Linkage of modules over Cohen–Macaulay rings

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\section*{A B S T R A C T}

Inspired by the theory of linkage for ideals, the concept of sliding depth of a finitely generated module over a Noetherian local ring is defined in terms of its Ext modules. As a result, in the module-theoretic linkage theory of Martsinkovsky and Strooker, one proves the Cohen–Macaulayness of a linked module if the base ring is Cohen–Macaulay (not necessarily Gorenstein). Some interplay is established between the sliding depth condition and other module-theoretic notions such as the G-dimension and the property of being sequentially Cohen–Macaulay.

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\section*{1. Introduction}

The new progress in the linkage theory is the recent work of Martsinkovsky and Strooker, \cite{12}, which established the concept of linkage of modules. This paper began to attract some interest. The importance of this work is not only recovering some of the known theorems in the theory of linked ideals but also presenting a new conceptual perspective in classification.

Recall that two ideals \( I \) and \( J \) in a Cohen–Macaulay local ring \( R \) are said to be linked if there is a regular sequence \( \alpha \) in their intersection such that \( I = (\alpha) : J \) and \( J = (\alpha) : I \). The first main theorem in the theory of linkage is the following \cite{14}.

**Theorem A.** If \((R, m)\) is a Gorenstein local ring, \( I \) and \( J \) are two linked ideals of \( R \) then \( R/I \) is Cohen–Macaulay if and only if \( R/J \) is so.

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According to [14, Contre-exemple 1.8.] Theorem A is no longer true if the base ring \( R \) is only Cohen–Macaulay (from now on CM). The concepts of Strongly Cohen–Macaulay (SCM), [9], and Sliding Depth condition, [10], fill the gap in CM rings as: in a Cohen–Macaulay local ring any ideal linked to an ideal that satisfies the sliding depth condition (or SCM) is Cohen–Macaulay. Recall that the ideal \( I \) satisfies sliding depth condition if \( \text{depth} H_i \geq \dim(R) - r + i \), for \( i > 0 \), where \( H_i \) is the \( i \)th homology of the Koszul complex with respect to some generating set of \( I \), and \( r \) is the number of elements of this generating set. The extreme case when all of the non-zero Koszul homologies are CM dubbed SCM.

In this paper, inspired by the works in the ideal case, we propose the strongly Cohen–Macaulay and sliding depth condition for modules; so that we can state Theorem A for linked modules in CM local rings. In Section 2, we define the new conditions for modules so-called SDE (Sliding Depth on Ext’s) or CME (Cohen–Macaulay Ext’s). Some sufficient conditions for being SDE or CME are demonstrated, for example in Proposition 2.5 it is shown that Cohen–Macaulay \( R \)-modules with finite G-dimension are CME. As well it is proven that over a Cohen–Macaulay local ring \( R \), if \( M \) is SDE, then \( \lambda M \) is maximal Cohen–Macaulay (see Corollary 2.7).

Trying to detect some module-theoretic invariants with less ideal inscription in the theory of linkage of ideals, we encounter the combinatorial conception sequentially Cohen–Macaulay. We first present a computational criterion for this concept involving the ideas from linkage of module in Corollary 2.11. Finally in this section we pose an extension to a theorem of Foxby [7] for the class of CME modules and answer this question; so that in the Cohen–Macaulay local ring with canonical module \( \omega_R \), it is shown that \( M \) is CME if and only if \( M \otimes_R \omega_R \) is sequentially Cohen–Macaulay and \( \text{Tor}_i^R(M, \omega_R) = 0 \) for \( i > 0 \) (Theorem 2.13).

In Section 3, for a finite \( R \)-module \( M \) over a Cohen–Macaulay local ring \( R \) of dimension \( d \geq 2 \) with canonical module \( \omega_R \), we establish a duality between local cohomology modules of \( M \otimes_R \omega_R \) and those of \( \lambda M \) (Theorem 3.3) provided \( M \otimes_R \omega_R \) is generalized Cohen–Macaulay. This theorem is a generalization to [12, Theorem 10] and also [16] moreover for its proof instead of techniques in derived category we appeal to spectral sequences. Also whenever \( M \) is generalized Cohen–Macaulay, under some vanishing assumption on Tor-modules of \( M \) and \( \omega_R \) we show that \( H^i_m(\lambda M) \cong \text{Ext}^i_R(M, R) \) for \( i = 1, \ldots, d - 1 \) (Corollary 3.4).

### 2. SDE and CME modules

Linkage of modules is a combination of two ring-theoretic operations, passing through quotient rings and returning, and a module-theoretic construct which is called horizontal linkage. The latter offers possibilities for generalizations, whereas the formers provide obstructions. Horizontal linkage itself consists of two module-theoretic operations; Transpose and Syzygy as follow.

Throughout the paper, \( R \) is a Noetherian ring and \( M \) is a finitely generated \( R \)-module. We assume that \( M \) is a stable \( R \)-module that is \( M \) has no projective summand. Let \( P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0 \) be a finite projective presentation of \( M \). The transpose of \( M \), \( \text{Tr} M \), is defined to be \( \text{Coker} f^* \) where \((-)^* := \text{Hom}_R(-, R)\), it suits the exact sequence

\[
0 \rightarrow M^* \rightarrow P_0^* \rightarrow P_1^* \rightarrow \text{Tr} M \rightarrow 0;
\]

and is unique up to projective equivalence. Thus the minimal projective presentations of \( M \) represent isomorphic transposes of \( M \). By [2, Theorem 32.13] \( \text{Tr} M \) is a stable \( R \)-module, whenever \( M \) is so. The syzygy module of \( M \), denoted by \( \Omega M \), is the kernel of an epimorphism \( P \xrightarrow{g} M \), where \( P \) is a projective \( R \)-module. The kernel of \( g \) is unique up to projective equivalence, thus \( \Omega M \) is determined uniquely up to isomorphism if \( P \rightarrow M \) is a projective cover.

The operator \( \lambda = \Omega \text{Tr} \), introduced by Martinkovský and Strooker, enabled them to define horizontal linkage for modules. In [12, Lemma 3], it is shown that ideals \( a \) and \( b \) are linked if and only if \( R/a \) and \( R/b \) are related to each other through the operator \( \lambda \); more precisely, \( R/a \cong \lambda R/b \) and \( R/b \cong \lambda R/a \).
Definition 2.1. (See [12, Definition 3].) Two finitely generated $R$-modules $M$ and $N$ are said to be horizontally linked if $M \cong \lambda N$ and $N \cong \lambda M$. Thus, $M$ is horizontally linked (to $\lambda M$) if $M \cong \lambda^2 M$.

Having defined the horizontal linkage, the general linkage for modules is defined as follows.

Definition 2.2. (See [12, Definition 4].) Let $M$ and $N$ be two finitely generated $R$-modules. The module $M$ is said to be linked to $N$ by an ideal $\mathfrak{c}$ of $R$, if $\mathfrak{c} \subseteq \text{Ann}_R(M) \cap \text{Ann}_R(N)$ and $M$ and $N$ are horizontally linked as $R/\mathfrak{c}$-modules.

Usually in most theorems in the category of $R$-modules two objects read equal if they are isomorphic. On the other hand in the ideal-theoretic linkage, ideals of a fixed ring are not counted equal up to isomorphism, while "two ideals in a commutative ring are equal if and only if the corresponding cyclic modules are isomorphic". Therefore it seems natural to pass from ideals to modules by considering the cyclic modules in place of ideals. It is also worthwhile to mention that the advantage of embedding an ideal into a free module is reserved, since $R/(0 : b)$ embeds into a free $R$-module for any ideal $b$ of $R$.

The following definition is the proposed module-theoretic version of Strongly Cohen–Macaulay and Sliding Depth condition.

Definition 2.3. Let $R$ be a local ring of dimension $d$ and let $M$ be a finitely generated $R$-module. The module $M$ is called to be SDE, having Sliding Depth of Extension modules, if either $\text{Ext}_R^i(M, R) = 0$ or $\text{depth}_R(\text{Ext}_R^i(M, R)) \geq d - i$ for all $i = 1, \ldots, d - 1$. Also $M$ is called to be CME, having Cohen–Macaulay Extension modules, if either $\text{Ext}_R^i(M, R) = 0$ or $\text{Ext}_R^i(M, R)$ is Cohen–Macaulay of dimension $d - i$ for all $i = 1, \ldots, d - 1$.

In the ideal-theoretic case, if $I$ is linked to $J$ by $\alpha$ then $J/(\alpha)$ is isomorphic to the last non-zero homology module of the Koszul complex of $I$; hence forcing the Koszul homologies to have enough depth or to be CM seems to be a natural condition on $I$ whereby implies CM-ness of linked ideals. In the module-theoretic case, $\lambda M$ is obtained by applying $(-)^\ast$ to a minimal projective resolution of $M$. Accordingly we propose the above sliding conditions on the depth of extension modules $\text{Ext}_R^i(M, R)$.

Clearly any CME module is SDE. To see the ambiguity of CME modules, it is shown in the next proposition that any CM module with finite Gorenstein dimension ($G$-dimension) is CME. Let us recall the definition of $G$-dimension for the sake of completeness.

Definition 2.4. A finite $R$-module $M$ belongs to the $G$-class $G(R)$ if and only if

1. $\text{Ext}_R^m(M, R) = 0$ for $m > 0$;
2. $\text{Ext}_R^m(M^\ast, R) = 0$ for $m > 0$; and
3. the biduality map $M \rightarrow M^\ast\ast$ is an isomorphism.

A $G$-resolution of a finite $R$-module $M$ is a right acyclic complex of modules in $G(R)$ whose zeroth homology module is $M$. The module $M$ is said to have finite $G$-dimension, denoted by $G\text{-dim}_R(M)$, if it has a $G$-resolution of finite length.

The class $G(R)$ is quite large, it contains all finitely generated projective modules in general. For a Gorenstein ring $R$, $G(R)$ is the class of all maximal Cohen–Macaulay modules. For more information on Gorenstein dimension we refer to the lecture note [6].

Proposition 2.5. Let $R$ be a Cohen–Macaulay local ring of dimension $d$. Then any Cohen–Macaulay $R$-module with finite $G$-dimension is CME.

Proof. Let $M$ be a Cohen–Macaulay $R$-module with finite $G$-dimension. The Auslander–Bridger formula $G\text{-dim}_R(M) + \text{depth}_R(M) = \text{depth} R$ [6, Theorem 1.4.8] in conjunction with the Cohen–
Macaulayness of \(M\), imply that \(\text{grade}_R(M) = \text{G-dim}_R(M) =: g\). So that \(\text{Ext}^i_R(M, R) = 0\) for all \(i \neq g\). Choose \(x := x_1, \ldots, x_g\) to be a maximal \(R\)-sequence contained in \(\text{Ann}_R(M)\). We have \(\text{Ext}^i_R(M, R) \cong \text{Hom}_R/(\lambda)(M, R/(\lambda))\) and \(\text{Ext}^i_R(M, R/(\lambda)) = 0\) for all \(i > 0\). Since \(M\) is a maximal Cohen–Macaulay \(R/(\lambda)\)-module, by [5, Proposition 3.3.3], \(\text{Hom}_R/(\lambda)(M, R/(\lambda))\) is a maximal Cohen–Macaulay \(R/(\lambda)\)-module. Therefore \(\text{Ext}^g_R(M, R)\) is Cohen–Macaulay of dimension \(d - g\). \(\Box\)

Determining the depth of linked ideals is in the center of the questions on the arithmetic properties of ideals. About linkage of modules the depth of modules which are to SDE modules is rather under control.

**Proposition 2.6.** Let \((R, m)\) be a local ring and let \(M\) be an SDE \(R\)-module. Then \(\text{depth}_R(\lambda M) \geq \min(\text{depth}_R(M), \text{depth}_R R)\).

**Proof.** Set \(t = \min(\text{depth}_R(M), \text{depth}_R(R))\). For \(t = 0\) it is trivial. Suppose that \(t > 0\). Set \(X := \bigcup_{i=1}^{t-1} \text{Ass}_R(\text{Ext}^i_R(M, R)) \cup \text{Ass}_R(R) \cup \text{Ass}_R(R)\). As \(M\) is SDE and \(t > 0\), there is \(x \in m \setminus \bigcup_{p \in X} p\). Set \(M = M/xM\) and \(R = R/xR\). The exact sequence \(0 \rightarrow R \xrightarrow{x} R \rightarrow R \rightarrow 0\) implies the exact sequence

\[
0 \rightarrow M^* \xrightarrow{x} M^* \rightarrow \text{Hom}_R(M, \overline{R}) \rightarrow \text{Ext}^1_R(M, R) \rightarrow \text{Ext}^1_R(M, R) \rightarrow \cdots.
\]

As each map \(\text{Ext}^i_R(M, R) \xrightarrow{x} \text{Ext}^i_R(M, R)\) is an injection for all \(i = 0, \ldots, d - 1\), we have standard isomorphisms \(\text{Ext}^i_R(M, \overline{R}) \cong \text{Ext}^i_R(M, \overline{R}) \cong \text{Ext}^i_R(M, R/x \text{Ext}^i_R(M, R))\) for all \(i = 0, 1, \ldots, d - 2\). Therefore \(\overline{M}\) is SDE as \(\overline{R}\)-module. Let \(P_1 \rightarrow P_0 \rightarrow M \rightarrow 0\) be a minimal projective presentation of \(M\) and consider the exact sequence \(0 \rightarrow M^* \rightarrow P_0^* \rightarrow \lambda M \rightarrow 0\). As \(\lambda M\) is a syzygy module, \(x\) is also a non-zero-divisor on \(\lambda M\). Thus there is a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & M^*/xM^* & \rightarrow & P_0^*/xP_0^* & \rightarrow & \lambda M/x\lambda M & \rightarrow & 0 \\
\text{iso} & & \text{iso} & & \text{iso} & & \text{iso} & & \text{iso} \\
0 & \rightarrow & \text{Hom}_R(\overline{M}, \overline{R}) & \rightarrow & \text{Hom}_R(\overline{P_0}, \overline{R}) & \rightarrow & \lambda \overline{R} \overline{M} & \rightarrow & 0
\end{array}
\]

which implies that \(\lambda M/x\lambda M \cong \lambda \overline{R} \overline{M}\).

By induction \(\text{depth}_R(\lambda \overline{R} \overline{M}) \geq \min(\text{depth}_R(\overline{M}), \text{depth}_R(\overline{R})) = t - 1\). Thus \(\text{depth}_R(\lambda M) \geq t\). \(\Box\)

It is shown in [12, Proposition 8 in Section 4] that, over a Gorenstein local ring \(R\), a stable \(R\)-module \(M\) of maximum dimension is maximal Cohen–Macaulay if and only if \(\lambda M\) is maximal Cohen–Macaulay and \(M\) is unmixed. If the ring \(R\) is merely a Cohen–Macaulay local ring this statements is not true [12, Section 6]. Trying to generalize this proposition for Cohen–Macaulay rings, as a corollary of the above general proposition, we have the following generalization of Theorem A which is in fact the module-theoretic version of [9, Proposition 1.1].

**Corollary 2.7.** Let \(R\) be a Cohen–Macaulay local ring of dimension \(d\), and let \(M\) be a maximal Cohen–Macaulay and an SDE \(R\)-module. Then \(\lambda M\) is maximal Cohen–Macaulay.

The composed functors \(T_i := \text{Tr} \Omega^{-i-1}\) for \(i > 0\) have been already introduced by Auslander and Bridger, [1], and recently used by Nishida, [13], to relate linkage and duality. In the following result, over a Cohen–Macaulay local ring, we characterize an SDE module \(M\) in terms of depths of the \(R\)-modules \(T_i M\) and \(\lambda \Omega^i M\). Moreover, it follows that for \(\lambda M\) to be maximal Cohen–Macaulay we only need \(M\) to be SDE.
Theorem 2.8. Let \( R \) be a Cohen–Macaulay local ring of dimension \( d \geq 2 \), \( M \) a finitely generated \( R \)-module. The following statements are equivalent.

(i) \( M \) is SDE.
(ii) \( \text{depth}_{R}(T_{i}M) \geq d - i \) for all \( i = 1, \ldots, d - 1 \).
(iii) \( \text{depth}_{R}(\lambda \Omega^{j}M) \geq d - i \) for all \( i = 0, \ldots, d - 2 \).

Proof. For a stable finite \( R \)-module \( N \), there is an exact sequence [12, Section 5],

\[
0 \longrightarrow \text{Ext}_{R}^{1}(\text{Tr} M, R) \longrightarrow N \longrightarrow \lambda^{2}N \longrightarrow 0.
\]

Note that since the transpose of every finite \( R \)-module is either stable or zero, \( \text{Tr} T_{i}M \) is stably isomorphic to \( \Omega^{i-1}M \) for \( i > 0 \), and \( \text{Ext}_{R}^{1}(\Omega^{i-1}M, R) \cong \text{Ext}_{R}^{1}(M, R) \), so we have the exact sequence

\[
0 \longrightarrow \text{Ext}_{R}^{1}(M, R) \longrightarrow \Omega T_{i}M \longrightarrow \lambda^{2}T_{i}M \longrightarrow 0. \tag{2.1}
\]

Also, since \( \lambda^{2}T_{i}M \) is stably isomorphic to \( \Omega T_{i+1}M \) and \( R \) is Cohen–Macaulay, we have

\[
\text{depth}_{R}(\lambda^{2}T_{i}M) = \text{depth}_{R}(\Omega T_{i+1}M). \tag{2.2}
\]

(i) \( \implies \) (ii). We proceed by induction on \( i \). From the exact sequence (2.1) we have \( \text{depth}_{R}(T_{d-1}M) \geq 1 \). Now suppose that \( i \leq d - 2 \) and \( \text{depth}_{R}(T_{i+1}M) \geq d - i - 1 \), accordingly, \( \text{depth}_{R}(\Omega T_{i}M) \geq d - i \) which in turn implies \( \text{depth}_{R}(T_{i}M) \geq d - i \), using (2.2) and (2.1).

(ii) \( \implies \) (i). By (2.2) and the assumption, \( \text{depth}_{R}(\lambda^{2}T_{i}M) = \text{depth}_{R}(\Omega T_{i+1}M) \geq d - i \) for all \( i = 1, \ldots, d - 1 \). Using (2.1), we get either \( \text{Ext}_{R}^{1}(M, R) = 0 \) or \( \text{depth}_{R}(\text{Ext}_{R}^{1}(M, R)) \geq d - i \) for all \( i = 1, \ldots, d - 1 \).

(ii) \( \iff \) (iii). Note that \( \Omega T_{i+1}M = \lambda \Omega^{i}M \) for each \( i \). Thus \( \text{depth}_{R}(\Omega T_{i}M) \geq d - i \) for all \( i = 1, \ldots, d - 1 \) if and only if \( \text{depth}_{R}(\lambda \Omega^{i}M) = \text{depth}_{R}(\Omega T_{i+1}M) \geq d - i \) for all \( i = 0, \ldots, d - 2 \). \( \square \)

A shellable simplicial complex is a special kind of Cohen–Macaulay complex with a simple combinatorial definition. Shellability is a simple but powerful tool for proving the Cohen–Macaulay property. A simplicial complex \( \Delta \) is pure if each facet (= maximal face) has the same dimension (cf. [15, Section III]).

The concept of sequentially Cohen–Macaulay was defined by combinatorial commutative algebraists [15, 3.9] to answer a basic question to find a “non-pure” generalization of the concept of a Cohen–Macaulay module, so that the face ring of a shellable (non-pure) simplicial complex has this property.

This concept was then applied by commutative algebraists to study some algebraic invariants or special algebras come from graphs (cf. [3]). In what follows we see the relation between sequentially Cohen–Macaulay, SDE and CME as well as a way to construct a family of modules with these properties.

Definition 2.9. Let \( (R, m) \) be a local Noetherian ring and let \( M \) be a finitely generated \( R \)-module. A finite filtration \( 0 = M_{0} \subset M_{1} \subset M_{2} \subset \cdots \subset M_{r} = M \) of submodules of \( M \) is called a Cohen–Macaulay filtration, if each quotient \( M_{i}/M_{i-1} \) is Cohen–Macaulay, and \( \dim_{R}(M_{1}/M_{0}) < \dim_{R}(M_{2}/M_{1}) < \cdots < \dim_{R}(M_{r}/M_{r-1}) \). The module \( M \) is called sequentially Cohen–Macaulay if \( M \) admits a Cohen–Macaulay filtration.

A basic fact about sequentially Cohen–Macaulay modules is the following theorem of Herzog and Popescu [8, Theorem 2.4].

Theorem 2.10. Let \( R \) be Cohen–Macaulay local of dimension \( d \) with canonical module \( \omega_{R} \). The following conditions are equivalent.

(i) \( M \) is sequentially Cohen–Macaulay.
(ii) \( \text{Ext}_R^{d-i}(M, \omega_R) \) are either 0 or Cohen–Macaulay of dimension \( i \) for all \( i \geq 0 \).

Thus one observes that over a Gorenstein local ring the conditions SDE, CME and sequentially Cohen–Macaulay are equivalent. Hence Theorem 2.8 provides the following computable characterization of sequentially Cohen–Macaulay modules.

**Corollary 2.11.** Let \( R \) be a Gorenstein local ring of dimension \( d \geq 2 \) and \( M \) be a finitely generated \( R \)-module. The following conditions are equivalent.

(i) \( M \) is sequentially Cohen–Macaulay.
(ii) \( \text{depth}_R(\lambda \Omega^i M) \geq d - i \) for all \( i \), \( 0 \leq i \leq d - 2 \).

The following result shows that, over a Gorenstein local ring, each of the properties sequentially Cohen–Macaulay (or equivalently, SDE or CME) is preserved under evenly linkage.

**Proposition 2.12.** Let \( R \) be a Gorenstein local ring. Then the condition sequentially Cohen–Macaulay (or equivalently, SDE or CME) is preserved under evenly linkage by ideals.

**Proof.** Set \( d := \text{dim } R \). Let \( c_1 \) and \( c_2 \) be Gorenstein ideals. Assume that \( M_1, M, \) and \( M_2 \) are \( R \)-modules such that \( M_1 \) is linked to \( M \) by \( c_1 \) and \( M \) is linked to \( M_2 \) by \( c_2 \). For each \( i > 0 \), by [12, Lemma 11 and Proposition 16], we have

\[
\text{Ext}_R^{i+g}(M_1, R) \cong \text{Ext}_{R/c_1}^i(M_1, R/c_1) \cong \text{Ext}_{R/c_2}^i(M_2, R/c_2) \cong \text{Ext}_R^{i+g}(M_2, R)
\]

where \( g = \text{ht } c_1 = \text{ht } c_2 = \text{grade}_R M_1 = \text{grade}_R M_2 \).

Suppose that \( M_1 \) is sequentially Cohen–Macaulay. By Theorem 2.9, \( \text{Ext}_R^{d-i}(M_1, R) \) is either zero or Cohen–Macaulay of dimension \( i \) for each \( i \). Hence \( \text{Hom}_{R/c_1}(M_1, R/c_1)(\cong \text{Ext}_R^i(M_1, R)) \) is a Cohen–Macaulay \( R \)-module of dimension \( d - g \), and so it is a maximal Cohen–Macaulay \( R/c_1 \)-module. Note that, for \( j = 1, 2 \) there are exact sequences

\[
0 \rightarrow \text{Hom}_{R/c_j}(M_j, R/c_j) \rightarrow P_j \rightarrow \lambda_{R/c_j} M_j \rightarrow 0
\]

where \( P_j \) is a projective \( R/c_j \)-module. Thus we have \( \text{depth}_{R/c_j}(\lambda_{R/c_j} M_2) = \text{depth}_R(M) = \text{depth}_R(\lambda_{R/c_j} M_1) \geq d - g - 1 \). Again \( \text{Hom}_{R/c_2}(M_2, R/c_2) \) is a maximal Cohen–Macaulay \( R/c_2 \)-module, and so \( \text{Ext}_R^g(M_2, R) \) is a Cohen–Macaulay \( R \)-module of dimension \( d - g \). Hence \( M_2 \) is sequentially Cohen–Macaulay by Theorem 2.10. \( \square \)

As mentioned just after Theorem 2.10, over Gorenstein local rings, CME modules are exactly sequentially Cohen–Macaulay modules. On the other hand, when \( (R, \mathfrak{m}) \) is a Cohen–Macaulay ring with canonical module \( \omega_R \), it follows from the result [7, Theorem 2.5] of Foxby that \( \text{Tor}_i^R(M, \omega_R) = 0 \) for all \( i > 0 \), whenever \( G\text{-dim}_R M < \infty \). Moreover, Khatami and Yassemi in [11, Theorem 1.11] prove that under the same situation \( M \otimes_R \omega_R \) is Cohen–Macaulay if and only if \( M \) is Cohen–Macaulay. Note that by Proposition 2.5, if \( G\text{-dim}_R M < \infty \) and \( M \) is Cohen–Macaulay then \( M \) is CME, i.e. the class of CME modules contains the class of Cohen–Macaulay modules of finite G-dimensions. Hence the following question is naturally posed.

**What does happen if in results of Foxby, Yassemi and Khatami one replaces finite G-dimension and Cohen–Macaulay conditions of \( M \) with the condition that \( M \) is CME?**

The following theorem provides an answer to this question.
Theorem 2.13. Let $R$ be a Cohen–Macaulay local ring with the canonical module $\omega_R$, and let $M$ be a finitely generated $R$-module. Then the following two statements are equivalent.

(i) $M$ is CME.

(ii) $M \otimes_R \omega_R$ is sequentially Cohen–Macaulay and $\text{Tor}^R_i(M, \omega_R) = 0$ for all $i > 0$.

Proof. (i) $\Rightarrow$ (ii). Let $P_0 : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$ be a projective resolution of $M$, and let $I^* : 0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ be an injective resolution of $\omega_R$ and construct the third quadrant double complex $F := \text{Hom}_R(\text{Hom}_R(P_*, R), I^*)$. Let $^\vee E$ (resp. $^h E$) denote the vertical (resp. horizontal) spectral sequence associated to the double complex $F$. Then $^\vee E^i,j \cong \text{Ext}^i_R(\text{Ext}^j_R(M, R), \omega_R)$. Since $\text{Ext}^i_R(M, R)$ is either zero or is Cohen–Macaulay of dimension $d - i$, we have

\[
^\vee E^i,j \cong \begin{cases} 
\text{Ext}^i_R(\text{Ext}^j_R(M, R), \omega_R) & i = j, \\
0 & \text{otherwise}.
\end{cases}
\]

By using the equivalence of functors $\text{Hom}_R(\text{Hom}_R(X, Y), Y)$ and $X \otimes_R Y$, when $X$ (resp. $Y$) belongs to the subcategory of projective (resp. injective) $R$-modules, we find that the double complex $\text{Hom}_R(\text{Hom}_R(P_*, R), I^*)$ is isomorphic to the third quadrant double complex $P_0 \otimes_R I^*$. Now we may use this double complex to find that

\[
h^i,j \cong \begin{cases} 
\text{Tor}^R_i(M, \omega_R) & j = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

It follows that $h^E_\infty = h^E_2$ and $^\vee E_\infty = ^\vee E_2$. By comparing the two spectral sequences $^h E$ and $^\vee E$ we get $\text{Tor}^R_i(M, \omega_R) = 0$ for all $i > 0$. Thus there is a filtration $0 = \phi_{d+1} \subset \phi_d \subset \cdots \subset \phi_0 = M \otimes_R \omega_R$ of $M \otimes_R \omega_R$ such that $\text{Ext}^i_R(\text{Ext}^i_R(M, \omega_R)) \cong \phi_i/\phi_{i+1}$ for $i = 0, \ldots, d$. Note that, by [5, Theorem 3.3.10], $\text{Ext}^i_R(\text{Hom}_R(M, \omega_R))$ is either zero or Cohen–Macaulay of dimension $d - i$. In other words $M \otimes_R \omega_R$ is sequentially Cohen–Macaulay.

(ii) $\Rightarrow$ (i). Consider the third quadrant double complex $\text{Hom}_R(P_0 \otimes_R \omega_R, E^*)$. Using the same notation as before, let $^\vee E$ (resp. $^h E$) be the vertical (resp. horizontal) spectral sequences associated to the double complex $\text{Hom}_R(P_0 \otimes_R \omega_R, E^*)$. Then $^\vee E^i,j \cong \text{Ext}^i_R(\text{Hom}_R(M, \omega_R)) \cong 0$, for all $j > 0$, by our assumption. By using the equivalence of functors $\text{Hom}_R(X \otimes_R \omega_R, Y)$ and $\text{Hom}_R(X, \text{Hom}_R(\omega_R, Y))$ in the category of $R$-modules, we find the following isomorphism of double complexes $\text{Hom}_R(P_0 \otimes_R \omega_R, E^*) \cong \text{Hom}_R(P_0, \text{Hom}_R(\omega_R, E^*))$. Thus, we get $h^E^i,j \cong \text{Ext}^i_R(M, \text{Ext}^j_R(\omega_R, \omega_R))$, for all $i, j \geq 0$. As $\text{Ext}^i_R(\omega_R, \omega_R) = 0$ for $i > 0$ and $\text{Hom}_R(\omega_R, \omega_R) \cong R$, we get

\[
h^E^i,j \cong \begin{cases} 
\text{Ext}^i_R(M, R) & j = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

As the two spectral sequences $^\vee E$ and $^h E$ collapse, we have $h^E_\infty = h^E_2$ and $^\vee E_\infty = ^\vee E_2$ and so that $\text{Ext}^i_R(M, R) \cong \text{Ext}^i_R(M \otimes_R \omega_R, \omega_R)$. Since $M \otimes_R \omega_R$ is sequentially Cohen–Macaulay, $\text{Ext}^i_R(M, R) = 0$ or Cohen–Macaulay of dimension $d - i$ (see [5, Theorem 1.9]), i.e. $M$ is CME. □

Corollary 2.14. Let $(R, m)$ be a Cohen–Macaulay local ring and let $M$ be a finitely generated $R$-module. Set $\omega_R$ as the canonical module of $R$, the completion of $R$ with respect to the $m$-adic topology. Then the following are equivalent.

(i) $M$ is CME.

(ii) $\hat{M} \otimes_R \omega_R$ is sequentially Cohen–Macaulay and $\text{Tor}^R_i(\hat{M}, \omega_R) = 0$ for all $i > 0$. 

3. Local cohomology and linkage

The main purpose of this section is to give a generalization of [12, Theorem 10] which states that $H^i_m(\lambda M) \cong D(H^{d-i}_m(M))$ for $i = 1, \ldots, d - 1$, whenever $M$ is a generalized Cohen–Macaulay module over Gorenstein local ring $R$, where $D(-)$ is the Matlis duality functor. Here we assume that $R$ is Cohen–Macaulay with canonical module $\omega_R$ such that $M \otimes_R \omega_R$ is generalized Cohen–Macaulay; it is then shown that for each $i = 1, \ldots, d - 1$, $H^i_m(M \otimes_R \omega_R) \cong D(H^{d-i}_m(\lambda M))$. Also whenever $M$ is generalized Cohen–Macaulay, under some vanishing assumption on Tor-modules of $M$ and $\omega_R$, we show that $H^i_m(\lambda M) \cong \text{Ext}_R^i(M, R)$ for $i = 1, \ldots, d - 1$ (see Corollary 3.4).

The next proposition will lead to a “cohomologic criterion” for generalized Cohen–Macaulay modules to be linked (Corollary 3.2). This proposition has its own interest as it shows the exactness of the sequence (3.1). Although this exact sequence may be already known, but for the sake of a detailed proof and statement we mention it.

**Proposition 3.1.** Let $(R, m)$ be a Cohen–Macaulay local ring of dimension $d$ with the canonical module $\omega_R$. Assume that $M$ is a finitely generated $R$-module, $\text{Ass}_R(M) \subseteq \text{Ass} R \cup \{m\}$ and that $M$ satisfies the Serre condition $(S_2)$ on the punctured spectrum. Set $M^\vee = \text{Hom}_R(M, \omega_R)$. Let $\phi : M \rightarrow M^{\vee \vee}$ be the natural map, $K := \text{Ker}(\phi)$ and $C := \text{Coker}(\phi)$. The following statements hold true.

(i) If $d = 0$ then $K = 0$.
(ii) If $d \leq 1$ then $C = 0$.
(iii) If $d \geq 1$ then $K \cong \Gamma_m(M)$.
(iv) If $d \geq 2$ then $C \cong H^1_m(M)$ and so there is an exact sequence

$$0 \rightarrow \Gamma_m(M) \rightarrow M \rightarrow M^{\vee \vee} \rightarrow H^1_m(M) \rightarrow 0.$$  \hspace{1cm} (3.1)

**Proof.** If $d = 0$, it is clear by [5, Theorem 3.3.10] that $C = 0$ and $K = 0$. Assume that $d \geq 1$. One has $\text{depth}_R(M^{\vee \vee}) \geq \text{Min}[2, \text{depth}_R(\omega_R)] \geq 1$ and so $\Gamma_m(M^{\vee \vee}) = 0$. By applying $\Gamma_m(-)$ on the exact sequence

$$0 \rightarrow K \rightarrow M \rightarrow M^{\vee \vee} \rightarrow C \rightarrow 0,$$  \hspace{1cm} (3.2)

it follows that $\Gamma_m(K) = \Gamma_m(M)$. Taking $d = 1$, for each $p \in \text{Spec} R \setminus \{m\}$, $M_p \cong (M^{\vee \vee})_p \cong (M_p)^{\vee \vee}$, which implies that $\text{Supp}_R(K) \subseteq \{m\}$ i.e. $K = \Gamma_m(M)$. Hence we get the exact sequence

$$0 \rightarrow M/\Gamma_m(M) \rightarrow M^{\vee \vee} \rightarrow C \rightarrow 0,$$  \hspace{1cm} (3.3)

from which, by applying $\Gamma_m(-)$, we obtain the exact sequence

$$0 \rightarrow \Gamma_m(C) \rightarrow H^1_m(M) \rightarrow H^1_m(M^{\vee \vee}) \rightarrow H^1_m(C).$$  \hspace{1cm} (3.4)

As $\text{depth}_R(M^{\vee \vee}) \geq \text{Min}[2, \text{depth}_R(\omega_R)] \geq 1$, $M^{\vee}$ is maximal Cohen–Macaulay. Therefore the natural map $M^{\vee} \rightarrow M^{\vee \vee}$ is isomorphism. Using the local duality theorem functorially gives the commutative diagram

$$\begin{array}{ccc}
0 & \rightarrow & \Gamma_m(C) \\
\downarrow^{\cong} & & \downarrow^{\cong} \\
H^1_m(M) & \rightarrow & H^1_m(M^{\vee \vee}) \\
\downarrow & & \downarrow \\
D(M^{\vee}) & \rightarrow & D(M^{\vee \vee}),
\end{array}$$

\[\]
where \( D(-) = \text{Hom}_R(-, E(R/m)) \). Thus we get \( \Gamma_m(C) = 0 \). Note that if \( p \in \text{Spec } R \setminus \{m\} \), then \( \dim R_p = 0 \) and so \( C_p = 0 = K_p \) by [5, Theorem 3.3.10]. Hence \( C = \Gamma_m(C) = 0 \).

In case \( d \geq 2 \), \( \text{depth}_R(M^{\omega_i}) \geq \min\{2, \text{depth}_R(\omega_R)\} \geq 2 \) and (3.4) implies that

\[
\Gamma_m(C) \cong H^1_m(M). \tag{3.5}
\]

Finally, we prove by induction on \( d \geq 2 \) that \( K = \Gamma_m(M) \) and \( C = H^1_m(M) \). Assume that the statement is settled for rings with dimension smaller than \( d \). Let \( p \in \text{Supp}_R(M) \setminus \{m\} \). We first show that \( p \notin \text{Supp}_R(K) \cup \text{Supp}_R(C) \). If \( \text{ht } p = 0 \), the claim holds true as before. Assume that \( \text{ht } p \geq 1 \). As \( \dim R_p < d \), induction hypothesis for \( R_p \) implies that \( K_p = \Gamma_{pR_p}(M_p) \) and \( C_p = H^1_{pR_p}(M_p) \). Since \( p \notin \text{Ass}_R(R) \) and so \( p \notin \text{Ass}_R(M) \), we get \( \text{depth}_{R_p}(M_p) \geq 1 \) and thus \( K_p = 0 \), i.e. \( p \notin \text{Supp}_R(K) \). For the case \( \text{ht } p = 1 \), we already have, \( C_p = 0 \). Assume that \( \text{ht } p \geq 2 \). Again from the exact sequence (3.2) and the fact that \( \text{depth}_{R_p}(M_{\omega_i}) > 1 \), we get \( K = \Gamma_{m}(K) = \Gamma_{m}(M) \). As \( R \) is Cohen–Macaulay and \( \text{Ass}_R(M) \subseteq \text{Ass}(R) \cup \{m\} \), \( \dim R_p = \dim_{R_p}(R_p) = \text{ht } p \geq 2 \) and so \( \text{depth}_{R_p}(M_p) \geq 2 \) because \( M \) satisfies (S2). Hence \( H^1_{pR_p}(M_p) = 0 \). Hence \( C_p = 0 \), i.e. \( p \notin \text{Supp}_R(C) \). In particular, \( K = \Gamma_{m}(K) \) and \( C = \Gamma_{m}(C) \). Now \( K = \Gamma_{m}(M) \) by (3.2), and \( C = H^1_{m}(M) \) by (3.5). \( \square \)

**Corollary 3.2.** Let \( R \) be a Gorenstein local ring and let \( M \) be a generalized Cohen–Macaulay stable \( R \)-module with \( \dim_{R}(M) = \dim R \). A necessary and sufficient condition for \( M \) to be horizontally linked is that \( \Gamma_{m}(M) = 0 \).

**Proof.** Note that, by [4, Exercise 9.5.6], \( \text{Ass}_R(M) \subseteq \text{Ass}_R(R) \cup \{m\} \) and \( M \) satisfies (S2) on the punctured spectrum. As \( M \) is linked if and only if the natural map \( M \to M^{**} \) is one-to-one, the result follows by Proposition 3.1. \( \square \)

In the following result, we extend [12, Theorem 10 in Section 10] for Cohen–Macaulay rings with canonical module.

**Theorem 3.3.** Let \( (R, m) \) be a local Cohen–Macaulay ring of dimension \( d \geq 2 \) with canonical module \( \omega_R \). Let \( M \) be a finitely generated \( R \)-module of dimension \( d \), such that \( M \otimes_R \omega_R \) is generalized Cohen–Macaulay. Then for each \( i = 1, \ldots, d - 1 \), \( H^i_m(M \otimes_R \omega_R) \cong \text{Hom}_R(H^{d-i-1}_m(\lambda M), E(R/m)) \).

**Proof.** First we examine the general situation for an \( R \)-module \( N \) which is a generalized Cohen–Macaulay of dimension \( d \). Let \( 0 \to I^0 \to I^1 \to \cdots \) be an injective resolution of \( \omega_R \) and \( \cdots \to F_1 \to P_0 \to 0 \) be a projective resolution of \( N \) and construct the third quadrant double complex \( F := \text{Hom}_R(\text{Hom}_R(P_\bullet, \omega_R), I^\bullet) \). Let \( \text{^v E} \) (resp. \( \text{^h E} \)) denote the vertical (resp. horizontal) spectral sequence associated to the double complex \( F \). Then \( \text{^v E}^{i,j}_{2} \cong \text{Ext}^i_R(\text{Ext}^j_R(N, \omega_R), \omega_R) \). As \( N \) is generalized Cohen–Macaulay, by local duality theorem, \( \text{Ext}^i_R(N, \omega_R) \) is of finite length, for all \( i = 1, \ldots, d \). Therefore

\[
\text{^v E}^{i,j}_{2} \cong \begin{cases} \text{Ext}^i_R(N^{\omega_j}, \omega_R) & \text{if } j = 0, \\ H^d_{m-j}(N) & \text{if } j \neq 0, \ i = d, \\ 0 & \text{if } j \neq 0, \ i \neq d. \end{cases}
\]

As the map \( d^r \) is of bidegree \( (r, 1 - r) \), one can observe that \( \text{^v E}^{i,0}_{r} \cong \text{Ext}^{d-i}_R(N^{\omega_j}, \omega_R) \), for \( r \geq 2 \). Thus we have the following diagram:
To compute \( h^2 \), we change our double complex with the functorial isomorphisms \( \text{Hom}_R(\text{Hom}_R(P_i, \omega_R), I_j) \cong P_i \otimes_R \text{Hom}_R(\omega_R, I_j) \). Thus we get

\[
h^2 \cong \begin{cases} 
N & \text{if } i = 0, j = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

As \( \text{Ker} d^r \) and \( \text{Coker} d^r \) are isomorphic to \( V_{E_\infty} \), comparing the two spectral sequences, one gets isomorphisms \( d^r : \text{Ext}^{d-r}_R(N^v, \omega_R) \to \text{H}^{d-r+1}_m(N) \) for \( r = 2, \ldots, d - 1 \). Therefore \( \text{Ext}^{d-r}_R(N^v, \omega_R) \) is of finite length and so, by local duality theorem, \( \text{Ext}^{d-r}_R(N^v, \omega_R) \cong \text{D}(\text{H}^{d}_m(N^v)) \). Hence one obtains the isomorphisms \( \text{H}^{d-r+1}_m(N) \cong \text{D}(\text{H}^{d}_m(N^v)) \), for all \( r = 2, \ldots, d - 1 \).

Replacing \( N \) by \( M \otimes_R \omega_R \), gives

\[
\text{H}^{d-i+1}_m(M \otimes_R \omega_R) \cong \text{D}(\text{H}^i_m(M \otimes_R M \otimes_R \omega_R)) \cong \text{D}(\text{H}^i_m(M^*))
\]

for all \( i = 2, \ldots, d - 1 \). Consider the exact sequence \( 0 \to M^* \to P_0^* \to \lambda M \to 0 \). Applying \( \Gamma_m(-) \) we get \( \text{H}^{i+1}_m(M^*) \cong \text{H}^i_m(\lambda M) \) for \( i = 0, \ldots, d - 2 \). Therefore we have isomorphisms \( \text{H}^i_m(M \otimes_R \omega_R) \cong \text{D}(\text{H}^{d-i}_m(\lambda M)) \), for \( i = 2, \ldots, d - 1 \). Now it remains to prove the claim for \( i = 1 \). Applying Proposition 3.1 to \( M \otimes_R \omega_R \) and applying the functor \( \text{Hom}_R(-, \omega_R) \) on the exact sequence \( 0 \to M^* \to P_0^* \to \lambda M \to 0 \), we get the following commutative diagram with exact rows and columns.

\[
\begin{array}{ccccccc}
P_0 \otimes_R \omega_R & \rightarrow & M \otimes_R \omega_R & \rightarrow & 0 \\
\downarrow & & \downarrow & & \\
\text{Hom}_R(P_0^*, \omega_R) & \rightarrow & (M \otimes_R \omega_R)^{\vee v} & \rightarrow & \text{Ext}^1_R(\lambda M, \omega_R) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{H}^1_m(M \otimes_R \omega_R) & & & & & & 0 \\
\downarrow & & \downarrow & & & & \\
0 & & & & & & \\
\end{array}
\]

which implies that \( \text{H}^1_m(M \otimes_R \omega_R) \cong \text{Ext}^1_R(\lambda M, \omega_R) \cong \text{D}(\text{H}^{d-1}_m(\lambda M)) \). \( \square \)
As the final result, we state the next corollary of Theorem 3.3.

**Corollary 3.4.** Let $R$ be a Cohen–Macaulay ring of dimension $d \geq 2$ with canonical module $\omega_R$, and let $M$ be a finitely generated $R$-module. Suppose that $\text{Ext}^i_R(M, R)$ is of finite length for $i = 1, \ldots, d - 1$ and $\text{Tor}^i_R(M, \omega_R) = 0$ for $i > 0$. Then $H^i_m(\lambda M) \cong \text{Ext}^i_R(M, R)$, $i = 1, \ldots, d - 1$, and so $\lambda M$ is generalized Cohen–Macaulay.

**Proof.** Since $\text{Tor}^i_R(M, \omega_R) = 0$ for all $i > 0$, as in the proof of Theorem 2.13 ((ii) $\Rightarrow$ (i)), $\text{Ext}^i_R(M \otimes_R \omega_R, \omega_R) \cong \text{Ext}^i_R(M, R)$ for $i = 0, \ldots, d - 1$. Hence $\text{Ext}^i_R(M \otimes_R \omega_R, \omega_R)$ is of finite length for $i = 1, \ldots, d - 1$ and so that $M \otimes_R \omega_R$ is generalized Cohen–Macaulay. Therefore the result follows from local duality theorem and Theorem 3.3.

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**References**