An existence theorem for nonlinear complementarity problems

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\textbf{ABSTRACT}

We present in this work an existence theorem for nonlinear complementarity problems. The main result is based on the condition (S)\textsuperscript{1,+}, and on the notions of quasi-bounded operator and scalarly compact operator.

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1. Introduction

The Complementarity Theory is devoted to the study of complementarity problems. There exist several kinds of complementarity problems [2–4]. Each complementarity problem is a mathematical model related to some practical problems considered in optimization, economics, engineering, game theory, mechanics etc. Given a complementarity problem, we are not certain that the problem has a solution.

The solvability of complementarity problems has been considered by many authors. In relation to this we cite the papers [2–13] and the references therein.

The notion of REFE-acceptable mapping and the notion of regular exceptional family of elements for a continuous function were recently introduced [11]. We proved that if in a Hilbert space \( H \) a mapping \( f \) is REFE-acceptable and without an exceptional family of elements with respect to a closed convex cone \( K \subset H \), then the nonlinear complementarity problem \( \text{NCP}(f, K) \) has a solution [12].

Now, in this work we present an existence theorem based on a main result proved in [11]. Some consequences of this existence result will also be presented. We note that generally, the existence of a solution of a nonlinear complementarity problem is not evident.

2. Preliminaries

Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space and \( K \subset H \) a non-empty subset. We say that \( K \) is a closed pointed convex cone if the following properties are satisfied:

(i) \( K \) is a closed subset,
(ii) \( K + K \subseteq K \),
(iii) \( \lambda K \subseteq K \) for any \( \lambda \in \mathbb{R}_+ \),
(iv) \( K \cap (-K) = \{0\} \).

The dual of closed pointed convex cone \( K \) is
\[ K^* = \{ y \in H \mid \langle y, x \rangle \geq 0 \text{ for all } x \in K \} \].

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If \( A \subset H \) is a subset, we denote by \( \overline{A} \) the topological closure of \( A \). Let \( K \) be a closed pointed convex cone and \( f : H \rightarrow H \) a mapping. The nonlinear complementarity problem defined by \( f \) and \( K \) is

\[
\text{NCP} (f, K) : \begin{cases} 
\text{find } x_0 \in K \text{ such that } \\
f (x_0) \in K^* \text{ and } (x_0, f (x_0)) = 0.
\end{cases}
\]

The problem \( \text{NCP} (f, K) \) is a class of mathematical models used in optimization, economics, engineering, mechanics, elasticity, and robotics among other areas \([2–4,10]\).

We recall that \( f \) is \textit{completely continuous} if \( f \) is continuous and for any bounded set \( B \subset H, f (B) \) is a compact set, and we say that \( f \) is \textit{decominuous} if for any sequence \( \{x_n\}_{n \in \mathbb{N}} \subset H \) convergent in norm to an element \( x_* \in H \), we have that \( \{f (x_n)\}_{n \in \mathbb{N}} \) is weakly convergent to \( f (x_*) \). Also, we recall that \( f \) is a \textit{monotone mapping} if \( (x - y, f (x) - f (y)) \geq 0 \) for any \( x, y \in H \) and if \( (x - y, f (x) - f (y)) \leq 0 \) for any \( x, y \in H \), we say that \( f \) is \textit{antimonotone}.

3. Conditions \((S)_+\) and \((S)^1_+\)

Conditions \((S)_+\) and \((S)^1_+\) are well known in nonlinear analysis and are used to replace compactness when compactness is absent.

Condition \((S)_+\) was introduced in nonlinear analysis by F.E. Browder as a mathematical tool in the study of the solvability of nonlinear equations. We note that there exists a topological degree for mappings satisfying the condition \((S)_+\).

Condition \((S)^1_+\) was defined by Isac and Gowda \([9]\) as a useful condition in the study of solvability of complementarity problems and of variational inequalities.

The reader can find other applications of condition \((S)_+\) in \([3,5,6]\).

Let \( T : E \rightarrow E^* \) be a mapping. We say that \( T \) satisfies condition \((S)_+\) if for any sequence \( \{x_n\}_{n \in \mathbb{N}} \subset E \), weakly convergent to an element \( x_* \in E \) and for which \( \lim \sup_{n \rightarrow \infty} \langle x_n - x_*, T (x_n) \rangle \leq 0 \), we have that \( \{x_n\}_{n \in \mathbb{N}} \) is convergent in norm to \( x_* \).

Now, we cite a few examples of mappings satisfying condition \((S)_+\).

(a) If \( T : H \rightarrow H \) is a completely continuous mapping then \( f (x) = x - T (x) \) satisfies condition \((S)_+\). (\( H \) is a Hilbert space.)

(b) Let \( f : E \rightarrow E^* \) be a \( \rho \)-strongly monotone mapping, i.e., \( (x - y, f (x) - f (y)) \geq \rho (\|x - y\|) \) for any \( x, y \in H \), where \( \rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a continuous strictly increasing mapping such that \( \rho (0) = 0 \). In this case \( f \) satisfies condition \((S)_+\).

(c) If \( H \) is a Hilbert space and \( T : H \rightarrow H \) is a construction, then the mapping \( f (x) = x - T (x) \) satisfies condition \((S)_+\).

Let \( D \subset E \) be a non-empty subset and \( T : D \rightarrow E^* \) a mapping. We say that \( T \) satisfies condition \((S)^1_+\) if for any sequence \( \{x_n\}_{n \in \mathbb{N}} \subset D \) such that \( \{x_n\}_{n \in \mathbb{N}} \) is weakly convergent to an element, \( x_* \), \( \{T (x_n)\}_{n \in \mathbb{N}} \) is weakly star convergent to \( u \in E \) and \( \lim \sup_{n \rightarrow \infty} \langle x_n, T (x_n) \rangle \leq \langle x_*, u \rangle \), then \( \{x_n\}_{n \in \mathbb{N}} \) has a subsequence, norm convergent to \( x_* \).

It is known that any mapping satisfying condition \((S)_+\) satisfies also condition \((S)^1_+\) \([9]\). If \( (E, \|\cdot\|) \) is a Banach space which satisfies Kadec’s property and \( E^* \) is strictly convex, then any duality mapping \( J : E \rightarrow E^* \) associated with a weight \( \rho \) satisfies condition \((S)^1_+\). We can show that if \( T_1 : E \rightarrow E^* \) satisfies condition \((S)^1_+\) and is demicontinuous, then for any mapping \( T_2 : E \rightarrow E^* \) which is completely continuous, we have that \( T_1 + T_2 \) satisfies condition \((S)^1_+\).

4. Quasi-bounded operators

The notion of \textit{quasi-bounded} operator is due to Granas \([1]\). This notion was introduced as a mathematical tool for fixed point theory. We note that the notion of quasi-bounded operator has interesting applications in complementarity theory \([7,8]\).

Let \((E, \|\cdot\|)\) and \((F, \|\cdot\|)\) be Banach spaces and \( f : E \rightarrow F \) a mapping. We say that \( f \) is quasi-bounded if

\[
\lim_{\|x\| \rightarrow \infty} \sup_{\rho > 0} \|f (x)\| = \inf_{\rho > 0} \|f (x)\| < +\infty.
\]

If \( f \) is quasi-bounded the number

\[
\|f\|_{qb} := \lim_{\|x\| \rightarrow \infty} \|f (x)\|
\]

is called the \textit{quasi-norm} of \( f \).

If \( f = T \), where \( T \) is a bounded linear operator from \( E \) into \( F \), then \( T \) is quasi-bounded and \( \|T\|_{qb} = \|T\| = \sup_{\|x\| = 1} \|T (x)\| \). If \( f : E \rightarrow F \) is a mapping such that there exists \( M_0 > 0 \) and \( M_1 \geq 0 \) with the property that

\[
\|f (x)\| \leq M_0 \|x\| + M_1, \text{ for any } x \in E,
\]

then \( f \) is quasi-bounded.

There exists an interesting relation between quasi-bounded operators and the operators which have an asymptotic derivative.
We recall the definition of asymptotic derivative due to Krasnoselskii [14,15].

Let \((E, \| \cdot \|)\) and \((F, \| \cdot \|)\) be Banach spaces and \(f : E \to F\) a mapping. We say that \(f\) has an asymptotic derivative if there exists a linear and continuous operator \(S\) such that \(\lim_{|x| \to \infty} \frac{\|f(x) - S(x)\|}{|x|} = 0\). In this case we denote \(S\) by \(f_\infty\) and we say that \(f_\infty\) is an asymptotic derivative of \(f\).

For the properties of asymptotic derivatives and their estimations the reader is referred to [14,15].

It is known that if \(f\) has an asymptotic derivative \(f_\infty\) then in this case \(f\) is quasi bounded and \(\|f\|_{qb} = \|f_\infty\|\).

The quasi-norm of a quasi-bounded operator has the following properties.

(1) If \(f : E \to F\) is quasi-bounded then \(\lambda f\) is quasi-bounded and \(\|\lambda f\|_{qb} = |\lambda| \|f\|_{qb}\).

(2) If \(f_1, f_2 : E \to F\) are quasi-bounded then \(f_1 + f_2\) is quasi-bounded and we have \(\|f_1 + f_2\|_{qb} \leq \|f_1\|_{qb} + \|f_2\|_{qb}\).

(3) If \(f : E \to F\) is quasi-bounded and \(g : F \to G\) is linear and bounded then \(g \circ f\) is quasi-bounded.

5. Scalarly compact operators

Let \((E, \| \cdot \|)\) be a Banach space and \(D \subset E\) a closed convex subset.

We say that a mapping \(f : D \to E^*\) is scalarly compact if for any sequence \(\{x_n\}_{n \in \mathbb{N}} \subset D\) weakly convergent to an element \(x_0 \in D\), there exists a subsequence \(\{x_{n_k}\}_{k \in \mathbb{N}}\) of the sequence \(\{x_n\}_{n \in \mathbb{N}}\) such that

\[
\lim_{k \to \infty} \langle x_{n_k} - x_0, f(x_{n_k}) \rangle = 0.
\]

Now, we indicate some examples of scalarly compact mappings:

(a) If \(f : E \to E^*\) is completely continuous, then \(f\) is scalarly compact.

(b) If \(h : E \to E^*\) is completely continuous, \(g : E \to E^*\) is monotone, then we can prove that \(f(x) = h(x) - g(x)\) for any \(x \in E\) is scalarly compact.

(c) If \((H, \langle \cdot, \cdot \rangle)\) is a Hilbert space and \(h : H \to H\) is a completely continuous mapping then the mapping \(f(x) = h(x) - x\) is scalarly compact but it is scalarly compact.

(d) If \(f : E \to E^*\) is antimonotone, then \(f\) is scalarly compact.

(e) If \(h : E \to E^*\) is scalarly compact and \(g : E \to E^*\) is completely continuous, then \(f(x) = h(x) + g(x)\), for any \(x \in E\) is scalarly compact.

(f) If \(h, g : E \to E^*\) are scalarly compact mappings, then for any positive real numbers \(a, b\) the mapping \(f(x) = ah(x) + bg(x)\) for any \(x \in E\) is scalarly compact.

As regards the mappings satisfying condition \((S)_+^+\) and the scalarly compact mappings we remark the following interesting result:

If \(f : E \to E^*\) satisfies condition \((S)_+^+\) and \(g : E \to E^*\) is scalarly compact, then \(f + g\) satisfies condition \((S)_+\).

The notion of scalarly compact operator is due to G. Isac.

This notion will be studied in a future paper.

6. The main result

Our main result is an existence theorem for nonlinear complementarity problems in Hilbert spaces.

Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space, \(K \subset H\) a closed convex cone and \(f : H \to H\) a mapping.

We say that a family of elements \(\{x_t\}_{t > 0} \subset K\) is a regular exceptional family of elements (for short denoted by REFE) for \(f\) with respect to \(K\) if for every real number \(r > 0\), there exists a real number \(\mu_r > 0\) such that the vector \(u_t = \mu_r x_t + f(x_t)\) satisfies the following conditions:

(i) \(u_t \in K^*\),

(ii) \(\langle u_t, x_t \rangle = 0\),

(iii) \(\|x_t\| = r\).

As regards the notion defined above the reader is referred to [11].

We recall that a mapping \(f : H \to H\) is called REFE-acceptable if either the problem NCP \((f, K)\) has a solution, or the mapping \(f\) has a REFE with respect to \(K\) [11].

Examples of REFE-acceptable mappings are given in [11].

From this notion we deduce immediately the following result.

**Lemma 1.** If the mapping \(f : H \to H\) is REFE-acceptable and without REFE, with respect to \(K\), then the problem NCP \((f, K)\) has a solution.

The following result is a test for REFE-acceptability.

**Theorem 2** ([11]). Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space, \(K \subset H\) a closed convex cone and \(f : H \to H\) a mapping. If \(f\) has a decomposition of the form \(f(x) = T_1(x) - T_2(x)\) and the following assumptions are satisfied:

(1) \(T_1\) is demicontinuous, bounded (i.e., the image by \(T_1\) of a bounded set is bounded) and satisfies condition \((S)_+^1\),

(2) \(T_2\) is demicontinuous and scalarly compact with respect to \(K\),

then \(f\) is REFE-acceptable with respect to \(K\).
Proof. A proof of this result is given in [11]. ■

Our main result is also based on the following result.

Theorem 3 ([11]). Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space, \(K \subset H\) a closed convex cone and \(f : H \rightarrow H\) a mapping. A necessary and sufficient condition for the mapping \(f\) to have the property of being without a REFE with respect to \(K\) is the following.

There is a \(\rho > 0\) such that for any \(x \in K\) with \(\|x\| = \rho\) at least one of the following conditions holds:

1. \(\langle f(x), x \rangle \geq 0\),
2. \(\rho^2 \langle f(x), y \rangle < \langle x, y \rangle \cdot \langle f(x), x \rangle\).

Proof. A proof of this result is given in [11]. ■

Some applications of Theorem 3 are also given in [12].

The main result is the following theorem.

Theorem 4. Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space, \(K \subset H\) a closed convex cone and \(h : H \rightarrow H\) demicontinuous mappings.

If the following assumptions are satisfied:

1. \(g\) is bounded and satisfies condition \((S)\),
2. there exists four constants \(\alpha > 0, \rho_0 > 0, c_0 > 0\) and \(c_1 > 0\) such that \(g(x), x) \geq \alpha \|x\|^2 - c_1 \|x\| - c_0\), for any \(x \in K\) with \(\|x\| > \rho_0\),
3. \(h\) is quasi-bounded, scalarly compact with respect to \(K\) and \([h]_{qb} < \alpha\),

then the problem NCP \((f, K)\), where \(f = g - h\) has a solution.

Proof. Let \(\varepsilon > 0\) be such that \(0 < \varepsilon < \alpha - [h]_{qb}\) and we define \(M_1 = [h]_{qb} + \varepsilon\). We have that \(0 < M_1 < \alpha\).

Because \(h\) is quasi-bounded, there exists \(\rho_1 > \rho_0\) such that \(\|h(x)\| \leq M_1 \|x\| + M_2\), where \(M_2 > 0\).

We define \(\beta = \alpha - M_1 > 0\).

We have

\[
\langle f(x), x \rangle = \langle g(x), x \rangle - \langle h(x), x \rangle \geq \alpha \|x\|^2 - c_1 \|x\| - c_0 - M_1 \|x\|^2 - M_2 \|x\| = \|x\| (\alpha - M_1) \|x\| - (M_2 + c_1) - c_0 = \|x\| \beta \|x\| - (M_2 + c_1) - c_0.
\]

that is, we have for any \(x \in K\), with \(\|x\| > \rho_1\),

\[
\langle f(x), x \rangle \geq \|x\| \beta \|x\| - (M_2 + c_1) - c_0 > 0.
\]

If \(\rho_\alpha\) is a real number such that \(\rho_\alpha > \max \left\{ \rho_1, \frac{M_2 + c_1 + 1}{\beta}, c_0 \right\}\), then we have \(\beta \rho_\alpha - (M_2 + c_1) > 1\) and \(\rho_\alpha > c_0\) which implies \(\rho_\alpha \beta \rho_\alpha - (M_2 + c_1) > c_0\).

Therefore for any \(x \in K\) with \(\|x\| = \rho_\alpha\) we have

\[
\langle f(x), x \rangle \geq \|x\| \beta \|x\| - (M_2 + c_1) - c_0 > 0.
\]

Because the assumptions of Theorem 2 are satisfied, we have that \(f\) is REFE-acceptable. By Theorem 3 we have that \(f\) is without a regular exceptional family of elements with respect to \(K\) and by Lemma 1 we obtain that the problem NCP \((f, K)\) has a solution, and the proof is complete. ■

7. Remarks

(a) If the mapping \(g\) is strongly monotone with respect to \(K\) in the following sense:

there exists \(\alpha > 0\) such that \(\langle x - y, g(x) - g(y) \rangle \geq \alpha \|x - y\|^2\) for any \(x, y \in K\) and \(g(0) = 0\), then in this case we can show [6,9] that \(g\) satisfies condition \((S)\), and \(\langle g(x), x \rangle \geq \alpha \|x\|^2\). Also, in this case assumption (2) of Theorem 4 is satisfied with \(c_0 = c_1 = 1\).

(b) In some practical problems the assumption \([h]_{qb} < \alpha\) is not restrictive, for example, if \(g\) satisfies the condition \(\lim_{\|x\| \to \infty} \frac{\langle g(x), x \rangle}{\|x\|^2} = +\infty\). In this case, we take \(\alpha > [h]_{qb}\) and there exists \(\rho_0 > 0\) such that for any \(x \in K\) with \(\|x\| > \rho_0\) we have \(\langle g(x), x \rangle \geq \alpha \|x\|^2\).

(c) If we replace the inequality used in assumption (2) of Theorem 4 by

\[
\langle g(x), x \rangle \geq \alpha \|x\|^2 - \varphi (\|x\|),
\]

where \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\) and \(\lim_{t \to +\infty} \varphi(t) = +\infty\), then in this case if there exist \(r > 0\) and \(\rho_0 > 0\) such that for any \(x \in K\) with \(\|x\| > \rho_0\)

\[
\varphi (\|x\|) \leq (\alpha - r) \|x\|^2,
\]

\((\theta_2)\)
we have that our condition becomes
\[ (g(x), x) \leq r \|x\|^2. \]

If (\(\theta_2\)) is not satisfied, in this case to have that \(\langle f(x), x \rangle \geq 0\) for \(x \in K\) with \(\|x\|\) sufficiently big we must ask to have \(\varphi(\|x\|) \leq \delta \|x\|\) for a particular \(\delta > 0\) and \(x\) with \(\|x\|\) sufficiently big.

Indeed
\[ \langle f(x), x \rangle \geq (g(x), x) - \langle h(x), x \rangle \geq \alpha \|x\|^2 - \varphi(\|x\|) - M_1 \|x\|^2 - M_2 \|x\| = \|x\|^2 \left(\|x\| \varphi(\|x\|) - M_1 \|x\|^2 - M_2 \|x\| \right), \]
where \(\beta = \alpha - M_1.\)

If \(\delta > 0\) and \(\|x\| \geq \frac{\delta + M_2}{\beta}\) then we have \(\langle f(x), x \rangle \geq 0\) if we ask to have \(\varphi(\|x\|) \leq \delta \|x\|\), that is our condition becomes
\[ \langle g(x), x \rangle \geq \alpha \|x\|^2 - \delta \|x\|, \]
we have again assumption (2) of Theorem 4.

(d) Another interesting case is if we ask to have \(\langle g(x), x \rangle \geq \alpha \|x\|^2\), for any \(x \in K\) and
\[ \|h(x)\| \leq A \|x\|^2 + B \|x\| + C, \]
for any \(x \in K\).

We do not suppose that \(h\) is quasi-bounded.

In this case, if we have that \(A > 0, B + 1 < \alpha\) and \(0 < C < \frac{\alpha - B - 1}{A}\) then for any \(x \in K\) with \(\|x\| = \rho_0\) we can show that \(\langle f(x), x \rangle \geq 0\).

(e) It is known (see \([4]\)) that if \(H\) is the \(n\)-dimensional Euclidean space \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\), \(K \subset \mathbb{R}^n\) a closed convex cone and \(f: \mathbb{R}^n \to \mathbb{R}^n\) a mapping which is strongly monotone in the following sense:
\[ (f(x) - f(y), x - y) \geq \gamma(\|x - y\|) \|x - y\|, \]
for any \(x \in K\), where \(\gamma: \mathbb{R}_+ \to \mathbb{R}_+\) is continuous and \(\lim_{t \to +\infty} \gamma(t) = +\infty\), then the problem NCP \((f, K)\) has a solution.

To have this result in an infinite dimensional Hilbert space we must have satisfied some supplementary conditions.

In this sense Theorem 4 is useful. If we suppose in Theorem 4 that \(g\) and \(h\) are demicontinuous, \(g\) is bounded and satisfies condition \((S) \beta_1\), and \(h\) is scalarly compact but the mapping \(f = g - h\) satisfies condition \((\theta_3)\), then we obtain that the problem NCP \((f, K)\) has a solution.

Indeed, we take a real number \(\rho_0 > 0\) such that \(\gamma(\rho_0) \geq \|f(0)\|\) and we have
\[ \langle f(x), x \rangle = \langle f(x) - f(0), x \rangle + \langle f(0), x \rangle \geq \gamma(\|x\|) \|x\| + \|f(0), x\| \geq \|x\| \gamma(\|x\|) - \|f(0)\|, \]
for any \(x \in K\).

Therefore, for any \(x \in K\) with \(\|x\| = \rho_0\), we have
\[ \|f(x), x\rangle \geq \rho_0 \langle \gamma(\rho_0) - \|f(0)\| \rangle \geq 0 \]
and as in the proof of Theorem 4 we obtain that the problem NCP \((f, K)\) has a solution.

8. Comments

We presented in this work an existence theorem for nonlinear complementarity problems. Because in the main result the mapping \(f\) has the form \(f = g - h\), where \(g\) satisfies condition \((S) \gamma_1\), and \(h\) is quasi-bounded and scalarly compact, this result is applicable in the study of nonlinear complementarity problems defined by integro-differential operators.

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