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Iterated conditional expectations

Alexander Y. Gordon, Joseph E. Quinn *

Department of Mathematics and Statistics, University of North Carolina at Charlotte, 9201 University City Blvd, Charlotte, NC 28223, USA

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ABSTRACT

Consider the probability space $([0,1),\mathcal{B},\lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[0,1)$ and λ the Lebesgue measure. Let $f=\mathbf{1}_{[0,1/2)}$ and $g=\mathbf{1}_{[1/2,1)}$. Then for any $\varepsilon>0$ there exists a finite sequence of sub-σ-algebras $\mathcal{G}_j\subset\mathcal{B}$ $(j=1,\dots,N)$, such that putting $f_0=f$ and $f_j=\mathbf{E}(f_{j-1}|\mathcal{G}_j)$, $j=1,\dots,N$, we have $\|f_N-g\|_\infty<\varepsilon$; here $\mathbf{E}(\cdot|\mathcal{G}_j)$ denotes the operator of conditional expectation given σ-algebra $\mathcal{G}_j$. This is a particular case of a surprising result by Cherny and Grigoriev (2007) [1] in which f and g are arbitrary equidistributed bounded random variables on a nonatomic probability space. The proof given in Cherny and Grigoriev (2007) [1] is very complicated. The purpose of this note is to give a straightforward analytic proof of the above mentioned result, motivated by a simple geometric idea, and then show that the general result is implied by its special case.

1. Introduction

Cherny and Grigoriev [1] proved a law invariance theorem for L^∞ dilatation monotone maps (in the nonatomic setting) which has application in establishing the equivalence of a number of properties relevant to coherent risk measures.

This result was a consequence of the following statement, which is also in [1]:

Theorem 1. *Let $(\Omega,\mathcal{F},\mu)$ be a nonatomic probability space and let f, g be equidistributed $L^\infty(\Omega,\mathcal{F},\mu)$ functions. Given $\varepsilon>0$, there exists a finite sequence $\mathcal{F}_i$, $1\leqslant i\leqslant N$, of sub-σ-algebras of $\mathcal{F}$ such that, if $f_0=f$, $f_1=\mathbf{E}(f_0|\mathcal{F}_1)$, $f_2=\mathbf{E}(f_1|\mathcal{F}_2),\dots,f_N=\mathbf{E}(f_{N-1}|\mathcal{F}_N)$, then $\|f_N-g\|_\infty<\varepsilon$.*

Recall that if h is an $L^1(\Omega,\mathcal{F},\mu)$ function and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$, then the conditional expectation of h given $\mathcal{G}$, denoted as $\mathbf{E}(h|\mathcal{G})$, is the $L^1(\Omega,\mathcal{G},\mu)$ function h^*, such that

$$\int_A h^*\,d\mu=\int_A h\,d\mu\quad\text{for all } A\in\mathcal{G}. \tag{1}$$

The existence of such h^* (which is unique up to a zero μ-measure set) is guaranteed by the Radon–Nicodym theorem that is applicable whenever μ is a σ-finite measure on $(\Omega,\mathcal{F})$.

The following statement is a special case of one of the steps in the proof of the main result in [1].

* Corresponding author.
E-mail addresses: aygordon@uncc.edu (A.Y. Gordon), jequinn@uncc.edu (J.E. Quinn).

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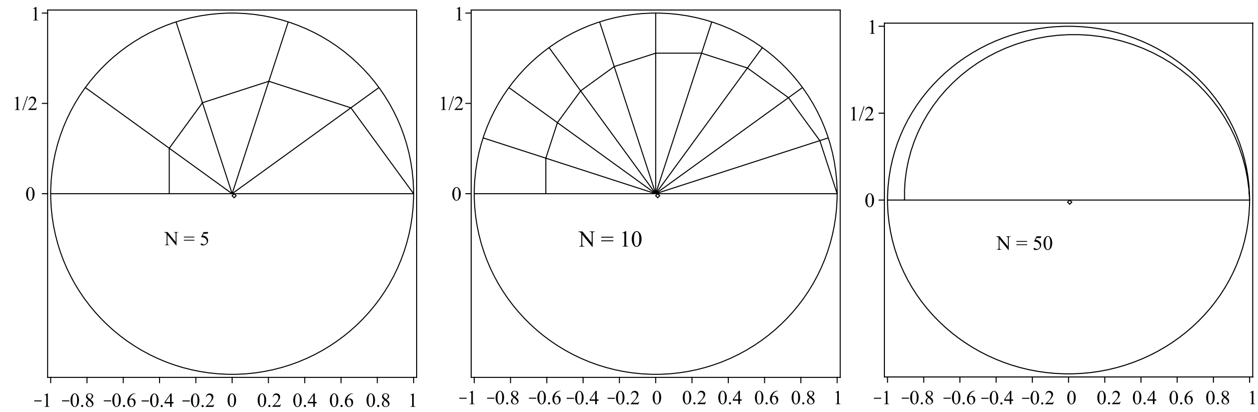

Fig. 1. P_j = projection onto the line $(\cos j\theta)y - (\sin j\theta)x = 0$, $\theta = \frac{\pi}{N}$.

Theorem 2. *Consider the probability space* $([0,1), \mathcal{B}, \lambda)$, *where* $\mathcal{B}$ *is the* σ*-algebra of Borel subsets of* $[0,1)$ *and* λ *is the Lebesgue measure on* $[0,1)$. *Let* $f = \mathbf{1}_{[0,1/2)}$ *and* $g = \mathbf{1}_{[1/2,1)}$. *Then it is possible, for any* $\varepsilon > 0$, *to choose an integer* $N > 0$ *and* N σ*-algebras* $\mathcal{F}_j \subset \mathcal{B}$ $(j = 1, 2, \dots, N)$ *in such a way that, putting* $f_0 = f$ *and* $f_j = \mathbf{E}(f_{j-1}|\mathcal{F}_j)$ $(j = 1, \dots, N)$, *we have*

$$\|f_N - g\|_\infty < \varepsilon. \tag{2}$$

Theorems 1 and 2 are striking and should be of interest to any mathematician and particularly any analyst, not just researchers in coherent risk theory. In addition, when properly interpreted physically, they have many counterintuitive consequences, which should make them of interest to an even broader group of scientists. For example, they imply that, if you have $2N$ balloons half of which are filled with air and the other half being flat, by merely connecting two balloons at a time and allowing pressures to equalize, if N is large enough, you can transfer almost all the air from the full balloons to the flat balloons.

However, the proof of Theorem 1 offered in [1] is very far from being transparent.

The purpose of this note is to give a straightforward analytic proof of Theorem 2, motivated by a simple geometric idea, and then show that Theorem 1 is implied by Theorem 2.

We prove Theorem 2 in Section 2. The derivation of Theorem 1 from its special case is presented in Section 3.

2. Proof of Theorem 2

Idea of the proof. Replace f and g by the functions $2f - 1$ and $2g - 1$, so that now $g = -f$. Both functions f and g belong to $L^2([0,1))$, and on that Hilbert space the operator of conditional expectation $\mathbf{E}(\cdot|\mathcal{G})$, where $\mathcal{G}$ is an arbitrary σ-algebra of Borel subsets of $[0,1)$, is an orthogonal projection (onto the subspace of $\mathcal{G}$-measurable L^2 functions). We need, therefore, to find a sequence of orthogonal projections $P_1, P_2, \dots, P_N$ (of a special form), such that

$$\left\| P_N P_{N-1} \dots P_2 P_1 f - (-f) \right\| < \varepsilon. \tag{3}$$

Forget, for the moment, that the projections P_j should be of a special form (operators of conditional expectation). Then inequality (3) can be easily realized in the Euclidean plane $\mathbf{R}^2$: let $f = (1, 0)$ and let P_j be the orthogonal projection onto the line L_j passing through the origin and the point $(\cos(\pi j/N), \sin(\pi j/N))$. Then $\|P_N P_{N-1} \dots P_2 P_1 f - (-f)\| \to 0$ as $N \to \infty$ (see Fig. 1).

The following proof mimics this two-dimensional construction in the infinite-dimensional Hilbert space $L^2([0,1))$.

Proof of Theorem 2. We start with transforming the problem. As was suggested above, replace f and g by the functions $2f - 1$ and $2g - 1$, respectively; furthermore, consider them as functions on the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ rather than on the interval $[0,1)$; and finally, rotate the circle $\mathbf{T}$ by $1/4$, having as a result $f = \mathbf{1}_{[-1/4,1/4)} - \mathbf{1}_{[1/4,3/4)}$, $g = -f$. Inequality (2), which is our goal, becomes

$$\left\| f_N - (-f) \right\|_\infty < 2\varepsilon. \tag{4}$$

Denote by $\mathcal{F}_{(\alpha)}$ $(\alpha \in \mathbf{T})$ the σ-algebra of all Borel subsets of $\mathbf{T}$ that are symmetric with respect to α (i.e., $\alpha + \gamma$ and $\alpha - \gamma$, for all $\gamma \in \mathbf{T}$, either both belong or both do not belong to such a set).

Let $h(x) = \cos 2\pi k(x - \delta)$. Then

$$\mathbf{E}\big(h(x) \big| \mathcal{F}_{(0)}\big) = \frac{1}{2}\big(h(x) + h(-x)\big) = \cos(2\pi k\delta)\cos(2\pi kx). \tag{5}$$

Rotation by α gives, for all $\alpha, \beta \in \mathbf{T}$,

$$\mathbf{E}\big(\cos 2\pi k(x-\beta)\big|\mathcal{F}_{(\alpha)}\big) = \cos\big(2\pi k(\beta-\alpha)\big)\cos\big(2\pi k(x-\alpha)\big). \tag{6}$$

Hence, putting $h_0^k(x) = \cos(2\pi kx)$ and $h_j^k = \mathbf{E}(h_{j-1}^k|\mathcal{F}_{j,n})$, where

$$\mathcal{F}_{j,n} = \mathcal{F}_{(j/2n)}, \quad j = 1, 2, \ldots, n, \tag{7}$$

we obtain

$$h_n^k(x) = \left(\cos\frac{k\pi}{n}\right)^n \cos\left(2\pi k\left(x - \frac{1}{2}\right)\right) = (1-\gamma_{n,k})(-1)^k h_0^k(x), \tag{8}$$

where

$$0 < \gamma_{n,k} < \frac{k^2\pi^2}{2n} \to 0 \quad \text{as } n \to \infty. \tag{9}$$

The function $f = \mathbf{1}_{[-1/4,1/4)} - \mathbf{1}_{[1/4,3/4)}$, as an element of $L^2(\mathbf{T})$, is even and consequently has a Fourier expansion of the form

$$f(x) = \sum_{k\geqslant 0} a_k \cos(2\pi kx), \tag{10}$$

which converges in $L^2(\mathbf{T})$. Moreover, $a_k = 0$ if k is even. Truncation gives

$$f(x) = \sum_{0\leqslant k\leqslant L,\ k \text{ odd}} a_k \cos(2\pi kx) + r(x), \tag{11}$$

where, for large enough L, $\|r\| < \theta$ ($\|\cdot\|$ denotes the L^2 norm) with arbitrarily small $\theta > 0$.

It is convenient to introduce a linear operator S_n in $L^2(\mathbf{T})$: for all $y \in L^2(\mathbf{T})$ we put

$$y_0 = y \quad \text{and} \quad y_j = \mathbf{E}(y_{j-1}|\mathcal{F}_{j,n}), \quad j = 1, \ldots, n; \qquad S_n y = y_n. \tag{12}$$

In view of (11) and (8),

$$(S_n f)(x) = -\sum_{0\leqslant k\leqslant L,\ k \text{ odd}} (1-\gamma_{n,k}) a_k \cos(2\pi kx) + (S_n r)(x). \tag{13}$$

Here $\|S_n r\| \leqslant \|r\| < \theta$ and $\gamma_{n,k} \to 0$ as $n \to \infty$ for each k $(1 \leqslant k \leqslant L)$. Hence, for large enough $n = n(\theta)$, we have

$$\big\|S_n f - (-f)\big\| < 3\theta. \tag{14}$$

Let $f_{n+1} = \mathbf{E}(S_n f|\mathcal{G})$, where $\mathcal{G} = \sigma([-1/4, 1/4))$ is the σ-algebra generated by the partition of the circle $\mathbf{T}$ into two intervals $[-1/4, 1/4)$ and $[1/4, 3/4)$. Then we have

$$\big\|f_{n+1} - (-f)\big\| < 3\theta \tag{15}$$

(we used the $\mathcal{G}$-measurability of f). Since the function $f_{n+1} - (-f)$ has constant values on the intervals $[-1/4, 1/4)$ and $[1/4, 3/4)$ and average 0 (which implies that its value on one of the intervals is the negative of the value on the other), its L^2 and L^∞ norms are equal, so that

$$\big\|f_{n+1} - (-f)\big\|_\infty < 3\theta. \tag{16}$$

Choosing $\theta = 2\varepsilon/3$ and putting $N = n+1$, we obtain (4). □

Remark. It can be shown that the above construction gives

$$\|f_N - g\|_\infty = O\big(N^{-1/2}\big) \quad \text{as } N \to \infty. \tag{17}$$

3. Proof of Theorem 1

We want to extend now the above result to equidistributed functions. First we clarify something about what has been proven above.

Denote by $f_{j,n}$ $(0 \leqslant j \leqslant n+1)$ the L^∞ functions we actually dealt with in the proof of Theorem 2 (except for the transformation $\varphi(\cdot) \mapsto 2\varphi(\cdot) - 1$, from which the proof started):

$$f_{0,n} = f = \mathbf{1}_{(-1/4,1/4]}; \qquad f_{j,n} = \mathbf{E}(f_{j-1,n}|\mathcal{F}_{j,n}), \quad j = 1, 2, \ldots, n+1. \tag{18}$$

Let $x_i = -1/4 + i/2n$ $(i = 0, 1, \ldots, 2n-1)$; these points divide the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ into $2n$ open intervals $\Delta_{i,n} = (x_i, x_{i+1})$, $i = 0, 1, \ldots, 2n-1$.

Each function $f_{j,n}$ $(0 \leqslant j \leqslant n+1)$ is constant on each open interval $\Delta_{i,n}$. Indeed, for $j = 0$ and $j = n+1$ this statement is obvious, while for $j = 1, 2, \ldots, n$ it follows by induction from the recursive relation defining $f_{j,n}$:

$$f_{j,n}(x) = \frac{1}{2}\left(f_{j-1,n}(x) + f_{j-1,n}\left(\frac{j}{n} - x\right)\right). \tag{19}$$

It follows now from (18) and the definition of the conditional expectation that if we remove from $\mathbf{T}$ a finite set $Z_n = \{x_0, x_1, \ldots, x_{2n-1}\}$ (which removal does not change anything, since Z_n is a zero measure set), then on the remaining set $\mathbf{T}_n^* = \bigsqcup_{i=0}^{2n-1} \Delta_{i,n}$ we have

$$f_{0,n} = f; \qquad f_{j,n} = \mathbf{E}(f_{j-1,n}|\mathcal{G}_{j,n}), \quad j = 1, 2, \ldots, n+1, \tag{20}$$

where

$$\mathcal{G}_{j,n} = \mathcal{F}_{j,n} \cap \mathcal{L}_n \tag{21}$$

and $\mathcal{L}_n$ is the finite σ-algebra of subsets of $\mathbf{T}_n^*$ generated by the intervals $\Delta_{i,n}$. (In Eqs. (20) and (21) by the symbols $f_{j,n}$ and $\mathcal{F}_{j,n}$ we denote the restrictions of the corresponding objects to $\mathbf{T}_n^*$; the second equality in (20) follows from the second equality in (18) and the fact that $f_{j,n}$, being $\mathcal{F}_{j,n}$- and $\mathcal{L}_n$-measurable, is, consequently, $\mathcal{G}_{j,n}$-measurable.)

Therefore, the proof of Theorem 2 actually implies the following statement.

Proposition 1. *For any $\varepsilon > 0$ there exist an integer $n \geqslant 1$ and $n+1$ σ-algebras $\mathcal{G}_{j,n} \subset \mathcal{L}_n$, $j = 1, 2, \ldots, n+1$, such that, if f and g are the restrictions of the functions $\mathbf{1}_{(-1/4,1/4]}$ and $\mathbf{1}_{(1/4,3/4]}$ to $\mathbf{T}_n^*$ and the functions $f_{j,n}$ are defined by Eqs.* (20)*, then*

$$\|f_{n+1,n} - g\|_\infty < \varepsilon. \tag{22}$$

Remark. The finite set Z_n is symmetric about the point $j/2n$, and so is the system of intervals $\Delta_{i,n}$. The σ-algebra $\mathcal{G}_{j,n}$ $(1 \leqslant j \leqslant n)$ is generated by the partition of $\mathbf{T}_n^*$ into the unions $\Delta_{i,n} \cup \Delta_{k,n}$, in each of which the intervals $\Delta_{i,n}$ and $\Delta_{k,n}$ are reflections of one another about the point $j/2n$, so the two intervals are either disjoint or identical. The corresponding operator $\mathbf{E}(\cdot|\mathcal{G}_{j,n})$, applied to the function $f_{j-1,n}$, replaces its (constant) values on each such pair of intervals by their arithmetic average (so that in the case of a single interval the value does not change). The σ-algebra $\mathcal{G}_{n+1,n}$ is generated by the partition of $\mathbf{T}_n^*$ into two sets: the union of intervals $\Delta_0, \ldots, \Delta_{n-1}$ and the union of intervals $\Delta_n, \ldots, \Delta_{2n-1}$; the corresponding operator $\mathbf{E}(\cdot|\mathcal{G}_{n+1,n})$, applied to the function $f_{n,n}$, replaces its constant values on the intervals of each group by their arithmetic average over the group.

The following statement is a consequence of Proposition 1.

Lemma 1. *Let $(\Omega, \mathcal{F}, \mu)$ be a nonatomic probability space; let f, g be two $\mathcal{F}$-measurable functions on Ω and $A, B \in \mathcal{F}$ two disjoint sets, such that $\mu(A) = \mu(B) > 0$ and*

$$f|_A \equiv c_1, \qquad f|_B \equiv c_2; \qquad g|_A \equiv c_2, \qquad g|_B \equiv c_1; \qquad f(x) = g(x) \quad \text{if } x \notin A \sqcup B. \tag{23}$$

Here $c_1, c_2 \in \mathbf{R}$ and $c_1 \neq c_2$. Then there exist $n \in \mathbf{N}$ and finite σ-algebras $\mathcal{F}_1, \ldots, \mathcal{F}_{n+1} \subset \mathcal{F}$, such that, denoting $\mathbf{E}(y|\mathcal{F}_j)$ by $E_j y$ and putting $v = E_{n+1}E_n \ldots E_1 f - g$, we have

$$\|v\|_\infty < \varepsilon \quad \text{and} \quad v(x) = 0 \quad \text{if } x \notin A \sqcup B. \tag{24}$$

Proof. It suffices to give a proof in the case, where $\Omega = A \sqcup B$, $\mu(A) = \mu(B) = 1/2$, and $c_1 = 1$, $c_2 = 0$ (the necessary changes in the general case are obvious). Choose the same n as in Proposition 1. Since the measure μ is nonatomic, Ω can be split into $2n$ disjoint sets $A_i \in \mathcal{F}$ $(i = 0, 1, \ldots, 2n-1)$, so that $\mu(A_i) = 1/2n$ for all i and $A = \bigsqcup_{i=0}^{n-1} A_i$, $B = \bigsqcup_{i=n}^{2n-1} A_i$. Use the one-to-one correspondence $A_i \leftrightarrow \Delta_{i,n}$ to carry over the σ-algebras $\mathcal{G}_{j,n}$ and the functions $f_{j,n}$ from $\mathbf{T}_n^*$ to Ω, thus obtaining

σ-algebras $\mathcal{F}_j$ and functions h_j (that are constant on the sets A_i). Then we have $h_0 = \mathbf{1}_A$, $h_j = E_j h_{j-1}$ $(j = 1, 2, \ldots, n+1)$, and $\|h_{n+1} - \mathbf{1}_B\|_\infty < \varepsilon$. $\square$

Proof of Theorem 1. It is sufficient to prove the theorem for *simple* (i.e., having finitely many values) functions f and g. (Indeed, for any $\delta > 0$ there exist equidistributed simple functions $\widehat{f}$, $\widehat{g}$, such that $\|\widehat{f} - f\|_\infty < \delta$ and $\|\widehat{g} - g\|_\infty < \delta$. If $\|E_N E_{N-1} \ldots E_1 \widehat{f} - \widehat{g}\|_\infty < \varepsilon$, then, since each operator E_j is a contraction in L^∞, we have $\|E_N E_{N-1} \ldots E_1 f - g\|_\infty < \varepsilon + 2\delta$.) Therefore, from now on we assume that f and g are simple functions.

Let $X = \{x \in \Omega\colon f(x) \neq g(x)\}$ (brief notation: $X = \{f \neq g\}$); let $\mathcal{G}$ and ν be the restrictions to X of the σ-algebra $\mathcal{F}$ and the measure μ, respectively. We may assume that $\nu(X) > 0$ – otherwise there is nothing to prove.

Lemma 2. *Let $(X, \mathcal{G}, \nu)$ be a nonatomic measure space $(0 < \nu(X) < \infty)$ and f, g be two equidistributed simple functions on it, such that $f(x) \neq g(x)$ for all $x \in X$. Then there exists an* (f, g)-chain, *by which we understand a finite sequence of disjoint sets $B_1, B_2, \ldots, B_k \in \mathcal{G}$ $(k \geqslant 2)$, such that*

(i) $\nu(B_1) = \nu(B_2) = \cdots = \nu(B_k) > 0$;
(ii) *both f and g are constant on each B_i;*
(iii) *the sequence $g_1, \ldots, g_k$ of values of g on the sets B_i is a re-arrangement of the similar sequence $f_1, \ldots, f_k$ for f.*

Proof. Let C be the finite set of all such $a \in \mathbf{R}$ that $\nu(\{f = a\}) > 0$. Pick $c_1 \in C$ arbitrarily. There exists c_2 $(c_2 \neq c_1)$, such that the set $A_1 = \{f = c_1,\ g = c_2\}$ has a positive ν-measure. Since f and g are equidistributed, the set $\{f = c_2\}$ also has a positive ν-measure, and the same is true for the set $A_2 = \{f = c_2,\ g = c_3\}$ with some c_3 $(c_3 \neq c_2)$. Continuing in the same manner, we will obtain a sequence of sets $A_i = \{f = c_i,\ g = c_{i+1}\} \in \mathcal{G}$ with $\nu(A_i) > 0$ and $c_{i+1} \neq c_i$. At some step r for the first time we will have $c_{r+1} = c_l$, where $1 \leqslant l < r$. All c_i with $l \leqslant i \leqslant r$ are distinct and, putting $s = \min_{l \leqslant i \leqslant r} \nu(A_i)$, we have $s > 0$. Since the measure ν is nonatomic, there exist $\mathcal{G}$-measurable sets $D_i \subset A_i$ $(l \leqslant i \leqslant r)$ with $\nu(D_i) = s$; they form an (f, g)-chain. $\square$

Remark. Note that, in the last step of the above proof, there is a k $(l \leqslant k \leqslant r)$ such that $D_k = A_k$ (up to a set of ν-measure 0) and, therefore, when the sets D_i are removed from X, we have $\nu\{f = c_k,\ g = c_{k+1}\} = 0$ on the remaining set.

Corollary 1. *Under the conditions of Lemma 2, there exist finitely many (f, g)-chains $B_1^{(j)}, B_2^{(j)}, \ldots, B_{k_j}^{(j)}$ $(j = 1, 2, \ldots, r)$, so that all the sets $B_i^{(j)}$ are disjoint and their union, up to a set of ν-measure 0, is X.*

Proof. Put $X_0 = X$. Using the construction just described and removing from X_0 the sets that form the resulting (f, g)-chain, we obtain a smaller set X_1 $(X_1 \in \mathcal{G})$, on which f and g are still equidistributed; so we can repeat the step, thus obtaining a smaller set X_2, etc. Due to the remark following Lemma 2, the finite set of all pairs $(a, b) \in C \times C$ for which the set $\{f = a,\ g = b\} \cap X_i$ has a positive ν-measure strictly decreases as i grows; therefore, at some step r $(r \geqslant 1)$ the remaining set X_r will have ν-measure 0 and the process will terminate. $\square$

Corollary 1 and the fact that any permutation is a product of transpositions imply that f can be transformed into g by a finite sequence of elementary transformations each of which exchanges constant values of a simple function on two disjoint sets having the same positive μ-measure. Denote those transformations by $T_1, T_2, \ldots, T_m$ and let

$$u_0 = f; \qquad u_k = T_k u_{k-1} \quad (1 \leqslant k \leqslant m), \tag{25}$$

so that $u_m = g$.

By Lemma 1, for any $\varepsilon > 0$ and any $k = 1, 2, \ldots, m$ there exists a finite sequence of operators of conditional expectation $E_j^{(k)}$, $j = 1, 2, \ldots, l_k$, such that their composition $\widetilde{T}_k$ satisfies inequality

$$\|\widetilde{T}_k u_{k-1} - T_k u_{k-1}\|_\infty < \varepsilon. \tag{26}$$

Let

$$\widetilde{u}_0 = f; \qquad \widetilde{u}_k = \widetilde{T}_k \widetilde{u}_{k-1}, \quad k = 1, 2, \ldots, m. \tag{27}$$

Equalities (25) and (27) imply the identity

$$\widetilde{u}_k - u_k = \widetilde{T}_k(\widetilde{u}_{k-1} - u_{k-1}) + (\widetilde{T}_k u_{k-1} - T_k u_{k-1}),$$

which, together with inequality (26) and the fact that each operator $\widetilde{T}_k$ is a contraction in L^∞, implies by induction that $\|\widetilde{u}_k - u_k\|_\infty \leqslant k\varepsilon$ for all k $(0 \leqslant k \leqslant m)$. For $k = m$ this gives

$$\|\widetilde{T}_m \widetilde{T}_{m-1} \dots \widetilde{T}_1 f - g\|_\infty \leqslant m\varepsilon; \tag{28}$$

since $\varepsilon > 0$ is arbitrary, this completes the proof of Theorem 1. □

Remark. We want to indicate how Theorem 1 is used in [1]. Let $(\Omega, \mathcal{F}, P)$ be a nonatomic probability space with σ-algebra $\mathcal{F}$. A function $f : L^\infty(\Omega, \mathcal{F}, P) \to \mathbf{R}$ is dilatation monotone if for each sub-σ-algebra Λ of $\mathcal{F}$ and each random variable X on Ω, $f(E(X|\Lambda)) \leqslant f(X)$. It is clear that Theorem 1 implies that every continuous dilatation monotone map $f : L^\infty(\Omega, \mathcal{F}, P) \to \mathbf{R}$ is law invariant. This is the main result in [1]. When combined with other results for risk measures, it implies that, on a nonatomic probability space, convex risk measures are law invariant if and only if they are dilatation monotone.

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