \( N = 0 \), it is demanded that \( X = X_0 = 0 \) and \( Y = Y_0 = 0 \). From Eq. (20), we have

\[
\phi = \phi_0 = a_0 f_{11}(y) + b_0 f_{01}(y).
\]  

(60)

Substituting it into the stress expressions in Eq. (4), we obtain

\[
\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = a_0 \frac{y}{s_{11}} + b_0 \frac{1}{s_{11}} \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0.
\]

(61)

This solution corresponds to a beam under simple tension and pure bending, as detailed in the following example.

**Example 6.1 (Beam under simple tension and pure bending)**. The axial force and bending moment in any section of the beam are

\[
T = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x \, dy = a_0 f_{10}\left(\frac{h}{2}\right) + b_0 f_{00}\left(\frac{h}{2}\right), \quad M = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_y \, dy = a_0 f_{20}\left(\frac{h}{2}\right) + b_0 f_{10}\left(\frac{h}{2}\right).
\]

(62)

If \( T_0 = T_f = \overline{T} \) and \( M_0 = M_f = \overline{M} \) are prescribed at the two ends of the beam (\( x = 0 \) and \( x = l \)), we obtain from Eq. (62)

\[
a_0 = (H_0 \overline{M} - H_1 \overline{T})/H_{02}, \quad b_0 = (H_2 \overline{T} - H_1 \overline{M})/H_{02},
\]

(63)

where

\[
H_0 = f_{00}(h/2), \quad H_1 = f_{10}(h/2), \quad H_2 = f_{20}(h/2), \quad H_{02} = H_0 H_2 - H_1^2.
\]

(64)

Substitution of Eq. (63) into Eq. (61) yields

\[
\sigma_x = \frac{1}{s_{11} H_{02}} [(H_0 \overline{M} - H_1 \overline{T}) y - H_1 \overline{M} + H_2 \overline{T}], \quad \sigma_y = \tau_{xy} = 0.
\]

(65)

When \( \overline{T} \neq 0, \overline{M} = 0 \), it is a tension or compression problem, and when \( \overline{T} = 0, \overline{M} \neq 0 \), it is a bending problem. For homogeneous anisotropic materials, \( s_{11}(y) = \text{const.} \), and it can be shown that \( H_1 = 0, H_0 = h/s_{11} \) and \( H_2 = h^2/(12s_{11}) \). Substituting these into Eq. (65), we obtain

\[
\sigma_x = \frac{\overline{M} y}{J} \frac{\overline{T}}{A}, \quad \sigma_y = 0, \quad \tau_{xy} = 0,
\]

(66)

where \( J = h^3/12 \) and \( A = h \). Eq. (66) coincides with the classical results of a homogeneous anisotropic beam (Gere and Timoshenko, 1984), for which the stresses are independent of material constants.

Secondly, we investigate the application of the stress function represented by Eq. (12) when \( N = 1 \). In this case, it is required that \( Y_N = Y_1 = 0 \) and \( Y_{N-1} = Y_0 = 0 \), implying

\[
X = x X_1(y), \quad Y = 0.
\]

(67)

The corresponding body forces are easily obtained \( F_x = X_1(y) \) and \( F_y = 0 \).

The stress function is

\[
\phi = \phi_0 + x \phi_1.
\]

(68)
From Eqs. (20) and (30), we have

\[
\begin{align*}
\phi_1 &= a_1 f_{11}(y) + b_1 f_{01}(y) + c_1 y + z_{01}^1(y), \\
\phi_0 &= a_1 g_{11}(y) + b_1 g_{01}(y) + c_1 f_{00}^1(y) + a_0 f_{11}(y) + b_0 f_{01}(y) + z_{11}^1(y) + z_{01}^0(y).
\end{align*}
\]  

(69)  

(70)  

Substituting Eqs. (67) and (68) into Eq. (4), and by virtue of Eqs. (69) and (70), we obtain

\[
\begin{align*}
\sigma_x &= a_1 \frac{s_{16}(y)f_{10}(y) + f_{00}^6(y)}{s_{11}(y)} + b_1 \frac{s_{16}(y)f_{00}(y) + f_{00}^6(y)}{s_{11}(y)} + c_1 \frac{s_{16}(y)}{s_{11}(y)} \\
&\quad + a_0 \frac{y}{s_{11}(y)} + b_0 \frac{1}{s_{11}(y)} + z_{00}^1(y) \frac{s_{16}(y)}{s_{11}(y)} + x \left[ a_1 \frac{y}{s_{11}(y)} + b_1 \frac{1}{s_{11}(y)} \right],
\end{align*}
\]  

(71)  

\[
\sigma_y = 0, \quad \tau_{xy} = -a_1 f_{10}(y) - b_1 f_{00}(y) - c_1 - z_{00}^1(y).
\]

Substitution of Eqs. (69) and (70) into Eqs. (57)–(59), yields

\[
\begin{align*}
Q_0 &= -a_1 f_{11}\left(\frac{h}{2}\right) - b_1 f_{01}\left(\frac{h}{2}\right) - c_1 h - z_{01}^1\left(\frac{h}{2}\right), \\
T_0 &= a_1 g_{11}\left(\frac{h}{2}\right) + b_1 g_{00}\left(\frac{h}{2}\right) + c_1 f_{00}^1\left(\frac{h}{2}\right) + a_0 f_{11}\left(\frac{h}{2}\right) + b_0 f_{01}\left(\frac{h}{2}\right) + z_{11}^1\left(\frac{h}{2}\right), \\
M_0 &= \left[ \frac{h}{2} g_{10}\left(\frac{h}{2}\right) - g_{11}\left(\frac{h}{2}\right) \right] a_1 + \left[ \frac{h}{2} g_{00}\left(\frac{h}{2}\right) - g_{01}\left(\frac{h}{2}\right) \right] b_1 + \left[ \frac{h}{2} f_{00}^1\left(\frac{h}{2}\right) - f_{01}^1\left(\frac{h}{2}\right) \right] c_1 \\
&\quad + \left[ \frac{h}{2} f_{10}\left(\frac{h}{2}\right) - f_{11}\left(\frac{h}{2}\right) \right] a_0 + \left[ \frac{h}{2} f_{00}\left(\frac{h}{2}\right) - f_{01}\left(\frac{h}{2}\right) \right] b_0 + \frac{h}{2} z_{11}^1\left(\frac{h}{2}\right) - z_{11}^1\left(\frac{h}{2}\right).
\end{align*}
\]  

(72)  

Example 6.2 (Cantilever beam subjected to a transverse force \( P \) at the free end). In absence of body force, we obtain from Eq. (71) and the boundary conditions \( \sigma_y = \tau_{xy} = 0 \) at \( y = \pm h/2 \)

\[
c_1 = 0, \quad a_1 f_{10}\left(\frac{h}{2}\right) + b_1 f_{00}\left(\frac{h}{2}\right) = 0.
\]  

(73)  

(74)  

By virtue of Eq. (72) and the boundary conditions at the free end, i.e. \( Q_0 = -P, \ T_0 = 0 \) and \( M_0 = 0 \), we obtain

\[
\begin{align*}
a_1 f_{11}\left(\frac{h}{2}\right) + b_1 f_{01}\left(\frac{h}{2}\right) &= P, \\
a_1 g_{11}\left(\frac{h}{2}\right) + b_1 g_{00}\left(\frac{h}{2}\right) + a_0 f_{11}\left(\frac{h}{2}\right) + b_0 f_{01}\left(\frac{h}{2}\right) &= 0, \\
a_1 \left[ \frac{h}{2} g_{10}\left(\frac{h}{2}\right) - g_{11}\left(\frac{h}{2}\right) \right] + b_1 \left[ \frac{h}{2} g_{00}\left(\frac{h}{2}\right) - g_{01}\left(\frac{h}{2}\right) \right] + a_0 \left[ \frac{h}{2} f_{10}^1\left(\frac{h}{2}\right) - f_{11}\left(\frac{h}{2}\right) \right] \\
&\quad + b_0 \left[ \frac{h}{2} f_{00}\left(\frac{h}{2}\right) - f_{01}\left(\frac{h}{2}\right) \right] &= 0.
\end{align*}
\]  

(75)  

(76)  

(77)  

From Eqs. (74)–(77), we obtain

\[
\begin{align*}
a_1 &= PA_1, \quad b_1 = -PD_1, \quad a_0 = PA_2, \quad b_0 = PD_2,
\end{align*}
\]  

(78)
where
\[
A_1 = H_0/H_{04}, \quad D_1 = H_1/H_{04}, \quad H_{04} = H_0H_4 - H_1H_3 = -H_{02},
\]
\[
H_3 = f_0(h/2) = H_0h/2 - H_1, \quad H_4 = f_1(h/2) = H_1h/2 - H_2,
\]
\[
A_2 = (H_0G_2 - H_1G_1)/H_{02}, \quad D_2 = (H_2G_1 - H_1G_2)/H_{02},
\]
\[
G_1 = D_1g_{01}(h/2) - A_1g_{10}(h/2) = [H_1g_{00}(h/2) - H_0g_{10}(h/2)]/H_{04},
\]
\[
G_2 = \frac{h}{2}G_1 - D_1g_{01}(h/2) + A_1g_{11}(h/2) = \frac{h}{2}G_1 - [H_1g_{01}(h/2) - H_0g_{11}(h/2)]/H_{04}.
\]

Substitution of Eq. (78) into Eq. (71) yields
\[
\sigma_x = \frac{P}{s_{11}(y)} [A_2y + D_2 + A_1[s_{16}(y)f_{10}(y) + f_{0}^{s}(y)] - D_1[x + s_{16}(y)f_{00}(y) + f_{00}^{s}(y)]],
\]
\[
\sigma_y = 0, \quad \tau_{xy} = P[D_1f_{00}(y) - A_1f_{10}(y)].
\]

If \(s_{16} = 0\), then we have \(f_{ij}^{s}(y) = 0\) and \(g_{ij}(y) = 0\), which in turn gives \(G_1 = 0\), \(G_2 = 0\), \(A_2 = 0\) and \(D_2 = 0\). Thus Eq. (80) can be simplified to
\[
\sigma_x = \frac{Px}{H_{04}s_{11}(y)} (H_0y - H_1), \quad \sigma_y = 0, \quad \tau_{xy} = \frac{P}{H_{04}} [H_1f_{00}(y) - H_0f_{10}(y)].
\]

Superposing the solution of Eq. (65) for \(T = 0\) on Eq. (81), and noting that \(H_{04} = -H_{02}\), we obtain
\[
\sigma_x = \frac{M - Px}{H_{02}s_{11}(y)} (H_0y - H_1), \quad \sigma_y = 0, \quad \tau_{xy} = \frac{P}{H_{02}} [H_0f_{10}(y) - H_1f_{00}(y)].
\]

Eq. (82) can be rewritten as
\[
\sigma_x = \frac{6(M - Px)}{S} E_1(y)(2S_1y - S_2), \quad \sigma_y = 0, \quad \tau_{xy} = \frac{6P}{S} \int_{-\frac{h}{2}}^{\frac{h}{2}} E_1(\xi)(2S_1\xi - S_2) d\xi,
\]
where \(E_1(y) = 1/s_{11}(y)\) and
\[
S_1 = H_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E_1(y) dy, \quad S_2 = 2H_1 = 2 \int_{-\frac{h}{2}}^{\frac{h}{2}} yE_1(y) dy,
\]
\[
S = 12H_{02} = 12 \left[ \int_{-\frac{h}{2}}^{\frac{h}{2}} E_1(y) dy \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 E_1(y) dy - \left( \int_{-\frac{h}{2}}^{\frac{h}{2}} yE_1(y) dy \right)^2 \right].
\]

Comparing Eq. (83) with Eqs. (19.8) and (19.9) in \textit{Lekhnitskii (1968)} shows a good agreement in form, and hence the two expressions of the location of neutral axes \(y_0 = H_1/H_0 = S_2/(2S_1)\) also have the same form. However, because the coordinate origin in our analysis is different from that in \textit{Lekhnitskii (1968)}, different values of \(y\) should be adopted in the two solutions for the same point in the beam. Furthermore, it should also be noticed that the only condition for Eq. (83) being valid is \(s_{16} = 0\), i.e. it is not necessary that the material is orthotropic, while the results of \textit{Lekhnitskii (1968)} were obtained for orthotropic materials.

The integral constants \(u_0, v_0, \omega\) can be determined from the boundary conditions at the fixed end: \(u = v = \theta = 0\), \(\partial u/\partial x = 0\) or \(\partial v/\partial y = 0\), at point \((l,0)\).

For homogeneous anisotropic beams, we can obtain \(G_1 = -2S_{16}/s_{11}, A_1 = -12S_{11}/h^3, D_1 = 0, A_2 = 0,\) and \(D_2 = -2S_{16}/h\) by virtue of the coefficients in Appendix. Substituting these into Eq. (80), gives
\[
\sigma_x = -\frac{P}{J} xy + \frac{P}{J} s_{16} \left( \frac{h^2}{12} - y^2 \right), \quad \sigma_y = 0, \quad \tau_{xy} = -\frac{P}{2J} \left( \frac{h^2}{4} - y^2 \right),
\]
which coincides with the solution given in \textit{Lekhnitskii (1968)}.
Next we investigate the application of the stress function in Eq. (12) when \( N = 2 \). In this case, in addition to \( X_0(y) = 0 \), we have, \( Y_N = Y_2 = 0 \) and \( Y_{N-1} = Y_1 = 0 \). Hence

\[
X = xX_1(y) + x^2X_2(y), \quad Y = Y_0(y).
\]  

(85)

The body forces corresponding to Eq. (85) are easily obtained as \( F_x = X_1(y) + 2xX_2(y) \) and \( F_y = Y_0(y) \).

The stress function takes the following form

\[
\phi = \phi_0 + x\phi_1 + x^2\phi_2.
\]  

(86)

From Eqs. (20), (30) and (42), we obtain

\[
\begin{align*}
\phi_2 &= a_2f_{11}(y) + b_2f_{01}(y) + c_2y + d_2 + z_{01}^2(y), \\
\phi_1 &= 2[a_2g_{11}(y) + b_2g_{01}(y) + c_2f_{01}^0(y)] + a_1f_{11}(y) + b_1f_{01}(y) + c_1y + 2z_{11}^2(y) + z_{01}^2(y), \\
\phi_0 &= 2[a_2g_{11}(y) + b_2g_{01}(y) + c_2h_{1}(y) - 2z_{01}^2(y)] + a_1g_{11}(y) + b_1g_{01}(y) + c_1f_{01}^0(y) + z_{11}^2(y) + a_0f_{11}(y) + b_0f_{01}(y) + 2z_{21}^2(y) + z_{01}^2(y).
\end{align*}
\]  

(87)

(88)

(89)

Eq. (4) gives

\[
\sigma_x = \phi''_0 + x\phi''_1 + x^2\phi''_2 - xX_1 - x^2X_2, \quad \sigma_y = 2\phi_2 - Y_0, \quad \tau_{xy} = -\phi'_1 - 2x\phi'_2.
\]  

(90)

It is obtained from Eqs. (57)–(59) that,

\[
\begin{align*}
Q_0 &= \phi_1 \left( -\frac{h}{2} \right) - \phi_1 \left( \frac{h}{2} \right) = -\frac{h}{2}c_1 - \phi_1 \left( \frac{h}{2} \right), \\
T_0 &= \phi_0 \left( \frac{h}{2} \right) - \phi_0 \left( -\frac{h}{2} \right) = \phi_0 \left( \frac{h}{2} \right), \\
M_0 &= \frac{h}{2} \left[ \phi'_0 \left( \frac{h}{2} \right) + \phi'_0 \left( -\frac{h}{2} \right) \right] - \phi_0 \left( \frac{h}{2} \right) + \phi_0 \left( -\frac{h}{2} \right) = \frac{h}{2}\phi'_0 \left( \frac{h}{2} \right) - \phi_0 \left( \frac{h}{2} \right).
\end{align*}
\]  

(91)

(92)

(93)

From Eq. (46) and by virtue of Eq. (47), we obtain

\[
\begin{align*}
&u = x(s_{11}\phi''_0 - s_{12}Y_0) + \frac{1}{2}x^2(s_{11}\phi''_1 - s_{11}X_1) + \frac{1}{3}x^3(s_{11}\phi''_2 - s_{11}X_2) \\
&\quad + 2xs_{12}\phi_2 - xs_{16}\phi'_1 - x^2s_{16}\phi'_2 - \int_{-\frac{1}{2}}^{y} F(\xi)\,d\xi + c_0y + u_0, \\
&v = \int_{-\frac{1}{2}}^{y} (s_{12}\phi''_0 - s_{22}Y_0)\,d\xi - \frac{x^2a_0}{2} + x\left[ \int_{-\frac{1}{2}}^{y} (s_{12}\phi''_1 - s_{12}X_1)\,d\xi - \frac{x^2a_1}{2} \right] + x^2 \left[ \int_{-\frac{1}{2}}^{y} (s_{12}\phi''_2 - s_{12}X_2)\,d\xi - \frac{x^2a_2}{3} - 4 \right] \\
&\quad + \int_{-\frac{1}{2}}^{y} s_{22}\phi_2\,d\xi - \int_{-\frac{1}{2}}^{y} s_{26}\phi'_1\,d\xi - 2x\int_{-\frac{1}{2}}^{y} s_{26}\phi'_2\,d\xi - c_0x + v_0,
\end{align*}
\]  

(94)

where

\[
F(y) = s_{66}\phi'_{11} - 2s_{26}\phi_2 - s_{16}\phi''_0 + s_{26}Y_0 + \int_{-\frac{1}{2}}^{y} [s_{12}\phi''_1 - 2s_{26}\phi'_2 - s_{12}X_1] \,d\xi.
\]  

(95)

In the following, we first present the analytical solution for a cantilever beam subjected to body forces, which is then followed by the solution for the beam subjected to surface loads.

**Example 6.3** (Cantilever beam subjected to body forces \( F_x = X_0(y) + 2xX_2(y) \), \( F_y = Y_0(y) \)). By virtue of the boundary conditions \( \tau_{xy} = \sigma_y = 0 \) at \( y = \pm h/2 \) we obtain from Eq. (90)

\[
\phi'_1(\pm h/2) = 0, \quad \phi'_2(\pm h/2) = 0, \quad 2\phi_2(\pm h/2) - Y_0(\pm h/2) = 0.
\]  

(96)
Substitution of Eqs. (87) and (88) into Eq. (96), yields
\begin{equation}
    c_2 = 0, \quad c_1 = 0,
\end{equation}
\begin{equation}
    a_2 f_{10}(h/2) + b_2 f_{00}(h/2) + z_{00}^2(h/2) = 0,
\end{equation}
\begin{equation}
    a_1 f_{10}(h/2) + b_1 f_{00}(h/2) + 2a_2 g_{10}(h/2) + 2b_2 g_{00}(h/2) + 2z_{10}^2(h/2) + z_{00}^2(h/2) = 0,
\end{equation}
\begin{equation}
    2d_2 - Y_0(-h/2) = 0,
\end{equation}
\begin{equation}
    2a_2 f_{11}(h/2) + b_2 f_{01}(h/2) + 2d_2 + 2z_{01}^2(h/2) - Y_0(h/2) = 0.
\end{equation}
Eq. (100) immediately determines \(d_2\), substitution of which into Eq. (101) gives
\begin{equation}
    2a_2 f_{11}(h/2) + b_2 f_{01}(h/2) + 2z_{01}^2(h/2) + Y_0(-h/2) - Y_0(h/2) = 0.
\end{equation}
From Eqs. (102) and (98), we can obtain \(a_2\) and \(b_2\). By virtue of the boundary conditions \(Q_0 = 0\), \(T_0 = 0\) and \(M_0 = 0\) at the free end \(x = 0\), the substitution of Eqs. (88) and (89) into Eqs. (91)–(93) yields
\begin{equation}
    a_1 f_{11}(h/2) + b_1 f_{01}(h/2) + 2a_2 g_{11}(h/2) + 2b_2 g_{01}(h/2) + 2z_{11}(h/2) + z_{01}(h/2) = 0,
\end{equation}
\begin{equation}
    a_0 f_{10}(h/2) + b_0 f_{00}(h/2) + a_1 g_{10}(h/2) + b_1 g_{00}(h/2) + 2a_2 g_{10}(h/2) + 2b_2 g_{00}(h/2) - 2d_2 f_{00}(h/2)
    + z_{10}^2(h/2) + 2z_{20}(h/2) + z_{00}^2(h/2) = 0,
\end{equation}
\begin{equation}
    a_0 f_{11}(h/2) + b_0 f_{01}(h/2) + a_1 g_{11}(h/2) + b_1 g_{01}(h/2) + 2a_2 g_{11}(h/2) + 2b_2 g_{01}(h/2)
    - 2d_2 f_{01}(h/2) + z_{11}(h/2) + 2z_{21}(h/2) + z_{01}^2(h/2) = 0.
\end{equation}
From Eqs. (99) and (103), we can obtain \(a_1\) and \(b_1\). Then by noticing Eq. (100), \(a_0\) and \(b_0\) can be determined from Eqs. (104) and (105). Hence we have completely derived the stress components in Eq. (90) as well as the displacements. The constants \(u_0\), \(v_0\) and \(\omega\) in the displacements can be determined from the fixed boundary conditions \(u = v = 0\), \(\partial u/\partial x = 0\) and \(\partial v/\partial y = 0\) at point \((l, 0)\).

Consider the case of \(X = x^2a/2\) and \(Y = by\), where \(a\) and \(b\) are known constants. This type of body force corresponds to the beam under tension and bending, induced by a centrifugal force in \(x\)-direction and gravity in \(y\)-direction. When \(s_{ij} = \text{const.}\), we obtain from Eqs. (97)–(105)
\begin{equation}
    a_2 = -6bs_{11}/h^2, \quad b_2 = -as_{11}/2, \quad c_2 = 0, \quad d_2 = -bh/4, \quad a_1 = as_{16}, \quad b_1 = s_{16}(ah - 4b)/2,
\end{equation}
\begin{equation}
    c_1 = 0, \quad a_0 = -4s_{16}^2b + \frac{6}{5}s_{66}b + \frac{7}{5}s_{12}b - \frac{s_{12}ah}{2}, \quad b_0 = \frac{1}{6}(3b - ah)s_{12} + \frac{hbs_{66}}{2}.
\end{equation}
Hence we obtain the stress and displacements as follows
\begin{equation}
    \sigma_x = -\frac{6x^2y}{h^2}b - \frac{x^2}{2}a + \left[\frac{2s_{11}s_{12} + s_{11}s_{66} - 4s_{16}^2y^2}{s_{11}h^2} - \frac{12s_{16}x}{10} - \frac{6s_{16}^2}{5s_{11}} + \frac{3s_{12}x}{2}\right]y + \frac{s_{16}x}{s_{11}}\frac{b}{s_{11}},
\end{equation}
\begin{equation}
    \tau_{xy} = \frac{2b}{h^2}y\left(\frac{b^2}{4} - y^2\right), \quad \tau_{yx} = -\frac{b(h - 2y)(h + 2y)(2s_{16}y + 3s_{11}x)}{2s_{11}h^2}.
\end{equation}
Now, we investigate the equilibrium of the beam subjected to a uniform load \(q\) applied on its upper surface. In absence of body force, i.e. \(X = Y = 0\), we have \(z_{0j}(y) = 0\). From the boundary conditions \(\tau_{xy} = 0\) at \(y = \pm h/2\), \(\sigma_y = 0\) at \(y = h/2\), and \(\sigma_x = -q\) at \(y = -h/2\), we obtain from Eq. (90) that
\begin{equation}
    \phi'_x(\pm h/2) = 0, \quad \phi'_n(\pm h/2) = 0, \quad \phi_x(h/2) = 0, \quad 2\phi_x(-h/2) = -q.
\end{equation}
The substitution of Eqs. (87) and (88) into Eq. (108) yields
\begin{equation}
    c_2 = 0, \quad c_1 = 0,
\end{equation}
\begin{equation}
    a_2 f_{10}(h/2) + b_2 f_{00}(h/2) = 0,
\end{equation}
\begin{equation}
    a_1 f_{10}(h/2) + b_1 f_{00}(h/2) + 2a_2 g_{10}(h/2) + 2b_2 g_{00}(h/2) = 0,
\end{equation}
\begin{equation}
    2d_2 = -q,
\end{equation}
\begin{equation}
    a_2 f_{11}(h/2) + b_2 f_{01}(h/2) + d_2 = 0.
\end{equation}
Obviously we can obtain $d_2$, $a_2$ and $b_2$ from Eqs. (109d), (109b) and (109e). Then we can obtain $a_1$, $b_1$, $a_0$ and $b_0$ from the boundary conditions at the two ends. In the following, we will consider three kinds of beams: the cantilever beam, the simply supported beam, the fixed–fixed beam.

**Example 6.4 (The cantilever beam).** The boundary conditions are, $T_0 = 0$, $M_0 = 0$ and $Q_0 = 0$ at $x = 0$, and $u = v = 0$, $\partial u/\partial x = 0$ or $\partial u/\partial y = 0$ at $(l, 0)$. From Eqs. (91)-(93), we obtain

\begin{align}
  a_0 f_{10}(h/2) + b_0 f_{00}(h/2) + a_1 g_{10}(h/2) + b_1 g_{00}(h/2) + 2a_3 g_{10}^p(h/2) + 2b_3 g_{00}^p(h/2) - 2d_2 f_{00}^2(h/2) &= 0, \\
  a_0 f_{11}(h/2) + b_0 f_{01}(h/2) + a_1 g_{11}(h/2) + b_1 g_{01}(h/2) + 2a_3 g_{11}^p(h/2) + 2b_3 g_{01}^p(h/2) - 2d_2 f_{01}^2(h/2) &= 0, \\
  a_1 f_{11}(h/2) + b_1 f_{01}(h/2) + 2a_3 g_{11}^p(h/2) + 2b_3 g_{01}^p(h/2) &= 0. 
\end{align}

(110a)

(110b)

(110c)

Obviously, $a_1$ and $b_1$ can be determined from Eqs. (109c) and (110c), while $a_0$ and $b_0$ from Eqs. (110a) and (110b). The expressions for these constants are obtained as

\begin{align}
  a_2 &= q A_2 / 2, \quad b_2 = -q D_2 / 2, \quad c_2 = 0, \quad d_2 = -q / 2, \\
  a_1 &= q A_3, \quad b_1 = q D_3, \quad c_1 = 0, \quad a_0 = q A_4, \quad b_0 = q D_4, 
\end{align}

where $A_1$ and $D_1$ were given in Eq. (79), and

\begin{align}
  A_3 &= (H_0 G_3 - H_3 G_1)/H_{04}, \quad D_3 = (H_4 G_1 - H_1 G_3)/H_{04}, \\
  A_4 &= (H_0 G_5 - H_5 G_4)/H_{04}, \quad D_4 = (H_4 G_4 - H_1 G_5)/H_{04}, 
\end{align}

where $H_i$ and $G_i$ were given in Eqs. (64) and (79), and

\begin{align}
  G_3 &= [H_1 g_{10}(h/2) - H_0 g_{11}(h/2)]/H_{04}, \\
  G_4 &= D_1 g_{10}^p(h/2) - A_1 g_{10}^p(h/2) - A_3 g_{10}(h/2) - D_3 g_{00}(h/2) - f_{00}^2(h/2), \\
  G_5 &= D_1 g_{11}^p(h/2) - A_1 g_{11}^p(h/2) - A_3 g_{11}(h/2) - D_3 g_{01}(h/2) - f_{01}^2(h/2). 
\end{align}

(111c)

Substituting Eq. (111a) into Eq. (90), and by virtue of Eqs. (87)-(89), we obtain

\begin{align}
  \sigma_x &= q \frac{x}{s_{11}} \{A_4 y + D_4 + A_3 [x y + s_{16} f_{10}(y) + f_{10}^p(y)] + D_3 [x + s_{16} f_{00}(y) + f_{00}^p(y)] \\
  &+ A_1 [x^2/2 + s_{16} x f_{10}(y) + x f_{10}^p(y) + s_{16} g_{10}(y) - s_{12} f_{11}(y) + B_1(y)] \\
  &- D_1 [x^2/2 + s_{16} x f_{00}(y) + x f_{00}^p(y) + s_{16} g_{00}(y) - s_{12} f_{01}(y) + B_0(y)] + s_{12}\}, \\
  \sigma_y &= q [A_1 f_{11}(y) - D_1 f_{01}(y) - 1], \\
  \tau_{xy} &= -q \{A_3 f_{10}(y) + D_3 f_{00}(y) + A_1 [x f_{10}(y) + g_{10}(y)] - D_1 [x f_{00}(y) + g_{00}(y)]\}. 
\end{align}

(112)

For homogeneous anisotropic materials, i.e. $s_{ij} =$ const., noticing the expressions in Appendix, we obtain

\begin{align}
  A_1 &= -12 s_{11}/h^3, \quad D_1 = 0, \quad A_3 = 0, \quad D_3 = -2 s_{16}/h, \\
  A_4 &= 2(6 s_{11} s_{12} + 3 s_{11} s_{66} - 2 s_{11}^2)/(5 s_{11} h), \quad D_4 = s_{66}/2. 
\end{align}

(113)

Substitution of Eq. (114) into Eq. (112) yields

\begin{align}
  \sigma_x &= -\frac{6 q x^3 y}{h^3} + q \left[\frac{s_{16}}{s_{11}} \left(1 - 12 \frac{y^2}{h^2}\right) x \right] + 2 \left(\frac{2 s_{12} + s_{66}}{4 s_{11}} - \frac{s_{16}^2}{s_{11}}\right) \left(\frac{4 y^3}{h} - \frac{3 y}{5h}\right), \\
  \sigma_y &= -\frac{q (y + h)(h - 2 y)^2}{2 h^3}, \quad \tau_{xy} = q (2 y - h)(2 y + h)(2 s_{16} y + 3 s_{11} x) \\
  &= \frac{2 s_{11} h^4}{2 s_{11}}. 
\end{align}

(114)

Eq. (114) coincides with the results of a homogeneous anisotropic beam presented by Lekhnitskii (1968) and Hashin (1967).
By virtue of the fixed boundary conditions \( u = v = 0 \) and \( \partial u / \partial x = 0 \) or \( \partial u / \partial y = 0 \) at \((l, 0)\), we can obtain the displacement components.

**Example 6.5 (The simply supported beam).** Because Eq. (53) is valid for the current problem of simply supported beam, we shall take Eqs. (54)–(56) as the boundary conditions, among which Eq. (54c) becomes \( Q_0 = ql/2 \). Calculating \( Q_0 \) with Eqs. (91) and (88), we obtain

\[
a_1 f_{11}(h/2) + b_1 f_{01}(h/2) + 2a_2 g_{11}(h/2) + 2b_2 g_{01}(h/2) = -ql/2. \tag{115}
\]

The following expressions for \( a_2, b_2, c_2, d_1, a_1, c_1, a_0 \) and \( b_0 \) then can be obtained from Eqs. (109), (110a), (110b) and (115),

\[
\begin{align*}
a_2 &= qA_1/2, & b_2 &= -qD_1/2, & c_2 &= 0, & d_2 &= -q/2, \\
a_1 &= qA_3', & b_1 &= qD_3', & c_1 &= 0, & a_0 &= qA_4', & b_0 &= qD_4',
\end{align*}
\tag{116a}
\]

where \( A_1 \) and \( D_1 \) were given in Eq. (79), and

\[
\begin{align*}
A_3' &= A_3 + \Delta A_3, & D_3' &= D_3 + \Delta D_3, & A_4' &= A_4 + \Delta A_4, & D_4' &= D_4 + \Delta D_4
\end{align*}
\tag{116b}
\]

with \( A_3, D_3, A_4 \) and \( D_4 \) shown in Eq. (111b), and

\[
\begin{align*}
\Delta A_3 &= -lH_0/(2H_{04}), & \Delta D_3 &= lH_1/(2H_{04}), \\
\Delta A_4 &= l[H_2 g_{11}(h/2) - H_3 g_{01}(h/2) - H_4 g_{10}(h/2) + H_5 g_{00}(h/2)]/(2H_{04}^2), \\
\Delta D_4 &= l[H_4 H_0 g_{10}(h/2) - H_4 H_0 g_{00}(h/2) - H_3 H_1 g_{11}(h/2) + H_5^2 g_{01}(h/2)]/(2H_{04}^2).
\end{align*}
\tag{116c}
\]

By virtue of Eqs. (87)–(89) and (116a), we readily obtain the stresses from Eq. (90)

\[
\begin{align*}
\sigma_x &= \frac{q}{s_{11}} [A_3' y + D_4' - A_3' x + s_{16} f_{01}(y) + f_{00}^0(y)] + D_3' [x + s_{16} f_{00}(y) + f_{00}^0(y)] \\
&\quad + A_1 [x^2 y/2 + s_{16} x f_{01}(y) + x f_{01}^0(y) + s_{16} g_{10}(y) - s_{12} f_{11}(y) + B_1(y)] \\
&\quad - D_1 [x^2 y/2 + s_{16} x f_{00}(y) + x f_{00}^0(y) + s_{16} g_{00}(y) - s_{12} f_{01}(y) + B_0(y)] + s_{12}, \\
\tau_{xy} &= -q[A_3' f_{10}(y) + D_4' f_{00}(y) + A_1 [x f_{10}(y) + g_{10}(y)] - D_1 [x f_{00}(y) + g_{00}(y)]].
\end{align*}
\tag{117}
\]

For homogeneous materials with \( s_{ij} = \text{const.} \), by using the formulations in Appendix, we obtain from Eq. (116c)

\[
\begin{align*}
\Delta A_3 &= 6ls_{11}/h^2, & \Delta D_3 &= 0, & \Delta A_4 &= 0, & \Delta D_4 &= ls_{16}/h.
\end{align*}
\tag{118}
\]

The stresses can be calculated from Eq. (117) by virtue of Eqs. (116b), (113) and (118) as

\[
\begin{align*}
\sigma_x &= \frac{6}{h} q(l - x)y + \frac{s_{16} q}{2 s_{11} h} (2x - l) \left( 1 - \frac{12 y^2}{h^2} \right) + 2q \left( \frac{2s_{12} + s_{66}}{4s_{11}} - \frac{s_{16}^2}{s_{11}^2} \right) \left( \frac{4y^3}{h^3} - \frac{3y}{5h} \right), \\
\sigma_y &= -\frac{q(y + h)(h - 2y)^2}{2h^3}, & \tau_{xy} &= \frac{q(4y^2 - h^2)(4s_{16} y - 3s_{11} l + 6s_{11} x)}{4s_{11} h^3}. 
\end{align*}
\tag{119}
\]

Eq. (119) coincides with the results for homogeneous anisotropic beams derived by Hashin (1967).

From the boundary conditions in Eqs. (55) and (56), the constants \( u_0, v_0 \) and \( \omega \) can be determined, thus the displacement expressions as in Eq. (94) are absolutely determined.

Substituting \( 2l \) for \( l \) and \( x + l \) for \( x \) in Eq. (119), we obtain the stresses in the coordinate system with the origin locating at the middle span of the beam with length \( 2l \). One stress component is

\[
\sigma_x = \frac{q}{2l} (l^2 - x^2) y + q \left[ \frac{s_{16} x}{h} \left( 1 - \frac{12 y^2}{h^2} \right) + 2 \left( \frac{s_{16}^2}{s_{11}^2} - \frac{2s_{12} + s_{66}}{4s_{11}} \right) \left( -\frac{4y^3}{h^3} + \frac{3y}{5h} \right) \right].
\tag{120}
\]
In the same coordinate system, the solution of Lekhnitskii (1968) is

\[
\sigma_x = \frac{q}{2\pi} (l^2 - x^2) y + \left( -\frac{s_{16}}{s_{11}} \frac{x}{h} \left( 1 - \frac{12y^2}{h^2} \right) + 2 \left( \frac{\pi_{16}}{s_{11}} - \frac{2s_{12} + s_{00}}{4s_{11}} \right) \left( -\frac{4y^3}{h^3} + \frac{3y^3}{5h} \right) \right). \tag{121}
\]

On substituting Eq. (121) and \( \tau_{xy} \) in Eq. (119) into Eq. (1), we find the equilibrium equations cannot be satisfied. Actually, by comparing with Eq. (120), it is seen that Eq. (121) contains an incorrect sign.

**Example 6.6** (Fixed–fixed beam). The boundary conditions are \( u = v = 0, \partial u/\partial x = 0 \) or \( \partial u/\partial y = 0 \) at \((0,0)\) and \((l,0)\). Obviously, \( c_1, c_2, d_2, d_2 \) and \( b_2 \) can be calculated from Eqs. (109a), (109b), (109d), and (109e). By virtue of the boundary conditions at the two ends, together with Eq. (109c), we have seven independent equations to solve for seven unknowns \( \alpha_1, b_1, a_0, b_0, u_0, v_0 \) and \( \omega \). Hence the stresses in Eq. (90) and displacements in Eq. (94) are determined. They will not be presented here because their expressions are too lengthy.

For homogeneous materials with \( s_{ij} = \text{const.} \), if the fixed boundary conditions at the two ends are adopted as \( u = v = 0 \) and \( \partial u/\partial x = 0 \), we can get

\[
\sigma_x = -\left( 6x^2 - 6xl + l^2 \right) \frac{qy}{h^3} + \frac{s_{11}s_{66} + 2s_{11}s_{12} - 4s_{12}}{2s_{11}^2 h^3} qy(4y^2 - h^2) + \frac{s_{16}}{2s_{11} h} q(l - 2x)(12y^2 - h^2) + \frac{s_{12}}{2s_{11} h} q(h - y) - \frac{s_{66}}{s_{11} h} qy,
\]

\[
\sigma_y = -\frac{q(y + h)(h - 2y)^2}{2h^3}, \quad \tau_{xy} = \frac{q}{4h} \left( 4y^2 - h^2 \right) \left[ \frac{4s_{16}}{s_{11}} y + 3(2x - l) \right], \tag{122}
\]

\[
u = \frac{3s_{11}s_{66} - 4s_{10}^2}{2s_{11}^2 h^3} + \frac{3s_{11}s_{10}}{2h^3} + \frac{3s_{11}^2}{2h^3} - \frac{s_{00}}{s_{11}^2 h} qy + \frac{s_{16} q}{s_{11} h} (l - x),
\]

\[
u = \frac{2s_{11}^2 + 2s_{11}s_{00} - 4s_{12}s_{12}^2}{2s_{11}^2 h^3} + \frac{2s_{11}^2}{2h^3} + \frac{2s_{11}^2}{2h^3} - \frac{s_{12}^2}{s_{11}^2 h^3} + \frac{3s_{12}}{2h^3} + \frac{3s_{12}s_{00}}{2h^3} + \frac{3s_{12}}{4h} qy^2 + \frac{s_{16}^2}{s_{11} h} (l - x)^2 - \frac{x}{2h^3} + \frac{s_{12}^2}{s_{11}^2 h^3} - \frac{s_{12}s_{12}}{2h^3} qy + \frac{s_{16} q}{s_{11} h} (l - x)^2 x^2. \tag{123}
\]

If \( \partial v/\partial x = 0 \) in the boundary conditions is replaced by \( \partial u/\partial y = 0 \), we can get

\[
\sigma_x = -\left( 6x^2 - 6xl + l^2 \right) \frac{qy}{h^3} + \left( \frac{s_{11}s_{66} + 2s_{11}s_{12} - 4s_{12}}{2s_{11}^2 h^3} \right) 2qy^3 + \frac{s_{16}}{s_{11} h^3} \left[ l - 2x + \frac{2s_{11}s_{26} - 2s_{12}s_{16}}{2s_{11}^2 h^2 - 3s_{12}^2 h^2 + 3s_{11}s_{66} h^2} \right] qy^2 - \frac{12x(s_{12}s_{16} - s_{11}s_{26})}{2s_{11}^2 l^2 - 3s_{11}^2 h^2 + 3s_{11}s_{66} h^2} qy^2 - \frac{s_{16} q}{s_{11} h} (2x - l) + \frac{s_{16} q}{s_{11} h} (2x - l) - \frac{6(s_{11}s_{26} - s_{12}s_{16})}{2s_{11}^2 l^2 + 3s_{16} h^2 s_{11} - 3s_{12} h^2} qy + \frac{s_{16} q}{s_{11} h} (2x - l)
\]

\[
\sigma_y = -\frac{q(y + h)(h - 2y)^2}{2h^3}, \quad \tau_{xy} = \frac{q}{4h} \left[ 3(2x - l) + 4 \frac{s_{16}}{s_{11}} y - \frac{6(s_{11}s_{26} - s_{12}s_{16})}{2s_{11}^2 l^2 - 3s_{12}^2 h^2 + 3s_{11}s_{66} h^2} \right]. \tag{124}
\]
$u = \frac{q y^4}{3} (3s_{11}s_{16}s_{66} - 4s_{16}^3 - 2s_{26}s_{s11} + 4s_{11}s_{12}s_{16}) + \frac{s_{11}s_{12} - 2s_{16}^2 + s_{11}s_{66}}{s_{11}h^3} \left[ 2x - l - \frac{2(s_{11}s_{26} - s_{12}s_{16})h^2}{s_{11}l^2 - 3s_{16}^2h^2 + 3s_{11}s_{66}h^2} \right] qy^3$

$$- \frac{3s_{16}}{h^3} x - \frac{3s_{16}l}{h^3} x - \frac{6s_{16}(s_{11}s_{26} - s_{12}s_{16})}{s_{11}l^2 - 3s_{16}^2h^2 + 3s_{11}s_{66}h^2} x + \frac{s_{11}l^2}{2h^3} - \frac{6s_{11}s_{26} + s_{16}(2s_{16}^2 - 5s_{11}s_{12} - 2s_{11}s_{66})}{4s_{11}h}$$

$$+ \frac{3s_{16}(s_{11}s_{26} - s_{12}s_{16})}{2s_{11}l^2 - 3s_{16}^2h^2 + 3s_{11}s_{66}h^2} qy^3 + \frac{s_{11}}{h^3} (l-x)(2x-l)xy$$

$$+ 6q_{s11}(l-x)xy \left( s_{12}s_{16} - s_{26}s_{11} \right) + \frac{q_{s16}}{4h} (l-x),$$

$$v = \frac{2s_{11}s_{12}^2 - s_{11}s_{22} + s_{11}s_{12}s_{66} + 2s_{11}s_{26} - 4s_{12}s_{16}}{2s_{11}h^3} qy^4 + \frac{(s_{11}s_{26} - 2s_{12}s_{16})}{s_{11}h^3}$$

$$\times \left( 2x - l + \frac{2(s_{11}s_{26} - s_{12}s_{16})h^3}{s_{11}l^2 + 3s_{11}s_{66}h^2 - 3s_{16}^2h^2} \right) qy^3 - \frac{3s_{12}}{s_{11}h^3} \left[ x - l - \frac{2(s_{11}s_{26} - s_{12}s_{16})}{s_{11}l^2 + 3s_{11}s_{66}h^2 - 3s_{16}^2h^2} \right] qxy^2$$

$$+ \frac{2s_{12}s_{16}^2 - s_{11}s_{12} + 3s_{11}s_{22} - 2s_{11}s_{12}s_{66}}{4s_{11}h} \left[ \frac{1}{2h^3} - \frac{3s_{12}(s_{11}s_{26} - s_{12}s_{16})}{s_{11}l^2 + 3s_{11}s_{66}h^2 - 3s_{16}^2h^2} \frac{s_{11}l^2}{2h^3} \right] qxy^2$$

$$+ \frac{s_{12}s_{16} - s_{11}s_{22}}{2s_{11}h^3} qxy + \frac{s_{11}l^2}{2h^3} \left[ \frac{1}{2h^3} + \frac{3(s_{11}s_{26} - s_{12}s_{16})h^2 l + (2s_{11}s_{26} - s_{12}s_{16})h^3}{4s_{11}h(2s_{11}l^2 + 3s_{11}s_{66}h^2 - 3s_{16}^2h^2)} \right] qy$$

$$- \frac{s_{11}qy^3(l-x)}{2h^3} + \frac{1}{2h^3} \left[ \frac{2s_{11}qy^3(l-x)}{2h^3} + \frac{3(s_{11}s_{26} - s_{12}s_{16})h^2 l + (2s_{11}s_{26} - s_{12}s_{16})h^3}{4s_{11}h(2s_{11}l^2 + 3s_{11}s_{66}h^2 - 3s_{16}^2h^2)} \right],$$

Equation (125)

If $s_{11} = 1/E$, $s_{12} = -\mu/E$, $s_{66} = 2(1+\mu)/E$ and $s_{16} = s_{26} = 0$, where $E$ denotes Young's modulus and $\mu$ Poisson's ratio, Eqs. (122)-(125) degenerate to the isotropic solution, which coincides with the results obtained by Ding et al. (2005). We note that the terms of $y^2$ in $\sigma_y$, $y^3$ in $\tau_{xy}$, $y^4$ and $y^5$ in $u$ and $y^3$ in $v$ appearing in the above anisotropic solution are not involved in the solutions for isotropic or orthotropic beams.

7. A numerical example

With the stress function represented by Eq. (12), we can solve beam problems with one or more elastic compliance constants depending on the thickness coordinate $y$. Let us consider, for example, an anisotropic FGM cantilever beam subjected to a shear force $P$ applied at the free end. We assume that $s_{11}$ is an exponential function of $y$, namely $s_{11}(y) = s_{11}^0 \exp \left( \lambda \frac{y+y/2}{h} \right)$, where $s_{11}^0$ is the value at $y = -h/2$, while all other elastic compliance parameters are constant, i.e. $s_{ij} = s_{ij}^0$.

Take the stress function in the form given by Eq. (68). The stresses have been given in Eq. (80). After being integrated, Eq. (80) can be written as

$$\sigma_x = \frac{P\lambda^2 y \left[ s_{11}^0 \left( e^{\frac{3\lambda y}{2h}} - e^{\frac{\lambda y}{2h}} \right) + 2s_{11}^0 h \left( e^{-\frac{\lambda y}{2h}} - e^{\frac{-\lambda y}{2h}} \right) \right]}{(\lambda^2 e^y - e^{2y} + 2e^y - 1)s_{11}^0 h^3}$$

$$+ \frac{\lambda^2 Pe}{2h^2(\lambda^2 e^y - e^{2y} + 2e^y - 1)} \left[ (\lambda - 2)e^{\frac{3\lambda y}{2h}} + (\lambda + 2)e^{\frac{-\lambda y}{2h}} \right]$$

$$- \frac{s_{11}^0 P e^y}{2s_{11}^0 h(\lambda^2 e^y - e^{2y} + 2e^y - 1)} \left( e^{\frac{-3\lambda y}{2h}} - e^{\frac{-\lambda y}{2h}} - 2\lambda e^{\frac{-\lambda y}{2h}} + 2\lambda e^{\frac{-3\lambda y}{2h}} + 2\lambda e^{\frac{-\lambda y}{2h}} \right),$$

$$\sigma_y = 0, \quad \tau_{xy} = \frac{Pe^y \left( e^{\frac{-3\lambda y}{2h}} - e^{\frac{-\lambda y}{2h}} \right)}{h^2(\lambda^2 e^y - e^{2y} + 2e^y - 1)} + \frac{Pe^y \left( e^{\frac{-\lambda y}{2h}} + e^{\frac{-3\lambda y}{2h}} - 2\lambda e^{\frac{-\lambda y}{2h}} \right)}{2h(\lambda^2 e^y - e^{2y} + 2e^y - 1)}.$$
When $\lambda = 0$, by using L’Hospital’s rule, we can obtain the stress components for a homogeneous beam from Eq. (126).

For numerical calculation, we take the shear force $P = 1000$ (N/m), the span of the beam $l = 1$ (m), the height $h = 0.1$ (m), and the material properties in Table 1.

The curves of $\sigma_x h/P$ versus $y/h$ at the middle span are shown in Fig. 2. The curve for $\lambda = 1$ is concave, the curve for $\lambda = 0$ tends to a straight line, while that for $\lambda = -1$ is convex.

The curves for the dimensionless stress $\tau_{xy} h/P$ versus $y/h$ at the middle span are shown in Fig. 3. The location of maximum shear stress changes with $\lambda$. When $\lambda = 0$, the curve tends to a parabola, and the maximum value of $\tau_{xy} h/P$ occurs at $y = 0$. When $\lambda = 1$, the maximum shear stress locates near $y = -0.1h$, while it is near $y = 0.1h$ when $\lambda = -1$.

The stress fields for different types of fixed-end boundary conditions are the same, whereas the displacement expressions are different. By applying these two fixed-end boundary conditions, we can determine the displace-

Table 1

<table>
<thead>
<tr>
<th>$s_{11}^0$ (m^2/N)</th>
<th>$s_{12}^0$ (m^2/N)</th>
<th>$s_{16}^0$ (m^2/N)</th>
<th>$s_{22}^0$ (m^2/N)</th>
<th>$s_{26}^0$ (m^2/N)</th>
<th>$s_{66}^0$ (m^2/N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.150e−9</td>
<td>−0.0259e−9</td>
<td>−0.0693e−9</td>
<td>0.01032e−9</td>
<td>−0.0144e−9</td>
<td>0.1464e−9</td>
</tr>
</tbody>
</table>

Fig. 2. $\sigma_x h/P$ versus $y/h$ ($x = 0.5l$).

Fig. 3. $\tau_{xy} h/P$ versus $y/h$ ($x = 0.5l$).
ment expressions from Eq. (46). The relationships between \( v(0,0)/h \) and \( \lambda \) for the two boundary condition are illustrated in Fig. 4, where BC1 denotes the result for boundary condition \( u = v = 0, \partial u/\partial x = 0 \) and BC2 for \( u = v = 0, \partial u/\partial y = 0 \). The two results are very close to each other for the beam with a span-to-height ratio of 10.

We also compare our analytical solution with the FEM solution by MSC.Nastran. The Quad4 element of 0.01 m \( \times \) 0.01 m is employed, i.e. there are totally 1000 elements for the whole beam. Since the beam is inhomogeneous, the material property of each element is set equal to that at the center of the element. The boundary conditions in the FEM model are \( u = v = 0 \) at \( x = l, -h/2 \leq y \leq h/2 \). The force \( P \) is directly applied to point (0,0). The FEM results are simultaneously presented in Figs. 2–4, where a good agreement can be observed between the two solutions.

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**Appendix**

Some functions and coefficients for homogeneous beams (\( s_{ij} = \text{const.} \))

A. \[
\begin{align*}
    f_{00}(y) &= \frac{1}{(n + 1)!2^{n+1}s_{11}}(2y + h)^{n+1}, \\
    f_{01}(y) &= \frac{1}{2s_{11}}(2y + h), \\
    f_{02}(y) &= \frac{1}{8s_{11}}(2y + h)^2, \\
    f_{03}(y) &= \frac{1}{48s_{11}}(2y + h)^3, \\
    f_{04}(y) &= \frac{1}{384s_{11}}(2y + h)^4.
\end{align*}
\]

B. \[
\begin{align*}
    f_{m0}(y) &= \frac{1}{(m + 1)!2^{m+1}s_{11}}[(2y)^{m+1} - (-h)^{m+1}], \\
    f_{10}(y) &= \frac{1}{8s_{11}}(4y^2 - h^2), \\
    f_{20}(y) &= \frac{1}{24s_{11}}(8y^3 + h^3).
\end{align*}
\]
C. \[ f_{16}(y) = \frac{1}{(n+2)!2^{n+3}} (2y + h)^{n+1} y \] 
\[ f_{11}(y) = \frac{1}{24s_{11}} (2y + h)^3 (y - h), \] 
\[ f_{12}(y) = \frac{1}{384s_{11}} (2y + h)^3 (2y - 3h), \] 
\[ f_{13}(y) = \frac{1}{1920s_{11}} (2y + h)^4 (y - 2h). \] 
\[ \text{D. } f_{mn}^6(y) = s_{16} f_{mn}(y). \] 
\[ \text{E. } f_{mn}^2(y) = s_{12} f_{mn}(y). \] 
\[ \text{F. } g_{mn}(y) = \frac{2s_{16}}{s_{11}} f_{mn+1}(y). \] 
\[ \text{G. } g_{mn}^6(y) = \frac{4s_{16}^2 - 2s_{11}s_{12} - s_{11}s_{66}}{s_{11}^2} f_{mn+2}(y). \] 
\[ \text{H. } B_m(y) = \frac{2s_{16}^2 - s_{11}s_{12} - s_{11}s_{66}}{s_{11}} f_{m1}(y). \] 
\[ \text{I. } B_{00}(y) = (s_{16}^2 - s_{11}s_{66}) f_{00}(y). \] 
\[ \text{J. } h_y(y) = \frac{1}{(n+2)!2^{n+3}} (2y + h)^{n+1}[(2s_{16}^2 - s_{11}s_{12} - s_{11}s_{66})(2y + h) + (n+2)s_{11}s_{12}h]. \] 
\[ \text{K. } H_0 = f_{00}(h/2) = h/s_{11}, \quad H_1 = f_{10}(h/2) = 0, \quad H_2 = f_{20}(h/2) = h^3/(12s_{11}), \] 
\[ H_3 = f_{01}(h/2) = h^2/(2s_{11}), \quad H_4 = f_{11}(h/2) = -H_2, \quad H_5 = H_0h/2 - H_3 = 0, \] 
\[ H_6 = H_4h/2 - H_4 = H_2, \] 
\[ H_{02} = H_0H_2 - H_1^2 = H_0H_2 = h^4/(12s_{11}^2), \quad H_{04} = H_0H_4 - H_1H_3 = H_0H_4 = -H_{02}, \] 
\[ H_{06} = H_0H_6 - H_1H_5 = H_0H_2 = H_{02}, \quad f_{02}(h/2) = h^4/(6s_{11}), \quad f_{03}(h/2) = h^4/(24s_{11}), \] 
\[ f_{12}(h/2) = -h^4/(24s_{11}), \quad f_{13}(h/2) = -h^5/(80s_{11}), \] 
\[ f_{06}(h/2) = \frac{s_{16}}{s_{11}} h, \quad f_{01}(h/2) = \frac{s_{16}}{2s_{11}} h^2, \quad f_{06}(h/2) = \frac{s_{16}}{s_{11}} h, \quad f_{01}(h/2) = \frac{s_{16}}{2s_{11}} h^2, \] 
\[ g_{00}(h/2) = \frac{s_{16}^2}{s_{11}^2} h^2, \quad g_{01}(h/2) = \frac{s_{16}^2}{3s_{11}^2} h^3, \quad g_{10}(h/2) = -\frac{s_{16}^2}{6s_{11}^2} h^3, \quad g_{11}(h/2) = -\frac{s_{16}^2}{12s_{11}^2} h^4, \] 
\[ g_{06}^6(h/2) = \frac{4s_{16}^2 - 2s_{11}s_{12} - s_{11}s_{66}}{6s_{11}^2} h^3, \quad g_{01}^6(h/2) = \frac{4s_{16}^2 - 2s_{11}s_{12} - s_{11}s_{66}}{24s_{11}} h^4, \] 
\[ g_{10}^6(h/2) = -\frac{4s_{16}^2 - 2s_{11}s_{12} - s_{11}s_{66}}{80s_{11}^2} h^5. \]