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Operators on the space of bounded strongly measurable functions

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ABSTRACT

Let $\mathcal{L}(X,Y)$ stand for the space of all bounded linear operators between real Banach spaces $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$, and let Σ be a σ-algebra of subsets of a non-empty set Ω. Let $\mathcal{L}^\infty(\Sigma,X)$ denote the Banach space of all bounded strongly Σ-measurable functions $f:\Omega\to X$ equipped with the supremum norm $\|\cdot\|$. A bounded linear operator T from $\mathcal{L}^\infty(\Sigma,X)$ to a Banach space Y is said to be σ-smooth if $\|T(f_n)\|_Y\to 0$ whenever $\|f_n(\omega)\|_X\to 0$ for all $\omega\in\Omega$ and $\sup_n\|f_n\|<\infty$. It is shown that if an operator measure $m:\Sigma\to\mathcal{L}(X,Y)$ is variationally semi-regular (i.e., $\widetilde{m}(A_n)\to 0$ as $A_n\downarrow\emptyset$, where $\widetilde{m}(A)$ stands for the semivariation of m on $A\in\Sigma$), then the corresponding integration operator $T_m:\mathcal{L}^\infty(\Sigma,X)\to Y$ is σ-smooth. Conversely, it is proved that every σ-smooth operator $T:\mathcal{L}^\infty(\Sigma,X)\to Y$ admits an integral representation with respect to its representing operator measure. We prove a Banach–Steinhaus type theorem for σ-smooth operators from $\mathcal{L}^\infty(\Sigma,X)$ to Y. In particular, we study the topological properties of the space $\mathcal{L}^\infty(\Sigma,X)^*_c$ of all σ-smooth functionals on $\mathcal{L}^\infty(\Sigma,X)$. We prove a form of a generalized Nikodým convergence theorem and characterize relative $\sigma(\mathcal{L}^\infty(\Sigma,X)^*_c,\mathcal{L}^\infty(\Sigma,X))$-sequential compactness in $\mathcal{L}^\infty(\Sigma,X)^*_c$. We derive a Grothendieck type theorem for $\mathcal{L}^\infty(\Sigma,X)^*_c$. The relationships between different classes of linear operators on $\mathcal{L}^\infty(\Sigma,X)$ are established.

1. Introduction and terminology

Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be real Banach spaces and let B_X stand for the closed unit ball in X. Let X^* and Y^* stand for the Banach duals of X and Y, respectively. Denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators between Banach spaces X and Y. The weak* operator topology (briefly, W*OT) is the topology on $\mathcal{L}(X,Y)$ defined by the family of seminorms $\{p_{y_*}:\ y^*\in Y^*\}$, where $p_{y^*}(U):=\|y^*\circ U\|_{X^*}$ for $U\in\mathcal{L}(X,Y)$.

By $\sigma(L,K)$ and $\tau(L,K)$ we will denote the weak topology and the Mackey topology on L with respect to a dual pair $\langle L,K\rangle$. For a topological vector space (L,τ) by $(L,\tau)^*$ we will denote its topological dual. Let $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural and real numbers.

Now we recall basic terminology concerning operator measures (see [12,5,6,18,19]). Let Σ be a σ-algebra of subsets of a non-empty set Ω. An additive mapping $m:\Sigma\to\mathcal{L}(X,Y)$ is called an operator-valued measure. We define the semivariation $\widetilde{m}(A)$ of m on $A\in\Sigma$ by $\widetilde{m}(A):=\sup\|\Sigma m(A_i)(x_i)\|_Y$, where the supremum is taken over all finite disjoint sequences (A_i) in Σ with $A_i\subset A$ and $x_i\in B_X$ for each i. By fasv$(\Sigma,\mathcal{L}(X,Y))$ we denote the set of all finitely additive measures $m:\Sigma\to\mathcal{L}(X,Y)$ with finite semivariation, i.e., $\widetilde{m}(\Omega)<\infty$.

For $y^*\in Y^*$ let $m_{y^*}:\Sigma\to X^*$ be a set function defined by $m_{y^*}(A)(x):=\langle m(A)(x),y^*\rangle$ for $x\in X$. Then m_{y^*} is an additive measure and $\widetilde{m}_{y^*}(A)=|m_{y^*}|(A)$, where $|m_{y^*}|(A)$ stands for the variation of m_{y^*} on $A\in\Sigma$. Moreover, for $A\in\Sigma$ we have

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$$\widetilde{m}(A) = \sup\{|m_{y^*}|(A): y^* \in B_{Y^*}\} \tag{1.1}$$

(see [5, Theorem 5]). Recall that an operator measure $m : \Sigma \to \mathcal{L}(X, Y)$ is said to be countably in W*OT if for each $y^* \in Y^*$, $\|m_{y^*}(A_n)\|_{X^*} \to 0$ whenever $A_n \downarrow \emptyset$ (see [5,6, p. 92]).

Following Lewis (see [18,19]) a measure $m : \Sigma \to \mathcal{L}(X, Y)$ is said to be *variationally semi-regular* if $\widetilde{m}(A_n) \to 0$ whenever $A_n \downarrow \emptyset$ and $(A_n) \subset \Sigma$. (Dobrakov [13] uses the term "continuous", Swartz [25,26] uses the term "strongly bounded"). Note that $m : \Sigma \to \mathcal{L}(X, Y)$ is variationally semi-regular if and only if $\widetilde{m}(\Omega) < \infty$ and the family $\{|m_{y^*}|: y^* \in B_{Y^*}\}$ is uniformly countably additive, i.e., the set $\{|m_{y^*}|: y^* \in B_{Y^*}\}$ in $\mathrm{ca}(\Sigma)$ (= the Banach space of all signed countably additive measures) is relatively weakly compact (see [10, Theorem 13, p. 92]).

Note that for a measure $\nu : \Sigma \to X^*(Y = \mathbb{R})$ we have $\widetilde{\nu}(A) = |\nu|(A)$ for $A \in \Sigma$. Hence $\nu \in \mathrm{fasv}(\Sigma, X^*)$ is variationally semi-regular if and only if $|\nu|(\Omega) < \infty$ and ν is countably additive, i.e., $\|\nu(A_n)\|_{X^*} \to 0$ whenever $A_n \downarrow \emptyset$ (see [11, Proposition 9, p. 3]). Let $\mathrm{bva}(\Sigma, X^*)$ stand for the Banach space of all vector measures $\nu : \Sigma \to X^*$ of bounded variation, equipped with the norm $\|\nu\| = |\nu|(\Omega)$. By $\mathrm{bvca}(\Sigma, X^*)$ we denote a linear subspace of $\mathrm{bva}(\Sigma, X^*)$ consisting of all those $\nu \in \mathrm{bva}(\Sigma, X^*)$ that are countably additive. For $\nu \in \mathrm{bvca}(\Sigma, X^*)$ and $x \in X$ let $\nu_x(A) = \nu(A)(x)$ for $A \in \Sigma$. Then $\nu_x \in \mathrm{ca}(\Sigma)$. Note that $\mathrm{bvca}(\Sigma, \mathbb{R})) = \mathrm{ca}(\Sigma)$.

By $\mathcal{S}(\Sigma, X)$ we denote the space of all X-valued Σ-simple functions $s = \sum_{i=1}^k (\mathbb{1}_{A_i} \otimes x_i)$, where $(A_i)_{i=1}^k$ is a disjoint sequence in Σ, $x_i \in X$ for $1 \leqslant i \leqslant k$ and $(\mathbb{1}_{A_i} \otimes x_i)(\omega) = \mathbb{1}_{A_i}(\omega)x_i$ for $\omega \in \Omega$. A function $f : \Omega \to X$ is said to be strongly Σ-measurable if there exists a sequence (s_n) in $\mathcal{S}(\Sigma, X)$ such that $\|s_n(\omega) - f(\omega)\|_X \to 0$ for all $\omega \in \Omega$. It is known that if $f : \Omega \to X$ is strongly Σ-measurable, then there exists a sequence (s_n) in $\mathcal{S}(\Sigma, X)$ such that $\|s_n(\omega) - f(\omega)\|_X \to 0$ for all $\omega \in \Omega$ and $\|s_n(\omega)\|_X \leqslant \|f(\omega)\|_X$ for $\omega \in \Omega$ and all $n \in \mathbb{N}$ (see [12, Theorem 1.6, p. 4]). By $\mathcal{L}^\infty(\Sigma, X)$ we denote the Banach space of all bounded strongly Σ-measurable functions $f : \Omega \to X$, equipped with the supremum norm $\|\cdot\|$. Let $\mathcal{L}^\infty(\Sigma, X)^*$ and $\mathcal{L}^\infty(\Sigma, X)^{**}$ stand for the Banach dual and the Banach bidual of $\mathcal{L}^\infty(\Sigma, X)$ respectively. For $f \in \mathcal{L}^\infty(\Sigma, X)$ and $A \in \Sigma$ let us put

$$\|f\|_A = \sup_{\omega \in A} \|f(\omega)\|_X.$$

Let $m \in \mathrm{fasv}(\Sigma, \mathcal{L}(X, Y))$. Then for $s = \sum_{i=1}^k (\mathbb{1}_{A_i} \otimes x_i) \in \mathcal{S}(\Sigma, X)$ and $A \in \Sigma$ we can define the integral by the equality

$$\int_A s\, dm := \sum_{i=1}^k m(A \cap A_i)(x_i).$$

The integral is independent of the representation chosen and is a linear operator from $\mathcal{S}(\Sigma, X)$ to Y. Moreover, for each $s \in \mathcal{S}(\Sigma, X)$ and $A \in \Sigma$ the following inequality holds:

$$\left\| \int_A s\, dm \right\|_Y \leqslant \|s\|_A \cdot \widetilde{m}(A).$$

Assume now that $m \in \mathrm{fasv}(\Sigma, \mathcal{L}(X, Y))$ is variationally semi-regular. Let $f \in \mathcal{L}^\infty(\Sigma, X)$ and $A \in \Sigma$, and choose a sequence (s_n) in $\mathcal{S}(\Sigma, X)$ such that $\|(\mathbb{1}_A s_n)(\omega) - (\mathbb{1}_A f)(\omega)\|_X \to 0$ for $\omega \in \Omega$ and $\sup_n \|s_n\|_A \leqslant \|f\|_A$. Then

$$\left\| \int_A s_n\, dm \right\|_Y \leqslant \|f\|_A \cdot \widetilde{m}(A).$$

It follows that the indefinite integrals $\int_{(\cdot)} s_n\, dm$ are uniformly countably additive measures on Σ. This means that f is *m-integrable* and the integral of f on a set A is defined by equality:

$$\int_A f\, dm := \lim_n \int_A s_n\, dm$$

(see [13, Definition 2, p. 523 and Theorem 5, p. 524]). Dobrakov [13, Example 7′, pp. 524–525] showed that the assumption of semi-regularity of m on Σ is necessary for every $f \in \mathcal{L}^\infty(\Sigma, X)$ to be m-integrable. Define the integration operator $T_m : \mathcal{L}^\infty(\Sigma, X) \to Y$ by

$$T_m(f) = \int_\Omega f\, dm.$$

In particular, for $\nu \in \mathrm{bvca}(\Sigma, X^*)$ the integration functional Φ_ν on $\mathcal{L}^\infty(\Sigma, X)$ is given by

$$\Phi_\nu(f) = \int_\Omega f\, d\nu.$$

For a bounded linear operator $T:\mathcal{L}^\infty(\Sigma,X)\to Y$ let $m_T:\Sigma\to\mathcal{L}(X,Y)$ stand for its *representing measure*, i.e.,

$$m_T(A)(x):=T(\mathbb{1}_A\otimes x)\quad\text{for } A\in\Sigma \text{ and } x\in X.$$

Then $\widetilde{m}_T(\Omega)\leqslant\|T\|<\infty$, i.e., $m_T\in\operatorname{fasv}(\Sigma,\mathcal{L}(X,Y))$. In particular, if $\Phi\in\mathcal{L}^\infty(\Sigma,X)^*$ and $\nu_\Phi(A)(x)=\Phi(\mathbb{1}_A\otimes x)$ for $A\in\Sigma$, $x\in X$, then $\nu_\Phi\in\operatorname{bva}(\Sigma,X^*)$. Then $(m_T)_{y^*}=\nu_{y^*\circ T}$ for each $y^*\in Y^*$.

Now we introduce a new class of linear operators from $\mathcal{L}^\infty(\Sigma,X)$ to Y.

Definition 1.1. A bounded linear operator $T:\mathcal{L}^\infty(\Sigma,X)\to Y$ is said to be *σ-smooth* if $\|T(f_n)\|_Y\to 0$ whenever $\|f_n(\omega)\|_X\to 0$ for all $\omega\in\Omega$ and $\sup_n\|f_n\|<\infty$.

By $\mathcal{L}^\infty(\Sigma,X)^*_c$ we will denote the space of all σ-smooth functionals on $\mathcal{L}^\infty(\Sigma,X)$.

Note that if $X=\mathbb{R}$ then the space $\mathcal{L}^\infty(\Sigma,\mathbb{R})$ coincides with the Dedekind σ-complete Banach lattice $\mathcal{L}^\infty(\Sigma)$ $(=B(\Sigma))$ of all bounded Σ-measurable real functions defined on Ω, and $\mathcal{L}^\infty(\Sigma,\mathbb{R})^*_c$ coincides with the σ-order continuous dual $\mathcal{L}^\infty(\Sigma)^*_c$ of $\mathcal{L}^\infty(\Sigma)$ (see [2, § 13.1]).

In Section 2 we show that if $m\in\operatorname{fasv}(\Sigma,\mathcal{L}(X,Y))$ is variationally semi-regular, then the corresponding integration operator $T_m:\mathcal{L}^\infty(\Sigma,X)\to Y$ is σ-smooth. Conversely, it is shown that every σ-smooth operator $T:\mathcal{L}^\infty(\Sigma,X)\to Y$ admits an integral representation with respect to its representing measure. We prove a Banach–Steinhaus type theorem for σ-smooth operators $T:\mathcal{L}^\infty(\Sigma,X)\to Y$. In Section 3 we study the topological properties of the space $\mathcal{L}^\infty(\Sigma,X)^*_c$. We prove a form of a generalized Nikodým convergence theorem for $\mathcal{L}^\infty(\Sigma,X)^*_c$. As an application we characterize relatively $\sigma(\mathcal{L}^\infty(\Sigma,X)^*_c,\mathcal{L}^\infty(\Sigma,X))$-sequentially compact subsets of $\mathcal{L}^\infty(\Sigma,X)^*_c$. We derive a Grothendieck type theorem saying that $\sigma(\mathcal{L}^\infty(\Sigma,X)^*,\mathcal{L}^\infty(\Sigma,X))$-convergent sequences in $\mathcal{L}^\infty(\Sigma,X)^*_c$ are $\sigma(\mathcal{L}^\infty(\Sigma,X)^*,\mathcal{L}^\infty(\Sigma,X)^{**})$-convergent. In Section 4 we establish the relationships between different classes of linear operators on $\mathcal{L}^\infty(\Sigma,X)$.

2. σ-smooth operators on $\mathcal{L}^\infty(\Sigma,X)$

In this section we establish the relationships between σ-smooth operators $T:\mathcal{L}^\infty(\Sigma,X)\to Y$ and their representing measures $m:\Sigma\to\mathcal{L}(X,Y)$.

Assume $m\in\operatorname{fasv}(\Sigma,\mathcal{L}(X,Y))$ be variationally semi-regular, and let

$$T_m(f)=\int_\Omega f\,dm\quad\text{for all } f\in\mathcal{L}^\infty(\Sigma,X).$$

For every $A\in\Sigma$ let us put

$$(T_m)_A(f)=T_m(\mathbb{1}_A f)=\int_A f\,dm,$$

and

$$\|(T_m)_A\|=\sup\left\{\left\|\int_A f\,dm\right\|_Y: f\in\mathcal{L}^\infty(\Sigma,X)\text{ and }\|f\|\leqslant 1\right\}.$$

Then for each $y^*\in Y^*$ we have

$$y^*\big(T_m(f)\big)=\int_\Omega f\,dm_{y^*}\quad\text{for all } f\in\mathcal{L}^\infty(\Sigma,X). \tag{2.1}$$

The following lemma will be useful.

Lemma 2.1. *Let $m\in\operatorname{fasv}(\Sigma,\mathcal{L}(X,Y))$ be variationally semi-regular. Then for every $A\in\Sigma$ we have $\widetilde{m}(A)=\|(T_m)_A\|$.*

Proof. Let $A\in\Sigma$. Then

$$\widetilde{m}(A)=\sup\left\{\left\|\int_A s\,dm\right\|_Y: s\in\mathcal{S}(\Sigma,X),\ \|s\|\leqslant 1\right\}.$$

Hence $\widetilde{m}(A)\leqslant\|(T_m)_A\|$. Now let $f\in\mathcal{L}^\infty(\Sigma,X)$ with $\|f\|\leqslant 1$. Then there exists a sequence (s_n) in $\mathcal{S}(\Sigma,X)$ such that $\|s_n(\omega)-f(\omega)\|_X\to 0$ for each $\omega\in\Omega$ and $\|s_n(\omega)\|_X\leqslant\|f(\omega)\|_X$ for each $\omega\in\Omega$ and $n\in\mathbb{N}$. Hence $\sup_n\|s_n\|\leqslant\|f\|\leqslant 1$ and $(T_m)_A(f)=\int_A f\,dm=\lim_n\int_A s_n\,dm$. Fix $\varepsilon>0$ and choose $n_\varepsilon\in\mathbb{N}$ such that $\|\int_A f\,dm-\int_A s_{n_\varepsilon}\,dm\|_Y\leqslant\varepsilon$. Hence

$$\left\| \int_A f\,dm \right\|_Y \leqslant \left\| \int_A f\,dm - \int_A s_{n_\varepsilon}\,dm \right\|_Y + \left\| \int_A s_{n_\varepsilon}\,dm \right\|_Y \leqslant \varepsilon + \widetilde{m}(A).$$

It follows that $\|(T_m)_A\| \leqslant \widetilde{m}(A)$. □

Now we are ready to prove our main results.

Proposition 2.2. *Assume that $m \in \mathrm{fasv}(\Sigma, \mathcal{L}(X,Y))$ is variationally semi-regular. Then the integration operator $T_m : \mathcal{L}^\infty(\Sigma, X) \to Y$ is σ-smooth.*

Proof. By Lemma 2.1 we have that $\|T_m\| = \widetilde{m}(\Omega) < \infty$. Since m is variationally semi-regular, in view of (1.1) the set $\{|m_{y^*}|: y^* \in B_{Y^*}\}$ is uniformly countably additive. Hence there exists $\mu \in \mathrm{ca}^+(\Sigma)$ such that the family $\{|m_{y^*}|: y^* \in B_{Y^*}\}$ is uniformly μ-continuous, i.e., $\widetilde{m}(A_n) \to 0$ whenever $\mu(A_n) \to 0$ (see [10, Theorem 13, p. 92]).

Assume that (f_n) is a sequence in $\mathcal{L}^\infty(\Sigma, X)$ such that $\|f_n(\omega)\|_X \to 0$ for all $\omega \in \Omega$ and $a = \sup_n \|f_n\| < \infty$, and let $\varepsilon > 0$ be given. For $\eta = \frac{\varepsilon}{2\|T_m\|} > 0$ and $n \in \mathbb{N}$ let us put

$$A_n(\eta) = \{\omega \in \Omega: \|f_n(\omega)\|_X \geqslant \eta\}.$$

Then $\mu(A_n(\eta)) \to 0$. Hence $\widetilde{m}(A_n(\eta)) \to 0$, and by Lemma 2.1 we have

$$\|(T_m)_{A_n(\eta)}\| = \sup\{\|T_m(\mathbb{1}_{A_n(\eta)} f)\|_Y: \|f\| \leqslant 1\} \underset{n}{\longrightarrow} 0.$$

Hence there exists $n_\varepsilon \in \mathbb{N}$ such that for $n \geqslant n_\varepsilon$ we get

$$\left\| T_m\left(\frac{1}{a}\mathbb{1}_{A_n(\eta)} f_n\right)\right\|_Y \leqslant \frac{\varepsilon}{2a}, \quad \text{i.e.,} \quad \|T_m(\mathbb{1}_{A_n(\eta)} f_n)\|_Y \leqslant \frac{\varepsilon}{2}.$$

Moreover, for $n \in \mathbb{N}$ we have

$$\|T_m(\mathbb{1}_{\Omega \setminus A_n(\eta)} f_n)\|_Y \leqslant \|T_m\| \cdot \|\mathbb{1}_{\Omega \setminus A_n(\eta)} f_n\| \leqslant \|T_m\| \cdot \eta = \frac{\varepsilon}{2}.$$

Hence for $n \geqslant n_\varepsilon$ we have

$$\|T_m(f_n)\|_Y \leqslant \|T_m(\mathbb{1}_{A_n(\eta)} f_n)\|_Y + \|T_m(\mathbb{1}_{\Omega \setminus A_n(\eta)} f_n)\|_Y \leqslant \varepsilon.$$

This means that T_m is σ-smooth. □

Proposition 2.3. *Assume that $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ is a σ-smooth operator. Then its representing measure $m_T \in \mathrm{fasv}(\Sigma, \mathcal{L}(X,Y))$ is variationally semi-regular and*

$$T(f) = T_{m_T}(f) = \int_\Omega f\,dm_T \quad \text{for all } f \in \mathcal{L}^\infty(\Sigma, X).$$

Moreover, $\|T\| = \widetilde{m}_T(\Omega)$.

Proof. Assume that $A_n \downarrow \emptyset$. Then for every $n \in \mathbb{N}$ there exist a Σ-partition $(A_{n,i})_{i=1}^{k_n}$ of A_n and $x_{n,i} \in B_X$, $1 \leqslant i \leqslant k_n$ such that

$$\widetilde{m}_T(A_n) \leqslant \left\| \sum_{i=1}^{k_n} m_T(A_{n,i})(x_{n,i}) \right\|_Y + \frac{1}{n}.$$

Let $s_n = \sum_{i=1}^{k_n} (\mathbb{1}_{A_{n,i}} \otimes x_{n,i})$ for $n \in \mathbb{N}$. Then $\|s_n(\omega)\|_X \leqslant \mathbb{1}_{A_n}(\omega) \leqslant \mathbb{1}_\Omega(\omega)$ for $\omega \in \Omega$ and all $n \in \mathbb{N}$, and $\mathbb{1}_{A_n}(\omega) \downarrow 0$ for $\omega \in \Omega$. Hence $\|s_n(\omega)\|_X \to 0$ for $\omega \in \Omega$ and $\sup_n \|s_n\| \leqslant 1$. Therefore

$$\|T(s_n)\|_Y = \left\| \sum_{i=1}^{k_n} m_T(A_{n,i})(x_{n,i}) \right\|_Y \underset{n}{\longrightarrow} 0,$$

so $\widetilde{m}_T(A_n) \to 0$, as desired.

Now let $f \in \mathcal{L}^\infty(\Sigma, X)$. Then there exists a sequence (s_n) in $\mathcal{S}(\Sigma, X)$ such that $\|s_n(\omega) - f(\omega)\|_X \to 0$ for $\omega \in \Omega$ and $\|s_n(\omega)\|_X \leqslant \|f(\omega)\|_X$ for $\omega \in \Omega$ and all $n \in \mathbb{N}$. Then $\sup_n \|s_n - f\| \leqslant 2\|f\| < \infty$. It follows that

$$T(f) = \lim_n T(s_n) = \lim_n \int_\Omega s_n\,dm_T = \int_\Omega f\,dm_T = T_{m_T}(f). \qquad \square$$

Remark 2.1. Note that some similar results concerning the problem of integral representation (with respect to operator-valued measures) of some class of linear operators on the Lebesgue–Bochner space $L^\infty(\mu, X)$ have been established in [15].

Let $\mathcal{L}(\mathcal{L}^\infty(\Sigma, X), Y)$ stand for the space of all bounded linear operators from $\mathcal{L}^\infty(\Sigma, X)$ to Y. The *topology $\mathcal{T}_s$ of simple convergence* is a locally convex topology on $\mathcal{L}(\mathcal{L}^\infty(\Sigma, X), Y)$ defined by the family of seminorms $\{p_f\colon f \in \mathcal{L}^\infty(\Sigma, X)\}$, where $p_f(T) = \|T(f)\|_Y$ for all $T \in \mathcal{L}(\mathcal{L}^\infty(\Sigma, X), Y)$. By $\mathcal{L}_c(\mathcal{L}^\infty(\Sigma, X), Y)$ we denote the set of all those $T \in \mathcal{L}(\mathcal{L}^\infty(\Sigma, X), Y)$ that are σ-smooth.

We will need the following Nikodým convergence type theorems (see [26, Proposition 13], [25, Proposition 11]).

Proposition 2.4. *Let $m_k \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ be variationally semi-regular for $k \in \mathbb{N}$. Assume that $T(f) = \lim_k \int_\Omega f\, dm_k$ exists in $(Y, \|\cdot\|_Y)$ for each $f \in \mathcal{L}^\infty(\Sigma, X)$. Then $m_T \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ is variationally semi-regular and $\sup_k \widetilde{m}_k(A_n) \underset{n}{\longrightarrow} 0$ as $A_n \downarrow \emptyset$.*

Proposition 2.5. *Let $m_k \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ be variationally semi-regular for $k \in \mathbb{N}$ and assume that $m(A)(x) := \lim_k m_k(A)(x)$ exists for each $A \in \Sigma$ and $x \in X$. If $m \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ is variationally semi-regular and $\sup_k \widetilde{m}_k(A_n) \underset{n}{\longrightarrow} 0$ as $A_n \downarrow \emptyset$, then $\lim_k \int_\Omega f\, dm_k = \int_\Omega f\, dm$ for each $f \in \mathcal{L}^\infty(\Sigma, X)$.*

Now we are ready to state the following Banach–Steinhaus type theorem for σ-smooth operators from $\mathcal{L}^\infty(\Sigma, X)$ to Y.

Theorem 2.6. *Let $T_k : \mathcal{L}^\infty(\Sigma, X) \to Y$ be σ-smooth operators for $k \in \mathbb{N}$. Assume that $T(f) := \lim_k T_k(f)$ exists in $(Y, \|\cdot\|_Y)$ for each $f \in \mathcal{L}^\infty(\Sigma, X)$. Then $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ is a σ-smooth operator and the family $\{T_k\colon k \in \mathbb{N}\}$ is uniformly σ-smooth, i.e., $\sup_k \|T_k(f_n)\|_Y \underset{n}{\longrightarrow} 0$ for any sequence (f_n) in $\mathcal{L}^\infty(\Sigma, X)$ such that $\|f_n(\omega)\|_Y \to 0$ for all $\omega \in \Omega$ and $\sup_n \|f_n\| < \infty$.*

Proof. Let $m_k \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ be the representing measures for T_k, $k \in \mathbb{N}$. By Proposition 2.3 $m_k \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ are variationally semi-regular for $k \in \mathbb{N}$. Then by Proposition 2.4 $m_T \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ is variationally semi-regular and $\sup_k \widetilde{m}_k(A_n) \underset{n}{\longrightarrow} 0$ as $A_n \downarrow \emptyset$. Since $m_T(A)(x) = \lim_k m_k(A)(x)$ for each $A \in \Sigma$ and $x \in X$, in view of Proposition 2.5 it follows that $\lim_k T_k(f) = \int_\Omega f\, dm_T$ for each $f \in \mathcal{L}^\infty(\Sigma, X)$. Hence $T = T_{m_T}$, and by Proposition 2.2 T is σ-smooth.

Now we shall show that the family $\{T_k\colon k \in \mathbb{N}\}$ is uniformly σ-smooth. Note first that if $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$, then by (1.1) we get

$$\sup_k \widetilde{m}_k(A_n) = \sup\bigl\{\bigl|(m_k)_{y^*}\bigr|(A_n)\colon y^* \in B_{Y^*},\ k \in \mathbb{N}\bigr\} \underset{n}{\longrightarrow} 0.$$

Moreover, since $\sup_k \widetilde{m}_k(\Omega) = \sup_k \|T_k\| = K < \infty$ (see Lemma 2.1), by (1.1) we have

$$\sup\bigl\{\bigl|(m_k)_{y^*}\bigr|(\Omega)\colon y^* \in B_{Y^*},\ k \in \mathbb{N}\bigr\} < \infty.$$

It follows that there exists $\mu \in \operatorname{ca}^+(\Sigma)$ such that the family $\{|(m_k)_{y^*}|\colon y^* \in B_{Y^*},\ k \in \mathbb{N}\}$ is uniformly μ-continuous (see [10, Theorem 13, p. 92]).

Now let (f_n) be a sequence in $\mathcal{L}^\infty(\Sigma, X)$ such that $\|f_n(\omega)\|_X \to 0$ for $\omega \in \Omega$ and $a = \sup_n \|f_n\| < \infty$. Let $\varepsilon > 0$ be given. For $\eta = \frac{\varepsilon}{2\max(a,K)} > 0$ and $n \in \mathbb{N}$ let us set

$$A_n(\eta) = \bigl\{\omega \in \Omega\colon \bigl\|f_n(\omega)\bigr\|_X > \eta\bigr\}.$$

Then $\mu(A_n(\eta)) \underset{n}{\longrightarrow} 0$, and in view of Lemma 2.1 it follows that

$$\sup_k \bigl\|(T_k)_{A_n(\eta)}\bigr\| = \sup_k \widetilde{m}_k\bigl(A_n(\eta)\bigr) = \sup\bigl\{\bigl|(m_k)_{y^*}\bigr|\bigl(A_n(\eta)\bigr)\colon y^* \in B_{Y^*},\ k \in \mathbb{N}\bigr\} \underset{n}{\longrightarrow} 0.$$

Hence there exists $n_\varepsilon \in \mathbb{N}$ such that for $n \geqslant n_\varepsilon$,

$$\sup_k \left\| T_k\left(\frac{1}{a}\mathbb{1}_{A_n(\eta)} f_n\right)\right\|_Y \leqslant \frac{\varepsilon}{2a},$$

i.e., for every $k \in \mathbb{N}$ and $n \geqslant n_\varepsilon$ we have

$$\bigl\|T_k(\mathbb{1}_{A_n(\eta)} f_n)\bigr\|_Y \leqslant \frac{\varepsilon}{2}. \tag{1}$$

Moreover, for every $k \in \mathbb{N}$ and $n \in \mathbb{N}$ we have

$$\bigl\|T_k(\mathbb{1}_{\Omega \setminus A_n(\eta)} f_n)\bigr\|_Y \leqslant \|T_k\| \cdot \|\mathbb{1}_{\Omega \setminus A_n(\eta)} f_n\| \leqslant K \cdot \eta \leqslant \frac{\varepsilon}{2}. \tag{2}$$

Hence by (1) and (2) for every $k \in \mathbb{N}$ and $n \geqslant n_\varepsilon$ we have

$$\bigl\|T_k(f_n)\bigr\|_Y \leqslant \bigl\|T_k(\mathbb{1}_{A_n(\eta)} f_n)\bigr\|_Y + \bigl\|T_k(\mathbb{1}_{\Omega \setminus A_n(\eta)} f_n)\bigr\|_Y \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $\sup_k \|T_k(f_n)\|_Y \leqslant \varepsilon$ for $n \geqslant n_\varepsilon$. Thus the proof is complete. □

Corollary 2.7. (i) *$\mathcal{L}_c(\mathcal{L}^\infty(\Sigma, X), Y)$ is a $\mathcal{T}_s$-sequentially closed subspace of $\mathcal{L}(\mathcal{L}^\infty(\Sigma, X), Y)$.*
(ii) *The space $(\mathcal{L}_c(\mathcal{L}^\infty(\Sigma, X), Y), \mathcal{T}_s)$ is sequentially complete.*

3. Topological properties of the space $\mathcal{L}^\infty(\Sigma, X)^*_c$

In this section we study the topological properties of the space $\mathcal{L}^\infty(\Sigma, X)^*_c$. We show that the classical theorems concerning the σ-order continuous dual $\mathcal{L}^\infty(\Sigma)^*_c$ of the Banach lattice $\mathcal{L}^\infty(\Sigma)$ (see [27,28]) continue to hold for the space $\mathcal{L}^\infty(\Sigma, X)^*_c$.

Applying Propositions 2.2 and 2.3 to linear functionals on $\mathcal{L}^\infty(\Sigma, X)$ we get:

Corollary 3.1. (i) *Assume that $\nu \in$ bvca(Σ, X^*). Then the functional Φ_ν on $\mathcal{L}^\infty(\Sigma, X)$ is σ-smooth, i.e., $\Phi_\nu \in \mathcal{L}^\infty(\Sigma, X)^*_c$, and $\|\Phi_\nu\| = |\nu|(\Omega)$.*
(ii) *Assume that $\Phi \in \mathcal{L}^\infty(\Sigma, X)^*_c$. Then $\nu_\Phi \in$ bvca(Σ, X^*) and*

$$\Phi(f) = \Phi_{\nu_\Phi}(f) = \int_\Omega f\, d\nu_\Phi \quad \text{for all } f \in \mathcal{L}^\infty(\Sigma, X).$$

Thus we have a dual pair $\langle \mathcal{L}^\infty(\Sigma, X), \mathcal{L}^\infty(\Sigma, X)^*_c \rangle$ with the duality

$$\langle f, \Phi_\nu \rangle = \int_\Omega f\, d\nu \quad \text{for all } f \in \mathcal{L}^\infty(\Sigma, X),\ \nu \in \mathrm{bvca}\big(\Sigma, X^*\big).$$

Note that for $Y = \mathbb{R}$, $\mathcal{T}_s$ on $\mathcal{L}^\infty(\Sigma, X)^*_c$ coincides with $\sigma(\mathcal{L}^\infty(\Sigma, X)^*_c, \mathcal{L}^\infty(\Sigma, X))$. Hence, as a consequence of Corollary 2.7, we get the following.

Corollary 3.2. (i) *$\mathcal{L}^\infty(\Sigma, X)^*_c$ is a sequentially closed subspace of $(\mathcal{L}^\infty(\Sigma, X)^*, \sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)))$.*
(ii) *The space $(\mathcal{L}^\infty(\Sigma, X)^*_c, \sigma(\mathcal{L}^\infty(\Sigma, X)^*_c, \mathcal{L}^\infty(\Sigma, X)))$ is sequentially complete.*

Now we state a form of generalized Nikodým convergence theorem for bvca(Σ, X^*). For a set $\mathcal{M}$ in bvca(Σ, X^*) let

$$|\mathcal{M}| = \big\{|\nu|: \nu \in \mathcal{M}\big\} \quad \text{and} \quad \mathcal{K}_\mathcal{M} = \big\{\Phi_\nu \in \mathcal{L}^\infty(\Sigma, X)^*_c: \nu \in \mathcal{M}\big\}.$$

Theorem 3.3. *Let $\mathcal{M}$ be a relatively $\sigma($*bvca$(\Sigma, X^*), \mathcal{L}^\infty(\Sigma, X))$*-sequentially compact set in* bvca(Σ, X^*). *Then the following statements hold*:

(a) $\sup_{\nu\in\mathcal{M}} |\nu|(\Omega) < \infty$.
(b) *$|\mathcal{M}|$ is uniformly countably additive.*
(c) *For each set $A \in \Sigma$ the set $\{\nu(A): \nu \in \mathcal{M}\}$ is relatively $\sigma(X^*, X)$-sequentially compact.*

Proof. (a) We will first show that $\mathcal{M}$ is $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathcal{L}^\infty(\Sigma, X))$-bounded. Assume on the contrary that $\mathcal{M}$ is not $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathcal{L}^\infty(\Sigma, X))$-bounded. Then there exists $f \in \mathcal{L}^\infty(\Sigma, X)$ such that $\sup_{\nu\in\mathcal{M}} |\int_\Omega f\, d\nu| = \infty$. Hence for each $n \in \mathbb{N}$ there exists $\nu_n \in \mathcal{M}$ such that $|\int_\Omega f\, d\nu_n| \geqslant n$. Choose a $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathcal{L}^\infty(\Sigma, X))$-Cauchy subsequence (ν_{k_n}) of (ν_n). It follows that a sequence $(\int_\Omega f\, d\nu_{k_n})$ is convergent and this leads to a contradiction. Hence $\mathcal{M}$ is $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathcal{L}^\infty(\Sigma, X))$-bounded, so $\mathcal{K}_\mathcal{M}$ is a $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X))$-bounded subset of $\mathcal{L}^\infty(\Sigma, X)^*$. By the uniform boundedness theorem, $\sup_{\nu\in\mathcal{M}} |\nu|(\Omega) = \sup_{\nu\in\mathcal{M}} \|\Phi_\nu\| < \infty$, as desired.

(b) Assume on the contrary that (b) does not hold. Then in view of [10, Theorem 10, p. 88] and the Rosenthal lemma (see [10, Chapter 7, p. 82]) there exist a pairwise disjoint sequence (A_n) in Σ, a positive number ε_0 and a sequence (ν_n), in $\mathcal{M}$ such that

$$|\nu_n|(A_n) > \varepsilon_0 \quad \text{and} \quad |\nu_n|\Big(\bigcup_{j\neq n} A_j\Big) < \frac{1}{8}\varepsilon_0 \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

In view of (1) for each $n \in \mathbb{N}$ there exists a finite Σ-partition $(A_{n,i})_{i=1}^{i_n}$ of A_n such that $\sum_{i=1}^{i_n} \|\nu_n(A_{n,i})\|_{X^*} > \varepsilon_0$. Next, for each $i = 1, \dots, i_n$ there exists $x_{n,i} \in B_X$ such that

$$\nu_n(A_{n,i})(x_{n,i}) \geqslant \big\|\nu_n(A_{n,i})\big\|_{X^*} - \frac{1}{2^{i+1}}\varepsilon_0.$$

Let $s_n = \sum_{i=1}^{i_n} (\mathbb{1}_{A_{n,i}} \otimes x_{n,i}) \in \mathcal{S}(\Sigma, X)$. Then $s_n(\omega) = 0$ for $\omega \in \Omega \setminus A_n$ with $\|s_n\| \leqslant 1$ for $n \in \mathbb{N}$ and

$$\int_\Omega s_n\,d\nu_n=\sum_{i=1}^{i_n}\nu_n(A_{n,i})(x_{n,i})\geqslant\sum_{i=1}^{i_n}\big\|\nu_n(A_{n,i})\big\|_{X^*}-\sum_{i=1}^{i_n}\frac{1}{2^{i+1}}\varepsilon_0\geqslant\frac{1}{2}\varepsilon_0.$$

Let (ν_{k_n}) be any subsequence of (ν_n), and let $f(\omega)=\sum_{n=1}^\infty s_{k_{2n}}(\omega)$ for $\omega\in\Omega$. Clearly $f\in\mathcal{L}^\infty(\Sigma,X)$ with $\|f\|\leqslant 1$, $f(\omega)=s_{k_{2n}}(\omega)$ for $\omega\in A_{k_{2n}}$ and $f(\omega)=0$ for $\omega\in A_{k_{2n+1}}$ and all $n\in\mathbb{N}$. Hence by (1) for each $n\in\mathbb{N}$ we have

$$\begin{aligned}\int_\Omega f\,d\nu_{k_{2n}}&=\int_{A_{k_{2n}}} f\,d\nu_{k_{2n}}+\int_{\bigcup_{j\neq k_{2n}}A_j} f\,d\nu_{k_{2n}}\\&\geqslant\int_\Omega s_{k_{2n}}\,d\nu_{k_{2n}}-|\nu_{k_{2n}}|\Big(\bigcup_{j\neq k_{2n}}A_j\Big)\\&\geqslant\frac{1}{2}\varepsilon_0-\frac{1}{4}\varepsilon_0=\frac{1}{2}\varepsilon_0.\end{aligned}$$

Moreover, using (1) we get for each $n\in\mathbb{N}$,

$$\begin{aligned}\int_\Omega f\,d\nu_{k_{2n+1}}&=\int_{\bigcup_{j=1}^\infty A_{k_{2j}}} f\,d\nu_{k_{2n+1}}\\&\leqslant\|f\|\cdot|\nu_{k_{2n+1}}|\Big(\bigcup_{j=1}^\infty A_{k_{2j}}\Big)\\&\leqslant|\nu_{k_{2n+1}}|\Big(\bigcup_{j\neq k_{2n+1}}A_j\Big)<\frac{1}{4}\varepsilon_0.\end{aligned}$$

This means that (ν_{k_n}) is not a $\sigma(\mathrm{bvca}(\Sigma,X^*),\mathcal{L}^\infty(\Sigma,X))$-Cauchy sequence because for $f\in\mathcal{L}^\infty(\Sigma,X)$ the limit of $\langle\nu_{k_n},f\rangle$ $(=\int_\Omega f\,d\nu_{k_n})$ does not exist. It follows that $\mathcal{M}$ is not a relatively $\sigma(\mathrm{bvca}(\Sigma,X^*),\mathcal{L}^\infty(\Sigma,X))$-sequentially compact subset of $\mathrm{bvca}(\Sigma,X^*)$.

(c) Note that for each $A\in\Sigma$ the mapping: $\mathrm{bvca}(\Sigma,X^*)\ni\nu\mapsto\nu(A)\in X^*$ is $(\sigma(\mathrm{bvca}(\Sigma,X^*),\mathcal{L}^\infty(\Sigma,X)),\sigma(X^*,X))$-continuous. Hence for each $A\in\Sigma$, the set $\{\nu(A)\colon\nu\in\mathcal{M}\}$ is relatively $\sigma(X^*,X)$-sequentially compact. □

The following lemma will be useful.

Lemma 3.4. *Let (ν_n) be a sequence in $\mathrm{bvca}(\Sigma,X^*)$ such that*

(a) $\sup_n|\nu_n|(\Omega)<\infty$,
(b) $\{|\nu_n|\colon n\in\mathbb{N}\}$ *is uniformly countably additive,*
(c) *for each $A\in\Sigma$ the sequence $(\nu_n(A))$ is $\sigma(X^*,X)$-convergent to some element $\nu(A)$ of X^*.*

Then the set function $\nu:\Sigma\ni A\mapsto\nu(A)\in X^*$ belongs to $\mathrm{bvca}(\Sigma,X^*)$ and

$$\int_\Omega f\,d\nu_n\to\int_\Omega f\,d\nu\quad\textit{for each } f\in\mathcal{L}^\infty(\Sigma,X).$$

Proof. In view of (c) and the Nikodým convergence theorem, $\nu_x\in\mathrm{ca}(\Sigma)$ for each $x\in X$, i.e., ν is countably additive in $\sigma(X^*,X)$. To prove that $\nu\in\mathrm{bvca}(\Sigma,X^*)$ it is enough to show that $|\nu|(\Omega)<\infty$ (see [20, Theorem 6.1.3]). Note that for each $A\in\Sigma$ we have $\|\nu(A)\|_{X^*}\leqslant\liminf_n\|\nu_n(A)\|_{X^*}$. Now let $(A_i)_{i=1}^k$ be a Σ-partition of Ω. Then by (a) we have

$$\begin{aligned}\sum_{i=1}^k\big\|\nu(A_i)\big\|_{X^*}&\leqslant\sum_{i=1}^k\liminf_n\big\|\nu_n(A_i)\big\|_{X^*}\\&\leqslant\liminf_n\Big(\sum_{i=1}^k\big\|\nu_n(A_i)\big\|_{X^*}\Big)\\&\leqslant\liminf_n|\nu_n|(\Omega)\leqslant\sup_n|\nu_n|(\Omega)<\infty.\end{aligned}$$

It follows that $|\nu|(\Omega)<\infty$, i.e., $\nu\in\mathrm{bvca}(\Sigma,X^*)$, and in view of Proposition 2.5 the proof is complete. □

In the theory of Riesz spaces the problem of weak*-compactness in the order duals of Riesz spaces has been studied by many authors (see [21,22,9,4]).

Recall that a σ-algebra Σ is said to be countably generated if there exists a countable subset of Σ that generates Σ as a σ-algebra. In particular, if Ω is a compact metric space, then any countable base for the topology of Ω generates the Borel sets as a σ-algebra. Now we are in position to characterize relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*_c, \mathcal{L}^\infty(\Sigma, X))$-sequentially compact subsets of $\mathcal{L}^\infty(\Sigma, X)^*_c$ when Σ is countably generated.

Theorem 3.5. *Assume that a σ-algebra Σ is countably generated and let $\mathcal{M}$ be a subset of* $\mathrm{bvca}(\Sigma, X^*)$. *Then the following statements are equivalent*:

(i) $\{\Phi_\nu\colon \nu \in \mathcal{M}\}$ *is a relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*_c, \mathcal{L}^\infty(\Sigma, X))$-sequentially compact subset of $\mathcal{L}^\infty(\Sigma, X)^*_c$.*
(ii) $\mathcal{M}$ *is a relatively $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathcal{L}^\infty(\Sigma, X))$-sequentially compact subset of* $\mathrm{bvca}(\Sigma, X^*)$.
(iii) *The following conditions hold*:
  (a) $\sup_{\nu\in\mathcal{M}} |\nu|(\Omega) < \infty$.
  (b) $|\mathcal{M}|$ *is uniformly countably additive.*
  (c) *For each $A \in \Sigma$ the set $\{\nu(A)\colon \nu \in \mathcal{M}\}$ is relatively $\sigma(X^*, X)$-sequentially compact.*
  Moreover, if X^* has the Radon–Nikodým property, then the condition (c) *is superfluous.*

Proof. (i) $\Longleftrightarrow$ (ii) See Corollary 3.1.

(ii) $\Longrightarrow$ (iii) It follows from Theorem 3.3.

(iii) $\Longrightarrow$ (ii) Assume that the conditions (a), (b), (c) hold. Let $\mathcal{B}$ be a countable set in Σ that generates Σ as a σ-algebra. Then the algebra $\mathcal{A}$ generated by $\mathcal{B}$ is countable (see [14, Lemma 4, p. 167]).

Let (ν_n) be a sequence in $\mathcal{M}$. Then in view of (c) we can use a diagonal argument to select a subsequence (ν_{k_n}) of (ν_n) such that for each $A \in \mathcal{A}$, $(\nu_{k_n}(A))$ is a $\sigma(X^*, X)$-Cauchy sequence, i.e., for each $A \in \mathcal{A}$, $\lim_n (\nu_{k_n})_x(A)$ exists for each $x \in X$. Since for each $x \in X$, the family $\{\nu_x\colon \nu \in \mathcal{M}\}$ in $\mathrm{ca}(\Sigma)$ is uniformly countably additive, we conclude that for each $A \in \Sigma$, $\lim_n (\nu_{k_n})_x(A)$ $(= \lim_n \nu_{k_n}(A)(x))$ exists for each $x \in X$ (see [10, Lemma, p. 91]). This means that for each $A \in \Sigma$, $(\nu_{k_n}(A))$ is a $\sigma(X^*, X)$-Cauchy sequence. Since the space $(X^*, \sigma(X^*, X))$ is sequentially complete, it follows that for each $A \in \Sigma$ the sequence $(\nu_{k_n}(A))$ is $\sigma(X^*, X)$-convergent to some element $\nu(A) \in X^*$. By Lemma 3.4 we conclude that $\nu \in \mathrm{bvca}(\Sigma, X^*)$ and $\int_\Omega f\, d\nu_{k_n} \to \int_\Omega f\, d\nu$ for each $f \in \mathcal{L}^\infty(\Sigma, X)$.

Note that if X^* has the Radon–Nikodým property, then the closed unit ball in X^* is $\sigma(X^*, X)$-sequentially compact (see [16, Corollary 2]). Since $\|\nu(A)\|_{X^*} \leqslant |\nu|(A) \leqslant |\nu|$ for each $A \in \Sigma$, in view of (a) we conclude that $\{\nu(A)\colon \nu \in \mathcal{M}\}$ is a relatively $\sigma(X^*, X)$-sequentially compact subset of X^* for each $A \in \Sigma$, i.e., (c) holds. $\square$

Remark 3.1. (i) Some related results to Theorems 3.3 and 3.5 concerning relative $\sigma(L^\infty(\mu, X)^\sim_n, L^\infty(\mu, X))$-sequential compactness in the order continuous dual $L^\infty(\mu, X)^\sim_n$ of $L^\infty(\mu, X)$ can be found in [24, Theorem 2.3 and Corollary 3.2]. It is known that $L^\infty(\mu, X)^\sim_n$ can be identified through integration with the space $L^1(\mu, X^*, X)$ of the weak*-equivalence classes of all weak*-measurable functions $g : \Omega \to X^*$ for which $\vartheta(g) \in L^1(\mu)$, where $\vartheta(g) = \sup\{|g_x|\colon x \in B_X\}$ and the supremum is taken in $L^0(\mu)$ (here $g_x(\omega) = g(\omega)(x)$ for $x \in X$ and all $\omega \in \Omega$). For each $g \in L^1(\mu, X^*, X)$ one can define a vector measure $\nu_g : \Sigma \to X^*$ by setting $\nu_g(A)(x) = \int_A \langle x, g(\omega)\rangle\, d\mu$ for all $A \in \Sigma$, $x \in X$. One can show (see [24, Corollary 3.2]) that if Σ is countably generated, then a subset H of $L^1(\mu, X^*, X)$ is relatively $\sigma(L^1(\mu, X^*, X), L^\infty(\mu, X))$-sequentially compact if and only if the following conditions hold:

(a) $\sup\{\int_\Omega \vartheta(g)(\omega)\, d\mu\colon g \in H\} < \infty$.
(b) $\{\vartheta(g)\colon g \in H\}$ is uniformly integrable.
(c) For each $A \in \Sigma$ the set $\{\nu_g(A)\colon g \in H\}$ in X^* is relatively $\sigma(X^*, X)$-sequentially compact.

(ii) Batt (see [6, Theorems 1 and 2]) found a characterization of relatively $\sigma(\mathrm{bvca}(\Sigma, X), \mathcal{L}^\infty(\Sigma, X^*))$-sequentially compact sets in $\mathrm{bvca}(\Sigma, X)$ and a characterization of relatively $\sigma(L^1(\mu, X), L^\infty(\mu, X^*))$-sequentially compact sets in $L^1(\mu, X)$. Moreover, some related results concerning conditional $\sigma(\mathrm{bvca}(\Sigma, X), L^\infty(\mu, X^*))$-compactness in $\mathrm{bvca}(\Sigma, X)$ and conditional $\sigma(L^1(\mu, X), L^\infty(\mu, X^*))$-compactness in $L^1(\mu, X)$ can be found in [1, Theorems 2.4 and 2.5]).

For a subset $\mathcal{K}$ of $\mathcal{L}^\infty(\Sigma, X)^*_c$ let $\mathcal{M}_\mathcal{K} = \{\nu \in \mathrm{bvca}(\Sigma, X^*)\colon \Phi_\nu \in \mathcal{K}\}$.

Now we are in position to prove a vector-valued version of Theorem 1.1 of [28] and Theorem 8 of [27] when X is a reflexive Banach space.

Theorem 3.6. *Assume that X is a reflexive Banach space. Then for bounded subset $\mathcal{K}$ of $\mathcal{L}^\infty(\Sigma, X)^*_c$ the following statements are equivalent*:

(i) *$\mathcal{K}$ is relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**})$-compact.*
(ii) *$\mathcal{K}$ is relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**})$-sequentially compact.*

(iii) *$\mathcal{K}$ is relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X))$-sequentially compact.*
(iv) *The set $\{|\nu|: \nu \in \mathcal{M}_\mathcal{K}\}$ in $ca^+(\Sigma)$ is uniformly countably additive.*

Proof. (i) $\Longleftrightarrow$ (ii) It follows from the Eberlein–Šmulian theorem.

(ii) $\Longrightarrow$ (iii) It is obvious.

(iii) $\Longrightarrow$ (iv) Assume that $\mathcal{K}$ is relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X))$-sequentially compact. Since $\mathcal{L}^\infty(\Sigma, X)^*_c$ is sequentially closed in $(\mathcal{L}^\infty(\Sigma, X)^*, \sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)))$ (see Corollary 3.2), $\mathcal{K}$ is a relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*_c, \mathcal{L}^\infty(\Sigma, X))$-sequentially compact subset of $\mathcal{L}^\infty(\Sigma, X)^*_c$. In view of Theorem 3.3 the set $\{|\nu|: \nu \in \mathcal{M}_\mathcal{K}\}$ is uniformly countably additive.

(iv) $\Longrightarrow$ (i) Assume that the set $\{|\nu|: \nu \in \mathcal{M}_\mathcal{K}\}$ in $ca^+(\Sigma)$ is uniformly countably additive. Then by [8, Corollary 1] $\mathcal{M}_\mathcal{K}$ is a relatively $\sigma(\mathrm{bvca}(\Sigma, X^*), \mathrm{bvca}(\Sigma, X^*)^*)$-compact subset of $\mathrm{bvca}(\Sigma, X^*)$. This means that $\mathcal{K}$ is relatively compact set in $(\mathcal{L}^\infty(\Sigma, X)^*_c, \sigma(\mathcal{L}^\infty(\Sigma, X)^*_c, (\mathcal{L}^\infty(\Sigma, X)^*_c)^*))$. Moreover, by Corollary 3.2 we obtain that $\mathcal{L}^\infty(\Sigma, X)^*_c$ is a closed subset of the Banach space $\mathcal{L}^\infty(\Sigma, X)^*$. Hence $\mathcal{L}^\infty(\Sigma, X)^*_c$ is a closed set in $(\mathcal{L}^\infty(\Sigma, X)^*, \sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**}))$, so

$$\mathrm{cl}_{\sigma(\mathcal{L}^\infty(\Sigma,X)^*, \mathcal{L}^\infty(\Sigma,X)^{**})}\mathcal{K} \subset \mathcal{L}^\infty(\Sigma, X)^*_c.$$

Note that (see [17, Corollary 3.3.3])

$$\sigma\big(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**}\big)\big|_{\mathcal{L}^\infty(\Sigma,X)^*_c} = \sigma\big(\mathcal{L}^\infty(\Sigma, X)^*_c, \big(\mathcal{L}^\infty(\Sigma, X)^*_c\big)^*\big). \tag{1}$$

It follows that

$$\mathrm{cl}_{\sigma(\mathcal{L}^\infty(\Sigma,X)^*_c, (\mathcal{L}^\infty(\Sigma,X)^*_c)^*)}\mathcal{K} = \mathrm{cl}_{\sigma(\mathcal{L}^\infty(\Sigma,X)^*, \mathcal{L}^\infty(\Sigma,X)^{**})}\mathcal{K}. \tag{2}$$

Since $\mathrm{cl}_{\sigma(\mathcal{L}^\infty(\Sigma,X)^*_c, (\mathcal{L}^\infty(\Sigma,X)^*_c)^*)}\mathcal{K}$ is a $\sigma(\mathcal{L}^\infty(\Sigma, X)^*_c, (\mathcal{L}^\infty(\Sigma, X)^*_c)^*)$-compact subset of $\mathcal{L}^\infty(\Sigma, X)^*_c$, in view of (1) and (2) we see that $\mathrm{cl}_{\sigma(\mathcal{L}^\infty(\Sigma,X)^*, \mathcal{L}^\infty(\Sigma,X)^{**})}\mathcal{K}$ is a $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, (\mathcal{L}^\infty(\Sigma, X)^{**})$-compact subset of $\mathcal{L}^\infty(\Sigma, X)^*$, i.e., $\mathcal{K}$ is relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**})$-compact, as desired. □

As a consequence of Theorem 3.6 we obtain a Grothendieck type theorem saying that $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X))$-convergent sequences in $\mathcal{L}^\infty(\Sigma, X)^*_c$ are also $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**})$-convergent.

Corollary 3.7. *Assume that X is a reflexive Banach space. Let $\Phi_n \in \mathcal{L}^\infty(\Sigma, X)^*_c$ for $n \in \mathbb{N}$ and $\Phi \in \mathcal{L}^\infty(\Sigma, X)^*_c$. Then the following statements are equivalent*:

(i) *$\Phi_n \to \Phi$ for $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**})$.*
(ii) *$\Phi_n \to \Phi$ for $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X))$.*

Proof. (i) $\Longrightarrow$ (ii) It is obvious.

(ii) $\Longrightarrow$ (i) Assume that $\Phi_n \to \Phi$ for $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X))$ and let (Φ_{k_n}) be a subsequence of (Φ_n). Then $\mathcal{K} = \{\Phi_{k_n}: n \in \mathbb{N}\}$ is a relatively sequentially compact subset of $(\mathcal{L}^\infty(\Sigma, X)^*, \sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)))$. By Theorem 3.6 $\mathcal{K}$ is a relatively sequentially compact subset of $(\mathcal{L}^\infty(\Sigma, X)^*, \sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**}))$, so there exists a subsequence $(\Phi_{l_{k_n}})$ of (Φ_{k_n}) such that $\Phi_{l_{k_n}} \to \Phi$ for $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**})$. This means that $\Phi_n \to \Phi$ for $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**})$, as desired. □

Remark 3.2. Theorems 3.3 and 3.6 are modifications and corrections of Theorems 2.1 and 3.1 of [23], where we incorrectly considered the Banach space $B(\Sigma, X)$ of all X-valued totally Σ-measurable functions instead of the space $\mathcal{L}^\infty(\Sigma, X)$.

4. Relationships between operators on $\mathcal{L}^\infty(\Sigma, X)$

We start with the following useful result.

Proposition 4.1. *For a linear operator $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ the following statements are equivalent*:

(i) *$y^* \circ T \in \mathcal{L}^\infty(\Sigma, X)^*_c$ for each $y^* \in Y^*$.*
(ii) *T is $(\sigma(\mathcal{L}^\infty(\Sigma, X), \mathcal{L}^\infty(\Sigma, X)^*_c), \sigma(Y, Y^*))$-continuous.*
(iii) *T is $(\tau(\mathcal{L}^\infty(\Sigma, X), \mathcal{L}^\infty(\Sigma, X)^*_c), \|\cdot\|_Y)$-continuous.*

Proof. (i) $\Longleftrightarrow$ (ii) See [3, Theorem 9.26]; (ii) $\Longleftrightarrow$ (iii) See [3, Ex. 11, p. 149]. □

Note that every σ-smooth operator $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ is $(\tau(\mathcal{L}^\infty(\Sigma, X), \mathcal{L}^\infty(\Sigma, X)^*_c), \|\cdot\|_Y)$-continuous. On the other hand, since $\mathcal{L}^\infty(\Sigma, X)^*_c \subset \mathcal{L}^\infty(\Sigma, X)^*$, we derive that every $(\tau(\mathcal{L}^\infty(\Sigma, X), \mathcal{L}^\infty(\Sigma, X)^*_c), \|\cdot\|_Y)$-continuous linear operator $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ is bounded.

Proposition 4.2. *Let $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ be a $(\tau(\mathcal{L}^\infty(\Sigma, X), \mathcal{L}^\infty(\Sigma, X)^*_c), \|\cdot\|_Y)$-continuous linear operator. Then its representing measure $m_T \in \mathrm{fasv}(\Sigma, \mathcal{L}(X, Y))$ is countably additive in* W*OT *and for each $y^* \in Y^*$ we have*

$$\big(y^* \circ T\big)(f) = \int_\Omega f\, d(m_T)_{y^*} \quad \text{for all } f \in \mathcal{L}^\infty(\Sigma, X).$$

Proof. Let $y^* \in Y^*$ be given. Since $y^* \circ T \in \mathcal{L}^\infty(\Sigma, X)^*_c$ (see Proposition 4.1), by Corollary 3.1 there exists $\nu_{y^*} \in \mathrm{bvca}(\Sigma, X^*)$ such that $(y^* \circ T)(f) = \int_\Omega f\, d\nu_{y^*}$ for all $f \in \mathcal{L}^\infty(\Sigma, X)$. Hence for each $A \in \Sigma$, $x \in X$ we have

$$\begin{aligned}(m_T)_{y^*}(A)(x) &= y^*\big(m_T(A)(x)\big) = y^*\big(T(\mathbb{1}_A \otimes x)\big)\\ &= \int_\Omega (\mathbb{1}_A \otimes x)\, d\nu_{y^*} = \nu_{y^*}(A)(x).\end{aligned}$$

It follows that $(m_T)_{y^*} = \nu_{y^*} \in \mathrm{bvca}(\Sigma, X^*)$, i.e., m_T is countably additive in W*OT. □

Now using Theorem 3.3 we are ready to establish some relationships between different classes of operators on $\mathcal{L}^\infty(\Sigma, X)$.

Theorem 4.3. *Let $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ be a weakly compact and $(\tau(\mathcal{L}^\infty(\Sigma, X), \mathcal{L}^\infty(\Sigma, X)^*_c), \|\cdot\|_Y)$-continuous linear operator. Then T is σ-smooth.*

Proof. Since the conjugate operator $T^* : Y^* \to \mathcal{L}^\infty(\Sigma, X)^*$ is weakly compact, the set $\{y^* \circ T\colon\ y^* \in B_{Y^*}\}$ is relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**})$-compact, and hence by the Eberlein–Šmulian theorem, $\{y^* \circ T\colon\ y^* \in B_{Y^*}\}$ is relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X)^{**})$-sequentially compact in $\mathcal{L}^\infty(\Sigma, X)^*$. It follows that $\{y^* \circ T\colon\ y^* \in B_{Y^*}\}$ is a relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X))$-sequentially compact subset of $\mathcal{L}^\infty(\Sigma, X)^*$. Since $\{y^* \circ T\colon\ y^* \in B_{Y^*}\} \subset \mathcal{L}^\infty(\Sigma, X)^*_c$ (see Proposition 4.1) and $\mathcal{L}^\infty(\Sigma, X)^*_c$ is a sequentially $\sigma(\mathcal{L}^\infty(\Sigma, X)^*, \mathcal{L}^\infty(\Sigma, X))$-closed subset of $\mathcal{L}^\infty(\Sigma, X)^*$ (see Corollary 3.2), we derive that $\{y^* \circ T\colon\ y^* \in B_{Y^*}\}$ is a relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*_c, \mathcal{L}^\infty(\Sigma, X))$-sequentially compact subset of $\mathcal{L}^\infty(\Sigma, X)^*_c$. But $\{y^* \circ T\colon\ y^* \in B_{Y^*}\} = \{\Phi_{(m_T)_{y^*}}\colon\ y^* \in B_{Y^*}\}$ (see Proposition 4.2), so by Theorem 3.3 the set $\{|(m_T)_{y^*}|\colon\ y^* \in B_{Y^*}\}$ in $\mathrm{ca}(\Sigma)$ is uniformly countably additive. This means that m_T is variationally semi-regular and hence T_{m_T} is σ-smooth (see Proposition 2.2). In view of Proposition 4.2 and (2.1) for each $y^* \in Y^*$ we have

$$y^*\big(T(f)\big) = \int_\Omega f\, d(m_T)_{y^*} = y^*\big(T_{m_T}(f)\big) \quad \text{for all } f \in \mathcal{L}^\infty(\Sigma, X).$$

It follows that $T = T_{m_T}$, i.e., T is σ-smooth. □

Theorem 4.4. *Let $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ be a $(\tau(\mathcal{L}^\infty(\Sigma, X), \mathcal{L}^\infty(\Sigma, X)^*_c), \|\cdot\|_Y)$-continuous linear operator. Assume that either Y^* has the Radon–Nikodým property or Y contains no isomorphic copy of c_0. Then T is σ-smooth.*

Proof. Assume first that Y^* has the Radon–Nikodým property. By Proposition 4.1 T is $(\sigma(\mathcal{L}^\infty(\Sigma, X), \mathcal{L}^\infty(\Sigma, X)^*_c), \sigma(Y, Y^*))$-continuous. Let $T^* : Y^* \to \mathcal{L}^\infty(\Sigma, X)^*_c$ stand for the conjugate operator for T. Then T^* is $(\sigma(Y^*, Y), \sigma(\mathcal{L}^\infty(\Sigma, X)^*_c, \mathcal{L}^\infty(\Sigma, X)))$-continuous (see [7, Chapter IV, §6, Proposition 1]). Since B_{Y^*} is $\sigma(Y^*, Y)$-sequentially compact (see [16, Corollary 2]), we obtain that $T^*(B_{Y^*})$ is a relatively $\sigma(\mathcal{L}^\infty(\Sigma, X)^*_c, \mathcal{L}^\infty(\Sigma, X))$-sequentially compact subset of $\mathcal{L}^\infty(\Sigma, X)^*_c$. Note that in view of Proposition 4.2 we have that $T^*(B_{Y^*}) = \{\Phi_{(m_T)_{y^*}}\colon\ y^* \in B_{Y^*}\}$. Hence, by Theorem 3.3 the set $\{|(m_T)_{y^*}|\colon\ y^* \in B_{Y^*}\}$ is uniformly countably additive, i.e., m_T is variationally semi-regular. This means that T_{m_T} is σ-smooth (see Proposition 2.2). Now, arguing as in the proof of Theorem 4.3 we obtain that $T = T_{m_T}$, i.e., T is σ-smooth.

Now we assume that Y contains no isomorphic copy of c_0. In view of Proposition 4.2 $m_T \in \mathrm{fasv}(\Sigma, \mathcal{L}(X, Y))$ is countably additive in W*OT. Hence by [5,6, Theorem 6, Theorem 5 and Remark 7] we obtain that m_T is variationally semi-regular. Then by Proposition 2.2 $T_{m_T} : \mathcal{L}^\infty(\Sigma, X) \to Y$ is σ-smooth. Arguing as in the proof of Theorem 4.3 we derive that $T = T_{m_T}$, i.e., T is σ-smooth. □

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