Small Complete Caps in Spaces of Even Characteristic*

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In 1959, Segre constructed a complete $(3q+2)$-cap in $PG(3,q)$, $q$ even. This showed that the size of the smallest complete $k$-cap in $PG(3,q)$, $q$ even, is almost equal to the trivial lower bound which is of order $\sqrt{2q}$. Generalizing the construction of Segre, complete $(q^3+3(q^{n-1}+\cdots+q)+2)$-caps in $PG(2n,q)$, $q$ even, $q \geq 4$, and complete $(3(q^n+\cdots+q)+2)$-caps in $PG(2n+1,q)$, $q$ even, $q \geq 4$, are constructed. This shows that in all spaces $PG(2n+1,q)$, $q$ even, the size of the smallest complete $k$-cap is almost equal to the trivial lower bound which is of order $\sqrt{2^q}$.


INTRODUCTION

Let $PG(n,q)$ be the projective space of dimension $n$ over the finite field $\mathbb{F}_q$ of order $q$. A $k$-cap $K$ in $PG(n,q)$ is a set of $k$ points, no three of which are collinear. A $k$-cap of $PG(2,q)$ is also called a $k$-arc [9, p. 285]. A point $r$ of $PG(n,q)$ extends a $k$-cap $K$ to a $(k+1)$-cap if and only if $K \cup \{r\}$ is a $(k+1)$-cap. A point of $PG(n,q) \backslash K$ is saturated by $K$ if it belongs to a bisecant of $K$. A $k$-cap $K$ of $PG(n,q)$ is called complete if it is not contained in a $(k+1)$-cap.

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With respect to the possible sizes of complete $k$-caps, two numbers are of particular interest. The first number is the maximum value of $k$ for which there exists a $k$-cap in $\text{PG}(r, q)$. This number is denoted by $m_2(r, q)$ \cite[p. 285]{9} and is only known, for arbitrary $q$, when $r \in \{2, 3\}$. Namely, $m_2(2, q) = q + 1$ if $q$ is odd, $m_2(2, q) = q + 2$ if $q$ is even and $m_2(3, q) = q^2 + 1$, $q \geq 2$. Regarding the other values of $m_2(r, q)$, apart from $m_2(2, 2) = 2$, $m_2(4, 3) = 20$ and $m_2(5, 3) = 56$ \cite[p. 285]{9}, only upper bounds are known.

The second interesting number is the size of the smallest complete $k$-cap in $\text{PG}(r, q)$. This number is denoted by $n_2(r, q)$ and is only known in some small spaces.

The exact values $n_2(2, q)$, $q \leq 13$, are by \cite[p. 61]{16}, $n_2(2, 16)$ by \cite[p. 193]{10}, $n_2(2, q)$, $q = 17, 19$, by \cite[p. 12]{11}, $n_2(2, 23)$ by \cite[p. 3]{3}, $n_2(3, 2)$ by \cite[p. 96]{7}, $n_2(3, 3)$ by \cite[p. 104]{7}, $n_2(3, 4)$ by \cite[p. 12]{4}, $n_2(3, 5)$ by \cite[p. 12]{5}, $n_2(4, 2)$ by \cite[p. 222]{6}, $n_2(4, 3)$ by \cite[p. 12]{2} and $n_2(5, 2)$ by \cite[p. 222]{6}. The upper bound on $n_2(4, 4)$ is by \cite[p. 12]{12} and on $n_2(6, 2)$ by \cite[p. 15]{6} as indicated by \cite[p. 222]{6}. All other upper bounds are by \cite[1, 2, 4, 11]{1}.

It is however easy to obtain a trivial lower bound for $n_2(r, q)$. A $k$-cap has $k(k - 1)/2$ bisecants, each one of which contains $q + 1$ points. So in order to be complete, $k(k - 1)(q + 1)/2 \geq |\text{PG}(r, q)| = q^r + q^{r-1} + \cdots + q + 1$. Hence $k > \sqrt{2q^{r-1}}$, so $n_2(r, q) > \sqrt{2q^{r-1}}$.

A natural question is whether this is only a trivial lower bound or whether the size of the smallest complete $k$-cap is indeed of this order.

### Table I

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A first indication was given by B. Segre in 1959. In [13, Paragraph 3], he constructed a complete \((3q+2)\)-cap in \(PG(3, q)\), \(q\) even. Hence, \(\sqrt{2} q < n_d(3, q) \leq 3q + 2\), which gives a small interval containing \(n_d(3, q)\).

Based on this result, an inductive construction for complete caps in \(PG(n, q)\), \(q\) even, \(q \geq 4\), is presented. Generalizing the construction of the \((3q+2)\)-cap in \(PG(3, q)\), \(q\) even, complete \((q'' + 3(q'' - 1 + \cdots + q) + 2)\)-caps in \(PG(2n, q)\), \(q\) even, \(q \geq 4\), and complete \((3(q'' + \cdots + q) + 2)\)-caps in \(PG(2n+1, q)\), \(q\) even, \(q \geq 4\), are constructed. It then follows from the order of these caps that \(\sqrt{2q''} < n_d(2n+1, q) \leq 3(q'' + \cdots + q) + 2\), so that for all spaces \(PG(2n+1, q)\), \(q\) even, \(q \geq 4\), the lower bound \(n_d(2n+1, q) > \sqrt{2q''}\) is not that trivial. It gives a clear indication for the order of \(n_d(2n+1, q)\).

In the last section, a survey is given of other results on small complete caps. For \(q = 4\), the caps constructed in Section 3 have the same number of points as caps constructed by Segre in [14]. It is shown that the two constructions coincide in \(PG(n, 4)\), \(n = 1, 2, 3\), but not in \(PG(n, 4), n > 3\).

To conclude the introduction, we recall the standard notation for a conic. The conic \(C: X_1^2 = X_0 X_2\) in \(PG(2, q)\) is the set \(C = \{(t^2, t, 1) \mid t \in \mathbb{F}_q^*\}\) where \(\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}\), and where \(t = \infty\) defines the point \((1, 0, 0)\).

In \(PG(n, q)\), let \(e_0 = (1, 0, 0, \ldots, 0)\) and \(e_1 = (0, 1, 0, \ldots, 0)\).

2. A Complete \((3q+2)\)-Cap in \(PG(3, q)\), \(q\) Even

Theorem 2.1 (Segre [13, Paragraph 3]). (1) Let \(\Pi_1\) and \(\Pi_2\) be two distinct planes of \(PG(3, q)\), \(q\) even, intersecting in the line \(L\). Consider an irreducible conic \(C_1^\prime\) in \(\Pi_1\) and an irreducible conic \(C_2^\prime\) in \(\Pi_2\), both conics touching \(L\) at the same point \(p\) and having the same nucleus \(n\) (necessarily belonging to \(L\) and distinct from \(p\)).

Then \(C_1^\prime \cup C_2^\prime \cup \{n\}\) is a \((2q+2)\)-cap saturating all points in \(PG(3, q)\), except for some points in one plane \(\Pi_3\) through \(L\). In \(\Pi_3\) only \(L\) and one conic \(C\), tangent to \(L\) at \(p\) and having nucleus \(n\), are saturated.

(2) By taking a conic \(C_3\), \(C_3 \neq C\), in \(\Pi_3\) of the pencil of conics in \(\Pi_3\) determined by \(C\) and the line \(L\) counted twice, \(K_3 = C_1 \cup C_2 \cup C_3 \cup \{n\}\) is a complete \((3q+2)\)-cap.

Remark 2.2. (1) The plane \(\Pi_3\) and the conic \(C\) are easily characterized. The points of \(C \setminus \{p\}\) are the only points of \(PG(3, q)\) in \(\Pi_1 \cup \Pi_2\) belonging to at least two bisecants of \(C_1 \cup C_2 \cup \{n\}\). In fact, they belong to precisely \(q\) bisecants [13, Paragraph 3].

(2) In \(PG(1, q)\),

\[
K_1 = \{e_0, e_1\}
\] (1)
and in $PG(2, q)$, $q$ even,

$$K_3 = \{ (t^2, t, 1) | t \in \mathbb{F}_q \} \cup \{ e_0, e_1 \}$$

(2)

are classical examples of complete caps. We do mention them because they will serve as induction basis for the inductive construction of caps, presented in Section 3.

(3) We first describe explicitly the coordinates of the points of $K_3$. This will be a guide during the inductive construction of the complete caps.

Let $L: X_2 = X_3 = 0, n = e_1, p = e_0$ and consider the following two conics $C_1 = \{ (t^2, t, 1, 0) | t \in \mathbb{F}_q^+ \}$ in $X_3 = 0$ and $C_2 = \{ (t^2, t, 0, 1) | t \in \mathbb{F}_q^+ \}$ in $X_2 = 0$.

Then $C_1 \cup C_2 \cup \{ n \}$ saturates all points in $PG(3, q)$ apart from some points in a plane $\Pi_1$ through $L$. In $\Pi_1 \setminus L$, only the points of a certain conic $C$ are saturated (2.1 (1)).

As mentioned in 2.2 (1), the points of $C$ are the intersection of bisecants of $C_1 \cup C_2$. For instance, the bisecants $\langle (0, 0, \ldots, 0, 1, 0, 0, 1) \rangle$ and $\langle (1, 1, 0, 0, 0, 0, 0, 1) \rangle$ intersect in $(0, 0, 1, 1)$, so $\Pi_1: X_2 = X_3$, and $\langle (1, 1, 0, 0, 0, 0, 0, 1) \rangle \cap \langle (0, 0, 1, 0, 1, 1, 1, 1) \rangle = \langle (1, 1, 1, 1) \rangle$, so $C = \{ (t^2, t, 1, 1) | t \in \mathbb{F}_q^+ \}$ since $C$ contains $(0, 0, 1, 1), (1, 1, 1, 1)$, passes through $p = e_0$ and has nucleus $n = e_1$ (2.1 (1) and 2.2 (1)).

To construct $C_3$ (2.1 (2)), consider the pencil of conics in $\Pi_3$

$$X_1^2 + X_0 X_2 + \xi X_2^2 = 0, \quad \xi \in \mathbb{F}_q^+$$

$$X_3 = X_3,$$

defined by $C$ and $L$ counted twice.

Selecting for $C_3$ the conic $\{ (t^2 + 1, t, 1, 1) | t \in \mathbb{F}_q^+ \}$ with parameter $\xi = 1$, $C_1 \cup C_2 \cup C_3 \cup \{ n \}$ is the complete $(3q + 2)$-cap

$$K_3 = \{ (t^2, t, 1, 0) | t \in \mathbb{F}_q \} \cup \{ (t^2, t, 0, 1) | t \in \mathbb{F}_q \}$$

$$\cup \{ (t^2 + 1, t, 1, 1) | t \in \mathbb{F}_q \} \cup \{ e_0, e_1 \}.$$ (3)

**Lemma 2.3.** For a given point $p_3 = (t^2 + 1, l, l_{n-1}, l_{n-2}, \ldots, l_1, l, 1)$ in $PG(2n, q), q$ even, $q \geq 4$, there always exist two points $p_2 = (t^2, l, t^2, l_{n-1}, t_{n-2}, \ldots, l_1, l, 1)$ and $p_3 = (s_{n-1}, s_n, s_{n-1}, s_{n-2}, \ldots, s_1, s, t, s, s \in \mathbb{F}_q$, such that $p_3$ is linearly dependent on $p_2$ and $p_3$, such that $p_3$ is linearly dependent on $p_2$ and $p_3$.

**Proof.** So $p_3 = (a_{2n}, \ldots, a_1, 1)$ where $(a_2n, a_{2n-1}) = (t^2 + 1, l, l)$ and $(a_{2i}, a_{2i+1}) = (l_i, l_{i+1})$ for $1 \leq i \leq n - 1$.

Selecting $t_i = s_i = l, 1 \leq i < n, t_n \neq \{ l_n, 1 + l_n \}$, and $s_n = t_n + (1 + t^2 + t^2_n)/((1 + t^2 + t^2_n)+(1 + t^2 + t^2_n))$, $p_3$ is linearly dependent on $p_2$ and $p_3$.  


3. New Complete Caps in $\text{PG}(n, q)$, $q$ Even $q \geq 4$

Consider the caps $K_1$, $K_2$ and $K_3$ (Remark 2.2). These caps serve as induction basis for the following inductive construction of sets $K_n$ in $\text{PG}(n, q)$, $q$ even, $q \geq 4$.

**Definition 3.1.** Let $K_{n-2}$ be already constructed and suppose the coordinates of a point of $K_{n-2}$ are $(a_0, ..., a_{n-2})$ where it is always assumed that the last non-zero coordinate is equal to 1.

Then

$$K_n = \{(\lambda^2, \lambda, a_0, ..., a_{n-2}) \mid \lambda \in \mathbb{F}_q \text{ and } (a_0, ..., a_{n-2}) \in K_{n-2}\}$$

$$\cup \{e_0, e_1\} \cup \{ (\lambda^2 + 1, \lambda, 1, 0, ..., 0) \mid \lambda \in \mathbb{F}_q \}.$$  

**Remark 3.2.** (1) Considering the points of $K_n$ in $\Pi_1 = \langle e_0, e_1, (0, 0, a_0, ..., a_{n-2}) \rangle$ and $\Pi_2 = \langle e_0, e_1, (0, 0, b_0, ..., b_{n-2}) \rangle$, where $(a_0, ..., a_{n-2})$ and $(b_0, ..., b_{n-2})$ are distinct points of $K_{n-2}$, these are the points of two conics $C_1 = \{(\lambda^2, \lambda, a_0, ..., a_{n-2}) \mid \lambda \in \mathbb{F}_q^+\}$ and $C_2 = \{(\lambda^2, \lambda, b_0, ..., b_{n-2}) \mid \lambda \in \mathbb{F}_q^+\}$, together with their nucleus $n = e_1$.

So, $K_n \cap (\Pi_1 \cup \Pi_2)$ is the cap discussed by Segre in [13, Paragraph 3] (see also 2.1 (1)), with $p = e_0$. Hence, the conclusion of 2.1 (1) holds with respect to $K_n \cap (\Pi_1 \cup \Pi_2)$ and this will be the key element in showing that $K_n$ is a complete cap.

(2) The following lemmas and theorems will prove that $K_n$ is a complete cap of $\text{PG}(n, q)$.

In the proof, it will be assumed that if the cap $K_{n-2}$ in $\text{PG}(n-2, q)$ was used to construct the set

$$L_n = \{(\lambda^2, \lambda, a_0, ..., a_{n-2}) \mid \lambda \in \mathbb{F}_q, (a_0, ..., a_{n-2}) \in K_{n-2} \} \cup \{e_0, e_1\},$$

then the cap $L_n$ saturates the hole space $\text{PG}(n, q)$ apart from the plane

$$\mathbb{P}^n_L = \{ \begin{array}{c} X_2 = X_3 \\ X_4 = \cdots = X_n = 0 \end{array} \}$$

in which only $L = \langle e_0, e_1 \rangle$ and $C = \{(\lambda^2, \lambda, 1, 0, ..., 0) \mid \lambda \in \mathbb{F}_q^+\}$ are saturated.

**Lemma 3.3.** The assumption of Remark 3.2 (2) is valid when constructing $L_n$ in $\text{PG}(n, q)$, $n = 3, 4, q \geq 4$.

**Proof.** $n = 3$. Here $K_3 = \{(1, 0), (0, 1)\}$ (see (1)) and $L_3 = \{(\lambda^2, \lambda, 1, 0) \mid \lambda \in \mathbb{F}_q\} \cup \{(\lambda^2, \lambda, 0, 1) \mid \lambda \in \mathbb{F}_q\} \cup \{e_0, e_1\}$. 


As indicated by Segre [13, Paragraph 3] (2.1 (1) and 2.2 (3)), \(PG(3,q)\) is saturated apart from the plane \(X_2 = X_3\) in which only \(L\) and \(C = \{(\lambda^2, \lambda, 1, 1) | \lambda \in \mathbb{P}^2_q \}\) are saturated.

\(n = 4\). Consider \(K_2 = \{(t^2, t, 1) | t \in \mathbb{P}^2_q \} \cup \{(1, 0, 0), (0, 1, 0)\}\) (see (2)). Using \(K_2\), define the set of planes

\[T_4 = \{(e_0, e_1, (0, 0, a_0, a_1, a_2)) | (a_0, a_1, a_2) \in K_2\}\]

through \(L\).

The planes through \(L\) form a structure \(\Sigma\) isomorphic to a projective plane \(PG(2,q)\). In \(\Sigma\), \(T_4\) defines a complete \((q + 2)\)-arc. So no three planes of \(T_4\) belong to the same three-dimensional space through \(L\).

It then follows that \(L_4 = \{(\lambda^2, \lambda, a_0, a_1, a_2) | \lambda \in \mathbb{P}^2_q, (a_0, a_1, a_2) \in K_2\} \cup \{(e_0, e_1)\}\) is a \((q^2 + 2q + 2)\)-cap. No three points of \(L_4\) in the same plane of \(T_4\) are collinear and if three points \((\lambda^2, \lambda, a_0, a_1, a_2), (\lambda^2, \lambda_2, b_0, b_1, b_2)\) and \((\lambda^2, \lambda_3, e_0, e_1, e_2)\), with \((a_0, a_1, a_2), (b_0, b_1, b_2), (e_0, e_1, e_2)\) three distinct points of \(K_2\), would be collinear, then the corresponding planes of \(T_4\) would belong to a three-dimensional space through \(L\). This is impossible.

So \(L_4\) indeed is a cap.

For two distinct planes \(\Pi_1 = \langle e_0, e_1, (0, 0, t_1^2, t_1, 1) \rangle\), \(\Pi_2 = \langle e_0, e_1, (0, 0, t_2^2, t_2, 1) \rangle\), the set \(\{\lambda^2, \lambda, a_0, a_1, a_2) | \lambda \in \mathbb{P}^2_q\} \cup \{(e_0, e_1)\}\) is a \((2q + 2)\)-cap in \(\langle \Pi_1, \Pi_2 \rangle\), of the type described by Segre in [13, Paragraph 3] (3.2 (1)). Hence from 2.1 (1), all points in \(\langle \Pi_1, \Pi_2 \rangle\) are saturated apart from one plane \(\Pi_3\) in which only \(L = \langle e_0, e_1 \rangle = \Pi_1 \cap \Pi_2\) and one conic are saturated. As indicated in 2.2 (1), the points of the conic are the only points of \(\langle \Pi_1, \Pi_2 \rangle\), not in \(\Pi_1, \Pi_2\), belonging to two bisecants.

The bisecants \(\langle (0, 0, t_1^2, t_1, 1), (0, 0, t_2^2, t_2, 1) \rangle\) and \(\langle (1, 1, t_1^2, t_1, 1), (1, 1, t_2^2, t_2, 1) \rangle\) intersect in \((0, 0, t_1 + t_2, 1, 0)\) so

\[\Pi_4 = \Pi_{n \cdot n}.:\]

\[\begin{aligned}
X_2 + (t_1 + t_2)X_3 &= 0 \\
X_4 &= 0.
\end{aligned}\]

Since \(q \geq 4\), any plane through \(L\), not contained in \(X_4 = 0\), belongs to at least one three-dimensional space \(\langle \Pi_1, \Pi_2 \rangle\) determined by two planes \(\Pi_1\) and \(\Pi_2\), defined above. Hence, all non-saturated planes through \(L\) are planes \(\Pi_{n \cdot n}\) lying in \(X_4 = 0\). But the \((2q + 2)\)-cap \(L_4 \cap (X_4 = 0) = \{(\lambda^2, \lambda, 1, 0, 0) | \lambda \in \mathbb{P}^2_q\} \cup \{(\lambda^2, \lambda, 0, 1, 0) | \lambda \in \mathbb{P}^2_q\} \cup \{(e_0, e_1)\}\) saturates \(X_4 = 0\) apart from the plane

\[\gamma^4 : \begin{cases}
X_2 = X_3 \\
X_4 = 0,
\end{cases}\]

in which only \(L \cup \{(\lambda^2, \lambda, 1, 1, 0) | \lambda \in \mathbb{P}^2_q\}\) is saturated (2.2 (3)).
This plane is also the non-saturated plane \( \Pi_{t_1, t_2} \) if \( t_2 = t_1 + 1 \). For \( t_2 = t_1 + 1 \), the point \((0, 0, t_1 + t_2, 1, 0) = (0, 0, 1, 1, 0)\) is saturated, also \((1, 1, 1, 0, 0)\) is saturated since it belongs to the bisecants \( \langle (1, 1, t_1^2, t_2, 1), (0, 0, t_1^2 + 1, t_1 + 1, 1) \rangle \) and \( \langle (0, 0, t_1^2, t_1, 1), (1, 1, t_1^2 + 1, t_1 + 1, 1) \rangle \). So the saturated conic in \( \gamma_4 = \Pi_{t_1, n+1} \) is \( C = \{ (\lambda^2, \lambda, 1, 1, 0) \mid \lambda \in F_q^* \} \) since it passes through \((0, 0, 1, 1, 0)\), \((1, 1, 1, 1, 0)\), \(p = e_0\) and has nucleus \( n = e_1 \) (2.1 (1) and 2.2 (1)).

This shows that the whole space \( PG(4, q) \) is saturated apart from the plane \( \#4 \) in which only \( L \) and \( C \) are saturated.

We now will show that \( K_n \) is a complete cap. When constructing \( K_n \), we assume that it has already been shown that the caps \( K_i \) in \( PG(i, q), i < n \), are complete and that the assumption made in Remark 3.2 (9) was valid when constructing \( L_{n-1} \) from \( K_{n-3} \).

We first prove

**Lemma 3.4.** Using the notations of Remark 3.2 (2), the set \( L_n \) is a cap of \( PG(n, q) \), \( q \geq 4 \).

*Proof.* The planes \( \langle e_0, e_1, (0, 0, a_0, \ldots, a_{n-2}) \rangle, (a_0, \ldots, a_{n-2}) \in K_{n-2}, \) form a cap isomorphic to the complete cap \( K_{n-2} \) in the projective space \( PG(n-2, q) \) defined by the planes of \( PG(n, q) \) through \( L \). So no three of those planes belong to the same three-dimensional space through \( L \).

Proceeding as in the proof of Lemma 3.3 for \( L_4 \), it is proved that \( L_n \) is a cap.

**Theorem 3.5.** Points of \( PG(n, q), n \geq 4, q \geq 4, \) not saturated by \( L_n \) belong to \( X_n = 0 \) if \( n \) is even, or belong to \( X_n = 0, X_{n-1} = 0, \) or \( X_n = X_{n-1} = X_n \) if \( n \) is odd.

*Proof.* \( n \) even. Since the coordinates of the points of \( K_n \) are constructed inductively from those of the points of \( K_2 \) (2.2 (2) and 3.1), the coordinates of the points of \( L_n \) are

\[
(\lambda^2, \lambda, t_2^2, \ldots, t_{n-2}^2, t_{n-1}, 1) \tag{4}
\]

or

\[
(\lambda^2, \lambda, a_3, a_4, \ldots, a_{n-1}, a_n, 0) \tag{5}
\]

where all pairs \((a_{2i-1}, a_{2j}) = (t_{2i}^2, t_i)\), except for one pair \((a_{2j-1}, a_2) = (t_{2j}^2 + 1, t_j)\) or \((a_{2j-1}, a_2) = (t_{2j}^2, t_j + 1)\). The two cases \((t_{2j}^2 + 1, t_j)\) and \((t_{2j}^2, t_j + 1)\) are equivalent since substituting \( t_j = t_{2j}^2 + 1, (t_{2j}^2 + 1, t_j) = (t_{2j}^2, t_{2j}^2 + 1)\), so we only consider \((a_{2j-1}, a_2) = (t_{2j}^2 + 1, t_j)\).
As already indicated in the proof of Lemma 3.4, the planes \( \langle e_0, e_1, (0, 0, x_0, \ldots, x_{n-2}) \rangle \in K_{n-2} \), form a complete cap in the projective space \( P^2(n-2, q) \) defined by the planes through \( L \).

This shows that every plane through \( L \) belongs to a three-dimensional space determined by two planes \( \Pi_1 = \langle e_0, e_1, (0, 0, x_0, \ldots, x_{n-2}) \rangle \), \( \Pi_3 = \langle e_0, e_1, (0, 0, y_0, \ldots, y_{n-2}) \rangle \) where \( (x_0, \ldots, x_{n-2}), (y_0, \ldots, y_{n-2}) \in K_{n-2} \).

Considering the cap \( \{ (l^2, \lambda, x_0, \ldots, x_{n-2}) \mid l \in F_p^2 \} \cup \{ (l^2, \lambda, y_0, \ldots, y_{n-2}) \mid l \in F_p^2 \} \cup \{ e_0, e_1 \} \), from the result by Segre (Theorem 2.1), the whole space \( \langle \Pi_1, \Pi_2 \rangle \) is saturated apart from one plane \( L \) and a conic \( C \) are saturated. We now determine this plane and this conic. As before, the points of \( C \setminus \{ e_0 \} \) are the intersection of bisecants to the \((2q+2)\)-cap \( L_n \cap \langle \Pi_1, \Pi_2 \rangle \) (2.1 (1)).

First of all, the point \( r_1 = (0, 0, x_0+y_0, \ldots, x_{n-2}+y_{n-2}) \) is the intersection of the bisecants \( \langle (1, 1, x_0, \ldots, x_{n-2}), (1, 1, y_0, \ldots, y_{n-2}) \rangle \) and \( \langle (0, 0, x_0, \ldots, x_{n-2}), (0, 0, y_0, \ldots, y_{n-2}) \rangle \), while \( \langle (1, 1, x_0, \ldots, x_{n-2}), (0, 0, y_0, \ldots, y_{n-2}) \rangle \) and \( \langle (0, 0, x_0, \ldots, x_{n-2}), (1, 1, y_0, \ldots, y_{n-2}) \rangle \) intersect in \( r_2 = (1, 1, x_0+y_0, \ldots, x_{n-2}+y_{n-2}) \). So \( \Pi_3 = \langle e_0, e_1, (0, 0, x_0+y_0, \ldots, x_{n-2}+y_{n-2}) \rangle \) and in \( \Pi_3 \) only \( L \) and \( C = \{ (l^2, \lambda, x_0+y_0, \ldots, x_{n-2}+y_{n-2}) \mid l \in F_p^2 \} \), which is the conic passing through \( r_1, r_2, p = e_0 \) and having nucleus \( e_1 \), are saturated (2.1 (1), 2.2 (1)).

For two planes \( \Pi_1 \) and \( \Pi_2 \) determined by points of the same type (4) or (5), \( x_{n-2}+y_{n-2} = 0 \), so \( \Pi_3 \) lies in \( X_n = 0 \). For a plane \( \Pi_1 \) determined by a point of type (4), and a plane \( \Pi_3 \) determined by a point of type (5), the non-saturated plane is \( \Pi_3 = \langle e_0, e_1, (0, 0, t_2^1+a_3, t_2+a_4, \ldots, t_{n-2}+a_{n-1}, t_{n-2}+a_{n-1}) \rangle \). Knowing that all pairs \( (a_{2i-1}, a_{2i}) = (l_j^2+1, l_j) \), apart from one pair \( (a_{2i-1}, a_{2i}) = (l_j^2+1, l_j) \), also in \( (t_1^2+a_3, t_2+a_4, \ldots, t_{n-2}+a_{n-1}, t_{n-2}+a_{n-1}) \), all pairs \( (t_1^2+a_{2i-1}, t_i+a_{2i}) = ((t_i+l_j)^2, t_i+l_j) \), apart from one pair \( (t_1^2+a_{2i-1}, t_i+a_{2i}) = ((t_i+l_j)^2+1, t_i+l_j) \).

But Lemma 2.3 shows that this plane \( \Pi_3 \) is linearly dependent on two planes \( \Pi_1 = \langle e_0, e_1, (0, 0, s_{n-2}^2, s_{n-2}, 1) \rangle \), \( \Pi_2 = \langle e_0, e_1, (0, 0, u_{n-2}^2, u_{n-2}, u_{n-2}^2, n_{n-2}, 1) \rangle \). The nonsaturated plane in \( \langle \Pi_1, \Pi_2 \rangle \) lies in \( X_n = 0 \), so this is not \( \Pi_1 \). Hence \( \Pi_3 \) is saturated.

This shows that all non-saturated planes lie in \( X_n = 0 \).

\( n \) odd. Also here, the coordinates of the points of \( L_n \) are constructed inductively, starting from the coordinates of the points of \( K_1 \) and \( K_3 \) (2.2). Hence, for \( \{ (l^2, \lambda, x_0, \ldots, x_{n-2}) \in L_n, (x_0, \ldots, x_{n-2}) \in K_{n-2} \} \), the 2-tuple \( (x_{n-3}, x_{n-2}) \in F_p^2 \).

So, repeating the calculations for \( n \) even, for a non-saturated plane \( \Pi_3 = \langle e_0, e_1, (0, 0, x_0+y_0, \ldots, x_{n-2}+y_{n-2}) \rangle \), \( (x_{n-3}+y_{n-3}, x_{n-2}+y_{n-2}) \in F_p^2 \), which shows that this plane is contained in \( X_n = 0, X_{n-1} = 0 \) or \( X_{n-2} = 1 = X_n \).
Lemma 3.6. Using the notations of Remark 3.2 (2), in $X_n = 0$, $n \geq 4$, the cap $L_n$ saturates all points except for the plane $\gamma_n$, in which only $L$ and $C$ are saturated.

Proof. From the proof of 3.5, the points of $L_n$ in $\Pi_1 = \langle e_0, e_1, (0, 0, x_0, ..., x_{n-2}) \rangle$, $\Pi_2 = \langle e_0, e_1, (0, 0, y_0, ..., y_{n-2}) \rangle$, $(x_0, ..., x_{n-2}) \in \mathcal{K}_{n-2}$, $(y_0, ..., y_{n-2}) \in \mathcal{K}_{n-2}$, saturate all points in $\langle \Pi_1, \Pi_2 \rangle$, except for points in $\Pi_3 = \langle e_0, e_1, (0, 0, x_0 + y_0, ..., x_{n-2} + y_{n-2}) \rangle$ in which only $L$ and $C$ are saturated. This plane $\Pi_3$ is only contained in $X_n = 0$ if $x_{n-2} = y_{n-2}$.

$n$ even. Here the non-saturated planes are $\Pi_1 = \langle e_0, e_1, (0, 0, (t_2 + l_2)^2, t_2 + l_2, ..., (t_{n-2} + l_{n-2})^2, t_{n-2} + l_{n-2}, 0) \rangle$ if $\Pi_1$ and $\Pi_2$ are defined by points of type (4) with parameters $t_i$ and $l_i$, or $\Pi_3 = \langle e_0, e_1, (0, 0, a_1 + b_1, ..., a_6 + b_6, 0) \rangle$ if $\Pi_1$ and $\Pi_2$ are defined by points of type (5) with coordinates $a_i$ and $b_i$. For $t_2 = l_2 + 1$ and $t_2 = l_2 > 2$, $\Pi_3 = \gamma_n$ and then only $L$ and $C$ are saturated.

Consider now the points of $L_n$ in $X_n = 0$. These are in fact the points of the cap $L_{n-1}$ in $PG(n-1, q)$ constituted by $X_n = 0$. This is true for $n = 4$ if one considers the points of $L_4$ in $X_4 = 0$ and the points of $L_3$ in $PG(3, q)$ (2.2). This is then valid for all even $n$ since the coordinates are constructed inductively.

It then follows from the assumption made in Remark 3.2 (2) that in $X_n = 0$, all points are saturated apart from the points in $\gamma_n$ in which only $L$ and $C$ are saturated.

$n$ odd. We first look at the points of $L_n$ with last coordinate equal to zero. For $L_n$ in $PG(5, q)$, these points constitute the set $L_n$ in $PG(4, q)$. So inductively, $L_n \cap (X_n = 0)$ is the cap $L_{n-1}$ in $PG(n-1, q)$. It then follows from Remark 3.2 (2) that $L_n \cap (X_n = 0)$ saturates all points in $X_n = 0$, except for $\gamma_n$ in which only $L$ and $C$ are saturated.

But $\gamma_n$ could be a (non-)-saturated plane for other planes $\Pi_1, \Pi_2$ with $\gamma_n \in \langle \Pi_1, \Pi_2 \rangle$.

Consider the points of $L_n$ with last coordinate equal to one. Starting from the coordinates of $K_3$ (2.2 (3)), these points are

\[(\lambda^2, \lambda, t_2^2, t_2, ..., t_{n-1}^2, t_{n-1} + 1, 0, 1)\] (6)

or

\[(\lambda^2, \lambda, t_2^2, t_2, ..., t_{n-1}^2, t_{n-1}^2 + 1, t_{n-1}^2 + 1, 1).\] (7)

The non-saturated plane defined by $\Pi_3 = \langle e_0, e_1, (0, 0, t_2^2, t_2, ..., t_{n-1}^2, t_{n-1}^2 + 1, t_{n-1}^2 + 1, 1) \rangle$ is $\gamma_n$ when $t_2 = l_2 + 1$ and $t_i = l_i$ if $i > 2$. For these values of $t_i$ and $l_i$, the conic saturated in $\gamma_n$ again is $C$. 

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Considering two planes defined by two points of type (7) gives the same conclusion.

Considering now \( \Pi_1 = \langle e_0, e_1, (0, 0, t_2^1, t_2^2, \ldots, t_{(n-1)/2}^1, t_{(n-1)/2}^0, 0, 1) \rangle \) and \( \Pi_2 = \langle e_0, e_1, (0, 0, t_2^2, t_2^3, \ldots, t_{(n-1)/2}^2, t_{(n-1)/2}^3, 0, 1, 1) \rangle \) gives as non-saturated plane, \( \Pi_3 = \langle e_0, e_1, (0, 0, (t_2^1 + t_3^0), t_2^2, t_2^3, \ldots, t_{(n-1)/2}^1 + t_{(n-1)/2}^2, t_{(n-1)/2}^2 + 1, t_{(n-1)/2}^2 + 1, 1, 0) \rangle \) which is never \( \gamma_n \).

If \( \gamma_n \subseteq \langle \Pi_1, \Pi_2 \rangle \) where \( \Pi_1 \) and \( \Pi_2 \) are defined as above for \( x_{n-2} = 0 \) and \( x_{n-3} = 1 \), then \( \gamma_n = \Pi_1 \) which is impossible if one considers the inductive construction of \( L_n \).

The conclusion is that all points in \( X_n = 0 \) are saturated apart from points in \( \gamma_n \), in which only \( L \) and \( C \) are saturated.

**Lemma 3.7.** Using the notations \( \gamma_n \) and \( C \) of Remark 3.2 (2), for \( n \geq 5 \), in \( X_{n-1} = 0 \), all points are saturated apart from points in \( \gamma_n \), in which only \( L \) and \( C \) are saturated.

**Proof.** The coordinates for \( L_n \) in \( PG(n, q) \), \( n \) odd, are constructed inductively from the coordinates of \( K_n \) and \( K_3 \) (2.2). From this, it follows immediately that the mapping \( \sigma: (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{n-2}, x_n, x_{n-1}) \) fixes \( L_n \).

Since \( \sigma \) interchanges \( X_n = 0 \) and \( X_{n-1} = 0 \) and since \( \sigma \) fixes \( \gamma_n \) point by point, the conclusion of the preceding lemma also holds in this case.

**Lemma 3.8.** For \( n \geq 5 \), in \( X_{n-1} = X_n \), all points are saturated apart from points in \( \gamma_n \), in which only \( L \) and \( C \) are saturated.

**Proof.** We first look at the points of \( L_n \) in \( X_n = 0 \) and \( X_{n-1} = X_n \). Since their coordinates are constructed inductively from \( K_1 \) or \( K_3 \), these two sets both contain the points of \( L_n \cap (X_n = X_{n-1} = 0) \), while the remaining points of \( L_n \cap (X_n = 0) \) have coordinates \((x_0^2, x_1^2, t_2^1, t_2^2, \ldots, t_{(n-1)/2}^1, t_{(n-1)/2}^1, 1, 0) \), and the remaining points of \( L_n \cap (X_n = X_{n-1} = X_n) \) are \((x_0^2, x_1^2, t_2^2, t_2^3, \ldots, t_{(n-3)/2}^2, t_{(n-3)/2}^2 + 1, 1, 1) \).

So applying \( \sigma: (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{n-4}, x_{n-4} + x_n, x_{n-2}, x_{n-1}, x_n) \) fixes \( X_n = 0 \), fixes \( X_{n-1} = X_n \), fixes the plane \( \gamma_n \) point by point, and maps \( L_n \cap (X_{n-1} = X_n) \) onto a cap equivalent to \( L_n \cap (X_n = 0) \).

But, as already indicated in the proof of Lemma 3.6, \( L_n \cap (X_n = 0) \) is in fact the cap \( L_{n-1} \) in \( PG(n-1, q) \), so using 3.2 (2), all points in \( X_{n-1} = X_n \) are saturated by \( L_n \cap (X_{n-1} = X_n) \) apart from points in \( \gamma_n \) in which only \( L \) and \( C \) are saturated.

Since \( \gamma_n \subseteq (X_n = 0) \), we already know from Lemma 3.6 that in \( \gamma_n \) only \( L \) and \( C \) are saturated by \( L_n \). Hence, the lemma is proved.
Notice that Theorem 3.5 and the three preceding lemmas show that the assumption made in 3.2 (2) is valid when constructing $L_n$ from the complete cap $K_{n-2}$ in $PG(n-2, q)$.

**Theorem 3.9.** In $PG(n, q)$, $q$ even, $q \geq 4$, $K_n = L_n \cup \{ (\lambda^2 + 1, \lambda, 1, 1, 0, ..., 0) \mid \lambda \in \mathbb{F}_q \}$ is a complete cap.

**Proof.** It follows from the three preceding lemmas that only points of $\mathbb{F}_n$ can extend $L_n$ to a larger cap.

Reasoning as in $PG(3, q)$ (see 2.2 (3)) and adding the points $(\lambda^2 + 1, \lambda, 1, 1, 0, ..., 0), \lambda \in \mathbb{F}_q$, to $L_n$, a complete cap $K_n$ is obtained.

**Theorem 3.10.** In $PG(2n, q)$, $q$ even, $q \geq 4$, there exists a complete $(q^n + 3(q^{n-1} + \cdots + q) + 2)$-cap and in $PG(2n + 1, q)$, $q$ even, $q \geq 4$, there exists a complete $(3(q^n + \cdots + q) + 2)$-cap.

**Proof.** It is known that $K_1$, $K_2$ and $K_3$ (Remark 2.2) are complete. The completeness of $K_m$ in $PG(m, q)$, $m \geq 4$, $q \geq 4$, was proved in Theorem 3.9.

The order of the complete caps follows from the inductive construction of $K_m$ since $|K_1| = 2, |K_2| = q + 2$ and $|K_m| = q|K_{m-2}| + q + 2$ (see 3.1).

**Corollary 3.11.** In $PG(2n, q)$, $q$ even, $q \geq 4$, $\sqrt{2}qq^{n-1} < n_2(2n, q) \leq q^n + 3(q^{n-1} + \cdots + q) + 2$ and in $PG(2n + 1, q)$, $q$ even, $q \geq 4$, $\sqrt{2}q^n < n_2(2n+1, q) \leq 3(q^n + \cdots + q) + 2$.

**Remark 3.12.** The orders of these caps are comparable to the size of $K_2$ in $PG(2, q)$ and to the size of $K_3$, the cap by Segre, in $PG(3, q)$ (2.2 (3)).

In $PG(2, q)$, any point not on $K_2$ belongs to $q/2 + 1$ bisecants. Also for $PG(2n, q)$, a point not on $K_{2n}$ belongs, on average, to $q/2$ bisecants.

For the cap $K_3$, a point not on $K_3$ belongs to, on average, $9/2$ bisecants. This is also the average number of bisecants in $PG(2n + 1, q)$ through a point $r$ not belonging to $K_{2n+1}$.

### 4. Small Complete Caps in $PG(n, q)$, $q$ Even

We now compare the size of the complete caps constructed in Section 3 to the sizes of other known complete caps in $PG(n, q)$, $q$ even.

We first mention the following result by E. M. Gabidulin, A. A. Davydov and L. M. Tombak. Rephrasing their results on binary codes [6, Theorems 4 and 5]:

**Theorem 4.1.** In $PG(2m - 1, 2)$, $m \geq 5$, complete $(15 \cdot 2^{m-3} - 3)$-caps and in $PG(2m - 2, 2)$, $m \geq 6$, complete $(23 \cdot 2^{m-4} - 3)$-caps exist.
In [17], G. Tallini presented a complete $k$-cap in $PG(4, q)$, whose construction is somewhat related to that of the complete $(3q+2)$-cap in $PG(3, q)$, $q$ even, by B. Segre (2.1).

Starting from two elliptic quadrics $e_1, e_2$ in hyperplanes $\Pi_1, \Pi_2$ of $PG(4, q)$ intersecting in a conic of $\Pi_1 \cap \Pi_2$, he proved:

**Theorem 4.2.** A complete $k$-cap containing $e_1$ and $e_2$ can contain at most $2q^2 + q + 5$ points if $q > 3$ and at most $2q^2 + q + 3$ points if $q = 2, 3$.

If $q$ is even, $q > 2$, then there exist complete $(2q^2 + q + 5)$-caps in $PG(4, q)$ containing $e_1$ and $e_2$.

From the size $k = q^2 + 3q + 2$ of the complete caps constructed in Section 3, the newly constructed caps contain half as many points as the caps by Tallini.

The final construction on complete caps in $PG(n, q)$, $q$ even, we would like to present again is due to B. Segre [14, p. 93].

**Theorem 4.3.** Let $i \in F_4 \setminus \{0, 1\}$ and let $K_i$ be the set of points, distinct from the unit point, having coordinates $(x_0, \ldots, x_n)$ where $x_j \in \{1, i\}$, $j = 0, \ldots, n$.

Then $K_i$ is a complete $(2^n + 1 - 2)$-cap of $PG(n, 4)$.

**Remark 4.4.** Considering the order of the caps of Section 3, also these caps in $PG(n, 4)$ contain $2^n + 1 - 2$ points.

We now investigate whether the caps of Section 3 coincide with the caps $K_i$ when $q = 4$.

The following lemma presents a description of $K_i$ which will be useful in solving the equivalence problem.

**Lemma 4.5.** The cap $K_i$ of Theorem 4.3 is the set of points of a cone $p;\beta$, where $p = (1, \ldots, 1)$ and where $\beta$ is a subgeometry $PG(n-1, 2)$ in $X_0 = 0$, not belonging to a subgeometry $PG(n, 2)$ containing $\beta$ and $p$.

**Proof.** The unit point belongs to the $2^n - 1$ bisecants $\langle (x_0, \ldots, x_n), (y_0, \ldots, y_n) \rangle$ where $\langle x_0, \ldots, x_n \rangle, \langle y_0, \ldots, y_n \rangle \in K_i$, $x_j + y_j = 1 + i$, $j = 0, \ldots, n$.

Projecting the cap from $p$ onto $X_0 = 0$ gives a subgeometry $\beta \equiv PG(n-1, 2))$ in $X_0 = 0$. So $K_i \subseteq p\beta$.

Consider the subgeometry $PG(n, 2)$ of $PG(n, 4)$ defined by $\beta, e_0$ and $p$. Since this subgeometry $PG(n, 2)$ is contained in $p\beta$, but $PG(n, 2) \cap K_i = \emptyset$, $K_i = p\beta \setminus PG(n, 2)$. 

So if the two caps coincide, there exists a point $p$ in $PG(n, 4)$ projecting the cap $K_i$ of Theorem 3.9 onto a subgeometry $PG(n-1, 2)$. This point $p$ only belongs to bisecants of $K_i$. We first determine all points of $PG(n, 4)$ only belonging to bisecants of $K_i$. 
Lemma 4.6. Let \( K_n \) be the complete cap constructed in Theorem 3.9.

The points of \( PG(n, 4), n \geq 4 \), only belonging to bisecants of \( K_n \), are the points of \( L = \langle e_0, e_1 \rangle \) and \( C = \{ (\lambda^2, \lambda, 1, 1, 0, \ldots, 0) \| \lambda \in F_4 \} \).

Proof. The cap constructed in Theorem 3.9 consists of conics, in planes through \( L \), tangent to \( L \) in \( e_0 \) and having nucleus \( e_1 \). If a point \( p \) only belongs to bisecants of \( K_n \), then \( p \) also belongs to a bisecant through \( e_0 \), so \( p \) belongs to a plane \( \Pi_1 \) through \( L \) containing a conic of \( K_n \).

Suppose \( p \not\in L \). Consider a bisecant through \( p \) containing a point of a conic of \( K_n \) in a plane \( \Pi_2 \), with \( \Pi_2 \not\parallel \Pi_1 \). Suppose this bisecant contains a point of \( K_n \) in a third plane \( \Pi_3 \), then \( \Pi_1, \Pi_2, \Pi_3 \) are three planes through \( L \) containing conics of \( K_n \) and belonging to the same three-dimensional space through \( L \).

When constructing the cap \( K_n \) in \( PG(n, 4) \), first of all a complete cap, isomorphic to \( K_{n-2} \) in \( PG(n-2, 4) \), of planes through \( L \) was constructed. In these planes, points on conics determined the cap \( L_n \) (3.1 and 3.2) and only at the end, a conic in the plane \( \gamma_n \) was added to obtain \( K_n \) (3.9). So \( \gamma_n \in \{ \Pi_1, \Pi_2, \Pi_3 \} \).

The reasoning made above is valid for all planes \( \Pi_2 \), distinct from \( \Pi_1 \), containing a conic of \( K_n \), so \( \Pi_1 \) is linearly dependent on \( |K_{n-2}|/2 \) pairs \( \{ \Pi_2, \Pi_3 \} \) where \( \Pi_2, \Pi_3 \) contain conics of \( K_n \).

So \( \Pi_1 = \gamma_{n-1} \). In \( \gamma_{n-1} \), only \( L \) and \( C \) are saturated by \( L_n \) (3.2 (2)). So \( p \in C \).

Lemma 4.7. In \( PG(n, 4), n \geq 4 \), the points of \( L \setminus \{ e_0, e_1 \} \) do not project the cap \( K_n \), constructed in Theorem 3.9, onto a subgeometry \( PG(n-1, 2) \) of \( X_0 = 0 \).

Proof. \( n \) even. The points of \( K_n \) are (4), (5), \( e_0, e_1 \) and \((\lambda^2 + 1, \lambda, 1, 1, 0, \ldots, 0), \lambda \in F_4 \).

Projecting from a point of \( L \setminus \{ e_0, e_1 \} \) onto \( X_0 = 0 \) only affects the first two coordinates of a point. Consider therefore only the following last \( n-1 \) coordinates \((t_2, t_2, \ldots, t_{n/2}, t_{n/2}, t_{n/2}, 1), (a_3, a_4, \ldots, a_{n-1}, a_n, 0) \) and \((1, 1, 0, \ldots, 0)\) of points of \( K_n \), which are also the last \( n-1 \) coordinates of projected points of \( K_n \).

From the inductive construction starting with the cap \( K_2 \) in \( PG(2, 4) \) (2.2 (2)), there is a point \((a_3, a_4, \ldots, a_{n-1}, a_n, 0) \) with \((a_{n-1}, a_n) = (0, 1) \).

If the projection of \( K_n \) onto \( X_0 = 0 \) is a subgeometry \( PG(n-1, 2) \), then two projected points must be collinear with a third projected point, so starting with \((t_2, t_2, \ldots, t_{n/2}, t_{n/2}, 1) \) and \((a_3, a_4, \ldots, a_{n-2}, 0, 1, 0)\), for some \( \lambda \in F_4 \setminus \{ 0 \} \), \((t_2 + \lambda a_3, \ldots, t_{n/2} + \lambda a_{n/2}, t_{n/2} + \lambda, 1) \) are the last \( n-1 \) coordinates of a projected point of \( K_n \).

The only possible last \( n-1 \) coordinates for the third projected point must be of type \((m_2^2, m_2, \ldots, m_{n/2}, m_{n/2}, 1) \), so \( m_2^2 = t_{n/2}^2 \) and \( m_{n/2} = t_{n/2} + \lambda \).
Hence $\lambda = 0$, so these two projected points are not collinear with a third projected point.

$n$ odd. From the inductive construction, starting with $K_1$ in $PG(3, 4)$, $K_n$ contains the points $(\lambda^2, \lambda, t_2, t_2, \ldots, t_{(n-1)/2}, t_{(n-1)/2}, 1, 0)$, $(\lambda^2, \lambda, t_2, t_2, \ldots, t_{(n-1)/2}, 0, 1)$, and $(\lambda^2, \lambda, t_2, t_2, \ldots, t_{(n-1)/2}, t_{(n-1)/2} + 1, 1)$ while all other points of $K_n$ have last two coordinates equal to zero.

Projecting from a point of $L \setminus \{e_0, e_1\}$ onto $X_0 = 0$ again only affects the first two coordinates of a point. So, since $n + 1 \geq 6$, does not affect the last four coordinates.

As for $n$ even, the line through two projected points of $K_n$ with last four coordinates $(t_{(n-1)/2}^2, t_{(n-1)/2}, 1, 0)$ and $(t_{(n-1)/2}^2, t_{(n-1)/2}, 0, 1)$ cannot contain a third projected point.

So the projection of $K_n$ is a subgeometry $PG(n - 1, 2)$.

**Lemma 4.8.** In $PG(n, 4), n \geq 6$, the projection of $K_n$ from a point $(\lambda^2, \lambda, 1, 1, 0, \ldots, 0), \lambda \in \mathbb{F}_4$, onto $X_2 = 0$ never is a $PG(n - 1, 2)$ of $X_2 = 0$.

**Proof.** $n$ even. Since $n + 1 \geq 7$, projecting from $(\lambda^2, \lambda, 1, 1, 0, \ldots, 0), \lambda \in \mathbb{F}_4$, onto $X_2 = 0$ does not affect the last three coordinates of a point.

Hence the reasoning of the preceding lemma can be copied.

$n$ odd. Since $n + 1 \geq 8$, projecting onto $X_2 = 0$ does not change the last four coordinates. Once again, the conclusion of the preceding lemma holds.

**Lemma 4.9.** In $PG(n, 4), n = 4, 5$, the cap $K_\lambda$ of Theorem 3.9 and the cap $K_{\lambda'}$ of Theorem 4.3 are not equivalent.

**Proof.** For $n = 4$, the two caps can only be equal if the projection of $K_\lambda$ from a point of $(\lambda^2, \lambda, 1, 1, 0) \in \mathbb{F}_4$, onto $X_2 = 0$, is a $PG(3, 2)$ in $X_2 = 0$.

Consider the cap $K_\lambda = \{ (t_{(n-1)/2}^2, t_{(n-1)/2}, t_1, 1) | t_1, t_2 \in \mathbb{F}_4 \} \cup \{ (t_{(n-1)/2}^2, t_{(n-1)/2}, 1, 0, 0) \}

The projection of $K_\lambda$ onto $X_2 = 0$ is the set $\{ (t_{(n-1)/2}^2, t_2, 0, 1, 0) | t_2 \in \mathbb{F}_4 \} \cup \{ (t_{(n-1)/2}^2, 0, 1, 0, 0) | t_2 \in \mathbb{F}_4 \} \cup \{ (t_{(n-1)/2}^2 + 1, t_2, 1, 1, 0) | t_2 \in \mathbb{F}_4 \}

The line $\langle (t_{(n-1)/2}^2, t_2, 0, 1, 0) | t_2 \in \mathbb{F}_4 \rangle$ does not contain a third projected point, so the projection of $K_\lambda$ from this point is not a subgeometry $PG(3, 2)$ in $X_2 = 0$.

The other points $(\lambda^2, \lambda, 1, 1, 0)$ give the same conclusion, so $K_\lambda$ and $K_{\lambda'}$ do not coincide.

The non-equivalence of the two caps in $PG(5, 4)$ is proved in the same way.
Theorem 4.10. The constructions of the caps $K_n$ of Theorem 3.9 and the caps $K'_n$ of Theorem 4.3 only are equivalent in $PG(n, 4)$, $n = 1, 2, 3$.

Proof. The cases $PG(1, 4)$ and $PG(2, 4)$ are trivial since there is a unique 2-cap and 6-cap in respectively $PG(1, 4)$ and $PG(2, 4)$.

In $PG(3, 4)$, both caps are complete 14-caps. The uniqueness of the complete 14-cap in $PG(3, 4)$ was proved by J. W. P. Hirschfeld and J. A. Thas [8, Theorem 3.2].

The non-equivalence of $K_n$ and $K'_n$ in $PG(n, 4)$, $n \geq 4$, follows from Lemmas 4.5 to 4.9.

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