



# Coherent algebras and noncommutative projective lines

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## Abstract

A well-known conjecture says that every one-relator group is coherent. We state and partly prove a similar statement for graded associative algebras. In particular, we show that every Gorenstein algebra  $A$  of global dimension 2 is graded coherent. This allows us to define a noncommutative analogue of the projective line  $\mathbb{P}^1$  as a noncommutative scheme based on the coherent noncommutative spectrum  $\text{qgr } A$  of such an algebra  $A$ , that is, the category of coherent  $A$ -modules modulo the torsion ones. This category is always abelian Ext-finite hereditary with Serre duality, like the category of coherent sheaves on  $\mathbb{P}^1$ . In this way, we obtain a sequence  $\mathbb{P}_n^1$  ( $n \geq 2$ ) of pairwise non-isomorphic noncommutative schemes which generalize the scheme  $\mathbb{P}^1 = \mathbb{P}_2^1$ .

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## 1. Introduction

We will consider  $\mathbb{N}$ -graded algebras of the form  $A = A_0 \oplus A_1 \oplus \cdots$  over a fixed field  $k$ . All our algebras will be assumed to be connected (that is,  $A_0 = k$ ) and finitely generated. All vector spaces and modules are assumed to be  $\mathbb{Z}$ -graded, all their elements and maps of them are homogeneous.

Recall that an algebra (respectively, a group) is called coherent if every its finitely generated ideal (subgroup) is finitely presented, see Definition 2.1 below. A well-known conjecture says

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that every one-relator group is coherent [Ba]. An analogous statement for graded algebras also seems to be true.

**Conjecture 1.1.** *Every graded algebra with a single defining relation is graded coherent.*

(For the definition of coherence, see Section 2.1 below.) We will prove this conjecture provided that the relation is quadratic.

**Theorem 1.2 (Theorem 4.1).** *Every graded algebra defined by a single homogeneous quadratic relation is graded coherent.*

Note that there are non-coherent quadratic algebras with two relations, for example, the algebras  $k\langle x, y, z, t \mid tz - zy, zx \rangle$  [Pi2, Proposition 10] or even  $k\langle x, y, z \mid yz - zy, zx \rangle$  [Po, Example 2].

Recall [Z2] that a graded algebra  $A$  is called regular if it has finite global dimension (say,  $d$ ) and satisfies the following Gorenstein property:

$$\mathrm{Ext}_A^i(k_A, k_A) \cong \begin{cases} 0, & i \neq d, \\ k[l] \text{ for some } l \in \mathbb{Z}, & i = d. \end{cases}$$

The most important class of regular algebras is the class of Artin–Shelter (AS) regular algebras, that is, the ones of polynomial growth. A well-known conjecture [AS] claims that all these algebras are Noetherian.

The following conjecture is due to A. Bondal (unpublished).

**Conjecture 1.3.** *Every regular algebra is graded coherent.*

Regular algebras of global dimension 2 have been described in [Z2]. All these algebras are one-relator. If such an algebra is generated in degree one, then it is quadratic, but in general such algebra is only ‘generalized quadratic’—like, for example, the algebra  $k\langle x, y \mid xy - yx = x^3 \rangle$ .

**Theorem 1.4 (Theorem 4.3).** *Every regular algebra of global dimension two is graded coherent.*

Two abelian categories may naturally be associated to any graded coherent algebra  $A$ , that is, the category  $\mathrm{cmod} A$  of finitely presented (= graded coherent) right graded  $A$ -modules and its quotient category  $\mathrm{qgr} A = \mathrm{cmod} A / \mathrm{tors} A$  by the category  $\mathrm{tors} A$  of finite-dimensional modules. This category  $\mathrm{qgr} A$  plays a role of projective spectrum for noncommutative coherent algebras [Po, BVD], in generalization of the well-known construction (due to Artin and Zhang) of noncommutative schemes in the Noetherian case [AZ]. In this approach, a noncommutative projective scheme is a triple

$$(\mathrm{qgr} A, \mathcal{A}, s),$$

where  $A$  is a coherent algebra, noncommutative structural sheaf  $\mathcal{A}$  is the image of  $A$  in  $\mathrm{qgr} A$ , and  $s$  is the autoequivalence of  $\mathrm{qgr} A$  induced by the shift of grading. Some details will be given in Section 2.2.

The noncommutative schemes of (Koszul) Noetherian (AS-)regular algebras of global dimension  $n + 1$  are usually considered as noncommutative generalizations of  $\mathbb{P}^n$ . However, in the

case of the projective line  $\mathbb{P}^1$ , this Noetherian construction does not give any more than the standard commutative  $\mathbb{P}^1$  again. On the other hand, there are other Noetherian abelian categories whose properties are close to the ones of the category of coherent sheaves on  $\mathbb{P}^1$  (that is, they are hereditary Ext-finite with Serre duality) [RVdB], but the “coordinate rings” of the corresponding noncommutative schemes are far from being connected graded, in contrast to the coordinate ring  $k[x_1, x_2]$  of  $\mathbb{P}^1$ .

Here we will introduce another noncommutative generalization of  $\mathbb{P}^n$ , that is, the noncommutative projective schemes corresponding to (degree-one generated) coherent regular algebras of dimension  $n + 1$ . We will show in Proposition 5.1 that the corresponding qgr  $A$  is an Ext-finite category of cohomological dimension  $n$ , and the algebra  $A$  (its coordinate ring) may be recovered by this category via a suitable “representing functor.” In the case of the projective line  $\mathbb{P}^1$ , we can obtain an infinite sequence  $\{\mathbb{P}_n^1\}_{n \geq 2}$  of pairwise non-isomorphic noncommutative schemes analogous to  $\mathbb{P}^1 = \mathbb{P}_2^1$ , where the coordinate ring of each  $\mathbb{P}_n^1$  is a connected graded 2-dimensional algebra with  $n$  generators. The corresponding categories of coherent sheaves are Ext-finite hereditary and satisfy Serre duality and BGG-correspondence. This is shown in the following

**Proposition 1.5.** *Let  $A$  be a degree-one generated regular algebra of global dimension 2 with  $n \geq 2$  generators.*

- (a) *The categories  $\text{cmod } A$  and  $\text{qgr } A$  and the noncommutative scheme  $\mathbb{P}_n^1 = \mathbb{P}_n^1(k)$  constructed by  $A$  are defined (up to isomorphisms) by the ground field  $k$  and the number  $n$  only, and do not depend on the algebra  $A$  itself. All these noncommutative schemes  $\mathbb{P}_n^1$  are pairwise non-isomorphic, with  $\mathbb{P}_2^1 \cong \mathbb{P}^1$ .*
- (b) *The category  $\text{qgr } A$  is Ext-finite hereditary with Serre duality. If  $n \geq 3$ , then it is not Noetherian, hence it does not belong to the classification in [RVdB].*
- (c) *The bounded derived category  $\mathcal{D}^b(\text{qgr } A)$  is equivalent to the category of finite  $B$ -modules modulo projectives, where  $B$  is a commutative Artinian algebra*

$$k[x_1, \dots, x_n]/(x_i x_j, x_i^2 - x_j^2 \mid i \neq j).$$

- (d) *The bounded derived category  $\mathcal{D}^b(\text{qgr } A)$  is equivalent to the bounded derived category of the finite-dimensional representations of the generalized Kronecker quiver  $Q_n$  (that is, quiver with two vertices  $v_0, v_1$  and  $n$  arrows  $v_0 \rightarrow v_1$ ).*

The last statement (d) of this proposition shows that the category of coherent sheaves on our noncommutative projective line  $\mathbb{P}_n^1$  is derived equivalent to the category of coherent sheaves on the noncommutative projective space  $\mathbb{P}^{n-1}$  in the sense of Kontsevich and Rosenberg, see [KR, 3.3]. This statement (d) has been originally communicated me by Michel Van den Bergh [VdB]; another proof is given recently by Minamoto [M].

This paper is organized as follows. In Section 2, we will give a background on coherent algebras, regular algebras of global dimension 2, and (relative) noncommutative complete intersections. In Section 3, we will give the following criterion for coherence: if an algebra  $B = A/I$  is a *relative noncommutative complete intersection* of  $A$  (that is, the ideal  $I$  is generated by a strongly free set), and  $B$  is right Noetherian, then  $A$  is graded coherent. In Section 4, we will apply the above criterion in order to prove Theorems 1.2 and 1.4. Finally, in Section 5 we will

consider noncommutative schemes associated to coherent regular algebras. In particular, we will prove Proposition 1.5.

## 2. Background

### 2.1. Coherence

A finitely generated (f.g.) right module  $M$  is called *coherent* if every its finitely generated submodule is finitely presented (that is, presented by a finite number of generators and relations). Analogously, a graded f.g. module is called *graded coherent* if every its graded f.g. submodule is finitely presented. In fact, this notion had been introduced by Serre [S] in a more general case of coherent sheaves.

**Theorem–Definition 2.1.** *A (graded) algebra  $A$  is called (graded) right coherent, if the following equivalent conditions hold:*

- (i) *every (homogeneous) finitely generated right-sided ideal in  $A$  is finitely presented, that is,  $A$  is (graded) coherent as a right module over itself;*
- (ii) *every finitely presented (graded) right  $A$ -module is (graded) coherent;*
- (iii) *all finitely presented (graded) right  $A$ -modules form an abelian category.*

The proof of equivalence may be found in [C] (see also [F]). For example, every right Noetherian algebra is right coherent, as well as every free associative algebra and every path algebra.

Since all our algebras and modules are graded, by the word *coherent* we will mean *graded right coherent* algebras and modules. The idea of noncommutative geometry based on such algebras will be explained in the next subsection.

### 2.2. Noncommutative schemes

Let  $A$  be a graded algebra. By  $\text{Gr } A$  (respectively,  $\text{cmod } A$ ) we denote the abelian category of graded (respectively, coherent)  $A$ -modules. Let  $\text{Tors } A$  (respectively,  $\text{tors } A$ ) be the category of torsion  $A$ -modules (respectively, finite-dimensional modules), where a module  $M$  is called torsion if for every  $x \in M$  there is  $n > 0$  such that  $x A_{\geq n} = 0$ . Note that  $\text{Tors } A$  is a Serre subcategory of  $\text{Gr } A$ ; moreover, if  $A$  is coherent, then  $\text{Tors } A$  is also a Serre subcategory of  $\text{cmod } A$ . The quotient abelian categories  $\text{Qgr } A = \text{Gr } A / \text{Tors } A$  and  $\text{qgr } A = \text{cmod } A / \text{tors } A$  (for coherent  $A$ ) play the roles of the categories of (quasi)coherent sheaves on the projective scheme associated to  $A$ . Due to classical Serre theorem [S], these categories of modules are indeed equivalent to the respective categories of sheaves provided that  $A$  is commutative.

The image  $\mathcal{A}$  of  $A_A$  in  $\text{Qgr } A$  (or in  $\text{qgr } A$ ) plays the role of the structure sheaf, and the degree shift  $s : M \mapsto M[1]$  plays the role of the polarization. Thus, a noncommutative scheme is a triple

$$X = (\mathcal{C}, \mathcal{A}, s),$$

where  $\mathcal{C}$  is a suitable  $k$ -linear abelian category,  $\mathcal{A}$  is an object, and  $s$  is an autoequivalence of  $\mathcal{C}$ . For  $\mathcal{C} = \text{Qgr } A$  with an arbitrary connected graded algebra  $A$ , this definition is due to Verevkin [V] (a general scheme). For  $\mathcal{C} = \text{qgr } A$  (coherent scheme), this definition is due to Artin

and Zhang [AZ] in the case of Noetherian  $A$  (Noetherian scheme) and to Bondal and Van den Bergh [BVdB] and Polishchuk [Po] in a more general setting of coherent algebra  $A$ .

According to [AZ], a morphism  $f : X \rightarrow X'$  of two schemes  $X = (\mathcal{C}, \mathcal{A}, s)$  and  $X' = (\mathcal{C}', \mathcal{A}', s')$  is a  $k$ -linear functor  $f : \mathcal{C} \rightarrow \mathcal{C}'$  such that  $f(\mathcal{A})$  is isomorphic to  $\mathcal{A}'$  and there is an isomorphism of functors  $f s \cong s' f$ . A map of schemes is defined as an isomorphism class of morphisms. Such a morphism  $f$  (or a respective map) is called an isomorphism if it is an equivalence of categories  $f : \mathcal{C} \cong \mathcal{C}'$ .

Given such a triple  $X = (\mathcal{C}, \mathcal{A}, s)$ , we can apply an analogue of the Serre functor to define a connected graded algebra  $A := \Gamma_{\geq 0}(X) = \bigoplus_{i \geq 0} \text{Hom}(\mathcal{A}, s^i(\mathcal{A}))$  with the multiplication  $a \cdot b := s^j(a) \circ b$  for  $a : \mathcal{A} \rightarrow s^i(\mathcal{A}), b : \mathcal{A} \rightarrow s^j(\mathcal{A})$ . In some cases, this algebra  $A$  is coherent and the scheme  $X$  itself is isomorphic to the scheme  $(\text{qgr } A, \mathcal{A}, s)$ . This happens if the autoequivalence  $s$  is ample [AZ], that is, the shifts of  $\mathcal{A}$  form an ample sequence in  $\mathcal{C}$  [Po].

If two general schemes  $X$  and  $Y$  are isomorphic, then the algebra  $\Gamma_{\geq 0}(Y)$  is isomorphic to a Zhang twist of  $\Gamma_{\geq 0}(X)$ ; on the other hand, if a coherent algebra  $B$  is a Zhang twist of an algebra  $A$ , then the coherent (and general) schemes of these algebras are isomorphic [Z1, Theorem 1.4]. Here an algebra  $B$  is called a Zhang twist of  $A$  if there are  $k$ -linear bijections  $\tau_i : A_i \rightarrow B_i, i \geq 0$  such that  $\tau_{m+n}(yz) = \tau_m(y)\tau_{m+n}(z)$  for homogeneous  $y \in A_m, z \in A_n$  [Z1, Proposition 2.8]. For example, the projective scheme of the quantum polynomial algebra  $k\langle x, y \mid xy = qyx \rangle$  is isomorphic to  $\mathbb{P}^1$  for every  $q \neq 0$ .

Let  $A$  be a graded algebra, let  $M, N \in \text{Gr } A$  be two modules, and let  $\mathcal{M}$  and  $\mathcal{N}$  be their images in  $\text{Qgr } A$ . Let  $\underline{\text{Hom}}(\mathcal{M}, \mathcal{N}) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}(\mathcal{M}, s^i \mathcal{N})$ , and let  $\text{Ext}$  and  $\underline{\text{Ext}}$  be the derived functors of  $\text{Hom}$  and  $\underline{\text{Hom}}$ . Since the obvious functor  $\text{qgr } A \rightarrow \text{Qgr } A$  is fully faithful for a coherent algebra  $A$ , the functors  $\text{Ext}$  and  $\underline{\text{Ext}}$  on the category  $\text{qgr } A$  are restrictions of the respective functors on  $\text{Qgr } A$ .

Following [V,AZ], we define the cohomologies of objects of  $\text{Qgr } A$  as  $H^i(\mathcal{M}) = \text{Ext}^i(\mathcal{A}, \mathcal{M})$  and  $\underline{H}^i(\mathcal{M}) = \underline{\text{Ext}}^i(\mathcal{A}, \mathcal{M}) = \lim_{n \rightarrow \infty} \text{Ext}_A^i(A_{\geq n}, M)$ . According to [BVdB, Lemma 4.1.6], we have  $\underline{H}^i(\mathcal{M}) \cong \lim_{n \rightarrow \infty} \text{Ext}_A^i(A_{\geq n}, M)$  and  $H^i(\mathcal{M}) \cong \lim_{n \rightarrow \infty} \text{Ext}_A^i(A_{\geq n}, M)_0$ .

### 2.3. Regular algebras of global dimension 2

Let us recall some results of [Z2]. Let  $V$  be a vector space. A rank of an element  $b \in T(V)$  is defined as the minimal number  $r$  of elements  $l_1, \dots, l_r \in V$  such that  $b = l_1 a_1 + \dots + l_r a_r$  for some  $a_1, \dots, a_r \in T(V)$ .

**Theorem 2.2.** (See [Z2].) *A graded algebra  $A$  is regular of global dimension 2 if and only if it is isomorphic to the algebra  $k\langle x_1, \dots, x_n \rangle / (b)$ , where  $\text{rank } b = n > 1$ , or, equivalently, the following conditions hold:*

- (1)  $n \geq 2$ ;
- (2)  $1 \leq \deg x_1 \leq \dots \leq \deg x_n$  with  $\deg b = \deg x_i + \deg x_{n+1-i}$  for all  $i$ ;
- (3) for some graded automorphism  $\sigma$  of the free algebra  $k\langle x_1, \dots, x_n \rangle$  we have  $b = \sum_{i=1}^n x_i \sigma(x_{n+1-i})$ .

*In this case, the algebra  $A$  is Noetherian if and only if  $n = 2$ .*

In particular, it follows that a regular two-dimensional algebra is Koszul if and only if it is degree-one generated.

2.4. (Relative) noncommutative complete intersections

In the next definition, we will unite several statements from [A]. For discussions on strongly free sets as a noncommutative analogue of regular sequences and related topics, see also [Pi3,U]. Recall that a relative complete intersection, from an algebraic point of view, is a quotient of some graded or local commutative ring by an ideal generated by a regular sequence. Here we introduce a relative noncommutative complete intersection (RNCI) as a quotient of a graded algebra by an ideal generated by a strongly free set. It is analogous to the term ‘noncommutative complete intersection,’ that is, RNCI of a free algebra [A,G,EG].

**Theorem–Definition 2.3.** (See [A].) *Suppose that a set  $X$  of homogeneous elements in a graded algebra  $A$  minimally generates a two-sided ideal  $I$ . Let  $B = A/I$  be a quotient algebra. The set  $X$  is called strongly free, if the following equivalent conditions hold:*

(i) *there are isomorphisms of graded vector spaces*

$$\begin{aligned} \text{Tor}_i^A(k, k) &\simeq \text{Tor}_i^B(k, k) \quad \text{for all } i \geq 3 \quad \text{and} \\ \text{Tor}_1^A(k, k) \oplus \text{Tor}_2^B(k, k) &\simeq \text{Tor}_1^B(k, k) \oplus \text{Tor}_2^A(k, k) \oplus kX; \end{aligned}$$

(ii) *there is an isomorphism of graded vector spaces*

$$B\langle X \rangle \simeq A,$$

where  $B\langle X \rangle = B * k\langle X \rangle$  is a free product of  $B$  and a free algebra on  $X$ ;

(iii) *the Shafarevich complex  $Sh(X, A)$  is acyclic in positive degrees.*

In this situation, we refer to the algebra  $B = A/I$  as a relative noncommutative complete intersection (RNCI) of the algebra  $A$ .

In particular, it follows that if  $B = A/I$  and there are isomorphisms  $\text{Tor}_i^A(k, k) \simeq \text{Tor}_i^B(k, k)$  for all  $i \geq 2$ , then  $B$  is an RNCI of  $A$ .

3. A criterion for coherence

**Lemma 3.1.** *Let  $X$  be a strongly free set in a graded algebra  $A$  and let  $I$  be an ideal generated by  $X$ . Then  $I$  is free as right (and left)  $A$ -module.*

More precisely, let  $B = A/I$ , and let  $B'$  be any pre-image of  $B$  in  $A$  with the natural isomorphism of vector spaces  $B' \cong B$ . Then  $I$  as a free right  $A$ -module is minimally generated by the vector space  $B'X$ .

**Proof.** Obviously,  $I = B'XA$ . We have to show that the natural epimorphism  $\gamma : B'X \otimes_k A \rightarrow B'XA$  is an isomorphism. Following [A], there is an isomorphism of graded vector spaces  $\alpha : B\langle X \rangle \rightarrow A$  such that  $\alpha(B) = B'$ ,  $\alpha(BXB\langle X \rangle) = I$  (this follows also from the property (ii) of Theorem 2.3). The right  $B\langle X \rangle$ -module in the last equality is free, so, we get the desired isomorphisms of graded vector spaces  $B'XA = I \cong BXB\langle X \rangle = B'X \otimes B\langle X \rangle \cong B'X \otimes_k A$ . Therefore, the map  $\gamma$  is an isomorphism of graded vector spaces. So, it is an isomorphism of modules.  $\square$

The next statement is similar to [Po, Proposition 3.3].

**Proposition 3.2.** *Let  $B = A/I$ , where the algebra  $B$  is right Noetherian and the ideal  $I$  is free as a left  $A$ -module. Then the algebra  $A$  is right graded coherent.*

**Proof.** Let  $J$  be a proper finitely generated homogeneous right-sided ideal in  $A$ . We have to show that  $J$  is finitely presented, that is,  $\dim_k \text{Tor}_2^A(A/J, k) < \infty$ .

Consider a standard spectral sequence  $E_{p,q}^2 = \text{Tor}_p^B(\text{Tor}_q^A(A/J, B), k) \Rightarrow \text{Tor}_*(A/J, k)$ . Let  $N_q = \text{Tor}_q^A(A/J, B)$ . Since the projective dimension of the left module  ${}_A B$  is at most one (because it admits a free resolution  $0 \rightarrow I \rightarrow A$ ), we have  $N_q = 0$  for  $q > 1$ , hence  $E_{p,q}^2 = 0$  for  $q > 1$ . The right  $B$ -module  $N_0 = A/J \otimes_A B$  is obviously finitely generated. Moreover, the short exact sequence

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$$

gives, after tensoring by  $B$ , an exact sequence

$$0 \rightarrow N_1 \rightarrow J \otimes_A B \rightarrow A \otimes_A B \rightarrow N_0 \rightarrow 0.$$

Since  $N_1$  is a submodule of a finitely generated  $B$ -module  $J \otimes_A B$ , it is finitely generated as well. Therefore, we have  $\dim_k E_{p,q}^2 = \dim_k \text{Tor}_p^B(N_q, k) < \infty$  for all  $p, q$ . Thus,  $\dim_k \text{Tor}_2^A(A/J, k) \leq \dim_k E_{2,0}^2 + \dim_k E_{1,1}^2 < \infty$ .  $\square$

**Corollary 3.3.** *Let  $B$  be an RNCI of a graded algebra  $A$ . If the algebra  $B$  is right Noetherian, then the algebra  $A$  is right graded coherent.*

### 4. One-relator quadratic algebras

**Theorem 4.1.** *Every algebra defined by a single homogeneous quadratic relation is graded coherent.*

**Proof.** Let  $b \in V \otimes V$  be a quadratic element in the free algebra  $T(V)$ , where  $\dim V = n$ , and let  $A$  be a quotient algebra of  $T(V)$  by an ideal  $\text{id}(b)$  generated by  $b$ . If  $n = 1$ , then the algebra  $A$  is finite-dimensional, hence Artinian, hence Noetherian, and coherent. If  $n = 2$ , then either  $A$  is Noetherian or  $b$  has the form  $b = xy$ , where  $x, y \in V$  [AS, p. 172]. In the last case,  $A$  is coherent by [Pi1, Theorem 2].

Consider the case  $n \geq 3$ . If  $\text{rank } b = 1$ , then  $b = xy$  is a monomial on generators, and the algebra  $A$  is coherent, again by [Pi1, Theorem 2]. So, we can assume that  $\text{rank } b \geq 2$ .

**Lemma 4.2.** *Let  $V$  be an  $n$ -dimensional vector space with  $n \geq 2$ , and let  $b \in V \otimes V$ . Given an  $(n - 2)$ -dimensional subspace  $W \subset V$ , let  $b'$  be the image of  $b$  in  $(V/W) \otimes (V/W)$ . Then either  $b = xy$  for some  $x, y \in V$  or there exists  $W$  such that  $\text{rank } b' = 2$ .*

**Proof.** By the induction on  $n$ , we can assume that for every  $x \in V$ , the image  $b''$  of  $b$  in  $(V/kx) \otimes (V/kx)$  has  $\text{rank} \leq 1$ . Let  $\{x_1 = x, \dots, x_n\}$  be a basis of  $V$ .

If, for some  $x$ , we have  $b'' = 0$ , then  $b = \alpha x^2 + x l_1 + l_2 x$  with  $l_i \in k\{x_2, \dots, x_n\}$ ,  $\alpha \in k$ . If  $\text{rank } b \geq 2$ , then  $l_1 \neq 0$  and  $l_2 \neq 0$ . Now, if  $l_2 \neq \beta l_1$  for  $\beta \in k$ , then the image  $b = \alpha x^2 + x l_1 + l_1 x$

has rank 2—a contradiction. Hence,  $l_2 = \beta l_1$  for some  $0 \neq \beta \in k$ , and  $b = \alpha x^2 + x l_1 + \beta l_1 x$  has rank two.

So, we can assume that  $\text{rank } b'' = 1$  for every  $x$ . Then  $b'' = uv$  for some nonzero  $u, v \in k\{x_2, \dots, x_n\}$ . Hence  $b = uv + \alpha x^2 + x l_1 + l_2 x$  with  $l_i \in k\{x_2, \dots, x_n\}$ ,  $\alpha \in k$ . We can assume that either (1)  $u = x_2$ ,  $v = x_3$  or (2)  $u = v = x_2$ .

Suppose that  $l_1 \neq 0$  and  $l_2 \neq 0$ . The image of  $b$  under the factorization by  $l_1$  has unit rank, hence  $l_2 = \beta u$  for some  $\beta \in k$ . Similarly,  $l_1 = \gamma v$  with  $\gamma \in k$ . In the case (1), let  $W = k\{x_2 - x_3, x_4, \dots, x_n\}$ ; in the case (2), let us put  $W = k\{x_3, \dots, x_n\}$ . In both cases, the image  $b'$  of  $b$  in  $(V/W) \otimes (V/W)$  has the same rank as  $b$ .

Now, it remains to consider the case  $l_2 = 0$  (the case  $l_1 = 0$  is analogous). Then  $b = uv + x(\alpha x + l_1)$ . If  $\alpha = 0$  and  $l_1 = 0$ , then  $\text{rank } b = 1$ , and there is nothing to prove. In the case (1), the image of  $b$  under the factorization by  $(x_2 - x_3)$  must have rank one, hence  $\alpha = 0$ ,  $l_1 = \lambda v$  for some  $\lambda \in k$ , and  $\text{rank } b = 1$ . In the case (2), because either  $l_1 = 0$  or the image of  $b$  under the factorization by  $l_1$  has unit rank, we have  $l_1 = \lambda x_2$  for some  $\lambda \in k$ . Then  $b$  depends on the variables  $x_1$  and  $x_2$  only, hence we may put  $W = k\{x_3, \dots, x_n\}$ .  $\square$

Recall that  $\text{rank } b \geq 2$ . Let  $x_1, \dots, x_n$  be a basis of  $V$  such that  $W = k\{x_i \mid i = 3, \dots, n\}$  be as in this lemma. Then the image  $b'$  of  $b$  in  $(V/W) \otimes (V/W) = k\{x_1, x_2\}^{\otimes 2}$  has rank 2. By Theorem 2.2, the algebra  $B = A/\text{id}(x_3, \dots, x_n) = k\langle x_1, x_2 \mid b' = 0 \rangle$  is Noetherian. Now, the set  $X = \{x_3, \dots, x_n\}$  is strongly free in the algebra  $A$ , because  $A/\text{id}(X) = B$ , while  $\text{Tor}_1^B(k, k) \oplus kX \cong \text{Tor}_1^A(k, k)$ ,  $\text{Tor}_2^B(k, k) \cong \text{Tor}_2^A(k, k) \cong kb$ , and  $\text{Tor}_i^B(k, k) = \text{Tor}_i^A(k, k) = 0$  for all  $i \geq 3$ . Thus, it follows from Corollary 3.3 that the algebra  $A$  is coherent.  $\square$

**Theorem 4.3.** *Let  $A$  be a regular algebra of global dimension 2. Then  $A$  is graded coherent.*

**Proof.** According to Zhang’s Theorem 2.2, the algebra  $A$  has the form  $A = k\langle x_1, \dots, x_n \rangle / (b)$ , where  $n \geq 2$ ,  $1 \leq \text{deg } x_1 \leq \dots \leq \text{deg } x_n$  with  $\text{deg } b = \text{deg } x_i + \text{deg } x_{n+1-i}$  for all  $i$ , and for some graded automorphism  $\sigma$  of the free algebra  $k\langle x_1, \dots, x_n \rangle$  we have  $b = \sum_{i=1}^n x_i \sigma(x_{n-i})$ . If  $\text{deg } x_1 = \dots = \text{deg } x_n$ , then  $b \in k\{x_1, \dots, x_n\}^{\otimes 2}$ . Hence  $A$  is coherent by Theorem 4.1.

So, we can assume that  $\text{deg } x_1 = \dots = \text{deg } x_p < \dots < \text{deg } x_{n-p+1} = \dots = \text{deg } x_n$ . Since the definition of regular rings is left–right symmetric, it follows that there is another graded automorphism  $\tau$  of the free algebra  $k\langle x_1, \dots, x_n \rangle$  such that  $b = \sum_{i=1}^n \tau(x_{n-i})x_i$ . Let  $\tilde{b} = \sum_{i=n-p+1}^n (x_i \sigma(x_{n-i}) + \tau(x_{n-i})x_i)$ . Obviously, the element  $b - \tilde{b}$  does not depend on the variables  $x_{n-p+1}, \dots, x_n$ , hence  $\text{rank}(b - \tilde{b}) \leq n - p$  (where rank is defined as the minimal number  $r$  of elements  $l_1, \dots, l_r \in k\{x_1, \dots, x_n\}$  such that  $b - \tilde{b} = l_1 a_1 + \dots + l_r a_r$  for some  $a_1, \dots, a_r \in k\{x_1, \dots, x_n\}$ ). Now, we are interested in  $\text{rank } \tilde{b}$ .

Consider the case where  $\text{rank } \tilde{b} \leq 1$ . Then  $\text{rank } b \leq \text{rank}(b - \tilde{b}) + \text{rank } \tilde{b} \leq n - p + 1$ . Since  $\text{rank } b = n$  by Theorem 2.2, we have  $p = 1$ . Then  $\tilde{b}b = x_n \sigma(x_1) + \tau(x_1)x_n = \alpha x_n x_1 + \beta x_1 x_n$  for some nonzero  $\alpha, \beta \in k$ . Thus,  $\text{rank } \tilde{b} = 2$ —a contradiction.

So,  $\text{rank } \tilde{b} \geq 2$ . Note that  $\tilde{b} \in V \otimes V$ , where  $V = k\{x_1, \dots, x_p, x_{n-p+1}, \dots, x_n\}$ . According to Lemma 4.2, there is a  $(2p - 2)$ -dimensional subset  $W$  in  $V$  (say,  $W = k\{x_2, \dots, x_p, x_{n-p+1}, \dots, x_{n-1}\}$ ) such that the rank of the image  $b'$  of  $\tilde{b}$  in  $(V/W) \otimes (V/W)$  is 2. It follows from Theorem 2.2 that the algebra  $B = k\langle x_1, x_n \mid b' \rangle = A/\text{id}(x_2, \dots, x_{n-1})$  is Noetherian and has global dimension 2. By the same arguments as in the proof of Theorem 4.1, the set  $X = \{x_2, \dots, x_{n-1}\}$  is strongly free in  $A$ . In the view of Corollary 3.3, we conclude that the algebra  $A$  is coherent.  $\square$



**5. Non-Noetherian  $\mathbb{P}^1$**

A module  $M$  over an algebra  $R$  is said to satisfy condition  $\chi$  if  $\dim_k \text{Ext}^i(k, M) < \infty$  for all  $i \geq 0$ , see [AZ]. A coherent algebra  $A$  said to satisfy  $\chi$  if every finitely presented  $A$ -module  $M$  satisfies  $\chi$ .

The following proposition is similar to [AZ, Theorem 8.1]. The proof is more or less similar too.

**Proposition 5.1.** *Let  $A$  be a graded coherent regular algebra of global dimension  $d \geq 0$ . Then:*

- (1)  *$A$  satisfies the condition  $\chi$ ;*
- (2) *the algebra  $A$  may be recovered from its noncommutative scheme  $\text{proj } A := (\text{qgr } A, \mathcal{A}, s)$  as*

$$A \cong \Gamma_{\geq 0}(\text{proj } A);$$

- (3) *the category  $\text{qgr } A$  is Ext-finite and has cohomological dimension  $d - 1$ .*

Notice that the condition (2) here means that  $s$  is ample [AZ], that is, that the shifts of  $\mathcal{A}$  form an ample sequence in  $\text{qgr } A$  [Po].

**Proof.** (1) Using the induction on the projective dimension  $p$  of a coherent module  $M$ , we will show that  $\dim_k \text{Ext}^i(k, M) < \infty$  for all  $i \geq 0$ . If  $p = 0$ , then  $M$  is a finitely generated free  $A$ -module, so, all  $\text{Ext}^i(k, M)$  are bounded by the Gorenstein condition. If  $p > 0$ , then there is a short exact sequence (presentation)

$$0 \rightarrow N \rightarrow P \rightarrow M \rightarrow 0 \tag{5.1}$$

where  $P$  is a projective module and  $\text{gl.dim } N < p$ . By the induction assumption, the condition  $\chi$  holds for  $P$  and  $N$ ; by the exact triangle of Exts, it holds for  $M$  as well.

(2) Let  $\mathcal{M}$  be an image of some  $M \in \text{cmod } A$  in  $\text{qgr } A$ . For every  $n > 0$ , the short exact sequence  $0 \rightarrow A_{\geq n} \rightarrow A \rightarrow A/A_{\geq n} \rightarrow 0$  gives an exact sequence

$$0 \rightarrow \text{Hom}_A(A/A_{\geq n}, A) \rightarrow A \rightarrow \text{Hom}_A(A_{\geq n}, A) \rightarrow \text{Ext}_A^1(A/A_{\geq n}, A) \rightarrow 0.$$

Since  $A$  is regular, the left and the right terms are zeros. Hence  $A \cong \text{Hom}_A(A_{\geq n}, A) \cong \lim_{n \rightarrow \infty} \text{Hom}_A(A_{\geq n}, A) = \underline{H}^0(\mathcal{A})$ .

(3) Notice that  $\underline{H}^i(\mathcal{A}) = \lim_{n \rightarrow \infty} \text{Ext}_A^{i+1}(A_{\geq n}, A)$  for  $i \geq 1$ , hence  $\underline{H}^i(\mathcal{A}) = 0$  for  $i \neq 0, d - 1$  and  $\underline{H}^{d-1}(\mathcal{A}) = A^*[l]$  for some  $l \in \mathbb{Z}$ . It follows that  $\text{cd}(\text{qgr } A) \geq d - 1$  and that the cohomologies  $H^i(\mathcal{A})$  are finite-dimensional.

Moreover,  $\underline{H}^i(\mathcal{M}) = \lim_{n \rightarrow \infty} \text{Ext}_A^i(A_{\geq n}, M) = 0$  for all  $i \geq d$ . If  $\text{pd } M = 0$ , then  $M = \bigoplus_i A[l_i]$  is a finitely generated free module, hence  $\text{Ext}^i(\mathcal{A}[l], \mathcal{M})$  is finite-dimensional for every  $i \geq 0, l \in \mathbb{Z}$ . By the induction on  $\text{pd } M$ , it follows from the  $\text{Ext}(\mathcal{A}[l], -)$  triangle for the exact sequence (5.1) that the vector spaces  $\text{Ext}^i(\mathcal{A}[l], \mathcal{M})$  are finite-dimensional for all  $i, l, M$ .

Let  $M'$  be an image in  $\text{qgr } A$  of another coherent  $A$ -module  $M'$ . If  $M' = \bigoplus_{i=0}^t A[l_i]$  is a free module, we can apply the functor  $\text{Ext}^i(-, \mathcal{M})$  to the short exact sequence  $0 \rightarrow \bigoplus_{i=0}^{t-1} A[l_i] \rightarrow$

$\mathcal{M}' \rightarrow \mathcal{A}[I] \rightarrow 0$ . The derived exact triangle shows that the vector space  $\text{Ext}^i(\mathcal{M}', \mathcal{M})$  is finite-dimensional for every  $i$  and vanishes for  $i \geq d$ . For non-free modules  $M'$ , we proceed by induction on  $\text{pd } M'$ . Applying the same functor to the short exact sequence  $0 \rightarrow N \rightarrow F \rightarrow M' \rightarrow 0$  analogous to (5.1), we deduce that the vector spaces  $\text{Ext}^i(\mathcal{M}', \mathcal{M})$  are finite-dimensional for all  $i$  and vanish for  $i \geq d$  as well. It follows that the cohomological dimension of  $\text{qgr } A$  is  $d - 1$  and that the category  $\text{qgr } A$  is Ext-finite.  $\square$

**Proof of Proposition 1.5.** According to [Z1, Theorem 1.4] (see also Section 2.2 above), the coherent scheme of  $A$  is independent (up to isomorphism) on the choice of the automorphism  $\sigma$  in Theorem 2.2. On the other hand, if the regular algebras  $A$  and  $A'$  of global dimension two have different numbers of generators (say,  $m$  and  $n$ ), then they are not twists of each other because  $\tau_1 : A_1 \rightarrow A'_1$  cannot be an isomorphism of vector space. This proves (a).

Let us also give a direct proof of the last statement. Let be  $\mathcal{A}$  and  $\mathcal{A}'$  the images of  $A$  and  $A'$  in respective  $\text{qgr}$ , and let  $s$  and  $s'$  be the shifts if grading in these  $\text{qgr}$ . Assume that the schemes  $\mathbb{P}_n^1$  and  $\mathbb{P}_m^1$  with the underlying algebras  $A$  and  $A'$  are isomorphic. By definition [AZ], this means that there is an equivalence of categories  $F : \text{qgr } A \rightarrow \text{qgr } A'$  such that  $F(\mathcal{A}) \cong \mathcal{A}'$  and  $s'F \cong Fs$ . Then  $F$  maps the exact sequence

$$0 \rightarrow s^2\mathcal{A} \rightarrow s\mathcal{A}^n \rightarrow \mathcal{A} \rightarrow 0$$

to the exact sequence

$$0 \rightarrow s'^2\mathcal{A}' \rightarrow s'\mathcal{A}'^n \rightarrow \mathcal{A}' \rightarrow 0.$$

Taking the Euler characteristics for the second exact sequence, we deduce that the following equality of formal power series holds for some polynomial  $p(z) \in \mathbb{Z}[z]$  (because a pre-image of this sequence in  $\text{cmod } A'$  must be exact up to finite-dimensional modules):

$$A'(z)(1 - nz + z^2) = p(z),$$

where  $A'(z) := \sum_{i \geq 0} (\dim A'_i)z^i = (1 - mz + z^2)^{-1}$ . It follows that  $m = n$ .

(b) The Serre duality for  $\text{qgr } A$  follows from [MV]. The heredity (that is, that  $\text{qgr } A$  has cohomological dimension  $\leq 1$ ) and Ext-finiteness follows from Proposition 5.1.

If  $n \geq 3$ , then  $A$  is not Noetherian by Theorem 2.2. Let us show that the image  $\mathcal{A}$  of  $A$  in  $\text{qgr } A$  is not Noetherian as well. In the view of (a), we may assume that  $b = x_1x_2 + x_2x_3 + \dots + x_nx_1$ . Then  $b$  forms a Groebner basis of the ideal  $\text{id}(b) \subset k\langle x_1, \dots, x_n \rangle$  with respect to an arbitrary deg-lex order, therefore, there is a linear basis of  $A$  consisting of the monomials on the variables  $x_1, \dots, x_n$  which do not contain a subword  $x_1x_2$ . Now, it is easy to see that the monomials  $x_1x_3, x_1^2x_3, \dots, x_1^s x_3$  form a right Groebner basis of the right-sided ideal  $I_t \subset A$  generated by them. It follows that every quotient module  $I_t/I_{t-1}$  is infinite-dimensional (because it contains a sequence of linearly independent monomials  $x_1^s x_3, s \geq 1$ ), hence the image in  $\mathcal{A}$  of the chain  $I_1 \subset I_2 \subset \dots$  is strictly ascending. This proves (b).

The statement (a) allows us to choose any particular  $b$  of rank  $n$ ; let us choose  $b = x_1^2 + \dots + x_n^2$ . Then we have  $B = A^1$ —a Koszul dual algebra. Now, the claim (c) follows from the Koszul duality and a noncommutative analogue of the Bernstein–Gelfand–Gelfand correspondence, see [MVS, Proposition 4.1 and Corollary 4.5].

The claim (d) has been communicated me by Michel Van den Bergh [VdB]. In the view of (a), it is sufficient to show the derived equivalence for the same value  $x_1^2 + \cdots + x_n^2$  of  $b$ . In this case, this equivalence has been also shown in [M, Theorem 0.1].  $\square$

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