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Extensions of sufficient conditions for starlikeness and convexity of order α for multivalent function

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ABSTRACT

In this paper, we obtain extensions of sufficient conditions for analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ to be starlike and convex of order α. Our results unify and extend some starlikeness and convexity conditions for analytic functions discussed by Mocanu (1988) [4], Uyanik et al. (2011) [3] and others.

1. Introduction

Let $\mathcal{A}_p(n)$ be the class of functions of the form

$$f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\})$$

which are analytic in open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

In particular,

$$\mathcal{A}_1(n) = \mathcal{A}(n).$$

A function $f(z) \in \mathcal{A}_p(n)$ is said to be *starlike* of order α in $\mathbb{U}$ if and only if it satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U};\ 0 \leqslant \alpha < p).$$

We denote by $\mathcal{S}_p^*(n, \alpha)$, the subclass of $\mathcal{A}_p(n)$ consisting of all functions $f(z)$ which are starlike of order α in $\mathbb{U}$ and in particular, $\mathcal{S}_1^*(n, 0) \equiv \mathcal{S}^*(n, 0)$ and $\mathcal{S}_1^*(1, 0) \equiv \mathcal{S}^*$.

A function $f(z) \in \mathcal{A}$ is said to be *convex* of order α in $\mathbb{U}$ if and only if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{U}; 0 \leqslant \alpha < p).$$

Also we denote by $\mathcal{C}_p(n, \alpha)$, the subclass of $\mathcal{A}_p(n)$ consisting of all functions $f(z)$ which are convex of order α in $\mathbb{U}$ and in particular, $C_1(n, 0) \equiv \mathcal{C}(n, 0)$ and $C_1(1, 0) \equiv \mathcal{C}$.

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2. Conditions for starlikeness of order α

We need the following lemma due to Mocanu ([1]; see also [2]) in order to consider the starlikeness of order α for $f(z) \in \mathcal{A}_p(n)$.

Lemma. *If $f(z) \in \mathcal{A}(n)$ satisfies the condition*

$$|f'(z) - 1| < \frac{n+1}{\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U},\ n \in \mathbb{N}),$$

then

$$f(z) \in \mathcal{S}^*(n, 0).$$

Theorem 1. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left| \left(\frac{f(z)}{z}\right)^{\frac{1}{p-\alpha}} \left(z^{\frac{1-\alpha}{p-\alpha}} \frac{f'(z)}{f(z)} - \alpha z^{\frac{1-p}{p-\alpha}} \right) - p + \alpha \right| < \frac{n+1}{\sqrt{(n+1)^2+1}}(p-\alpha) \quad (z \in \mathbb{U})$$

for some real values of $\alpha (0 \leq \alpha < p)$, then $f(z) \in \mathcal{S}_p^*(n, \alpha)$.

Proof. Let us define a function $h(z)$ by

$$h(z) = \left(\frac{f(z)}{z^\alpha}\right)^{\frac{1}{p-\alpha}} = z + \frac{a_{p+n}}{p-\alpha} z^{n+1} + \cdots \tag{2.1}$$

for $f(z) \in \mathcal{A}_p(n)$. Then $h(z) \in \mathcal{A}(n)$.

Differentiating (2.1) logarithmically, we find that

$$\frac{h'(z)}{h(z)} = \frac{1}{p-\alpha}\left[\frac{f'(z)}{f(z)} - \frac{\alpha}{z}\right] \tag{2.2}$$

which gives

$$\left|h'(z) - 1\right| = \frac{1}{p-\alpha} \left| \left(\frac{f(z)}{z}\right)^{\frac{1}{p-\alpha}} \left(z^{\frac{1-\alpha}{p-\alpha}} \frac{f'(z)}{f(z)} - \alpha z^{\frac{1-p}{p-\alpha}} \right) - p + \alpha \right|. \tag{2.3}$$

Thus using the condition given with the theorem, we get

$$\left|h'(z) - 1\right| \leq \frac{n+1}{\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U}). \tag{2.4}$$

Hence using the lemma, we have $h(z) \in \mathcal{S}^*(n, 0)$.

From (2.2), we infer that

$$\frac{zh'(z)}{h(z)} = \frac{1}{p-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right). \tag{2.5}$$

Since

$$h(z) \in \mathcal{S}^*(n, 0) \Rightarrow \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) > 0,$$

therefore from (2.5), we get

$$\frac{1}{p-\alpha}\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \alpha\right) = \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) > 0,$$

or

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (0 \leq \alpha < p,\ z \in \mathbb{U}).$$

Thus $f(z) \in \mathcal{S}_p^*(n, \alpha)$. $\square$

Theorem 2. *Let a function $h(z)$ be defined by*

$$h(z) = \left(\frac{f(z)}{z^{\alpha}}\right)^{\frac{1}{p-\alpha}}, \quad (0 \leq \alpha < p,\ z \in \mathbb{U}) \tag{2.6}$$

for $f(z) \in \mathcal{A}_p(n)$. If $h(z)$ satisfies

$$|h''(z)| \leq \frac{n+1}{\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U},\ 0 \leq \alpha < p), \tag{2.7}$$

then $f(z) \in \mathcal{S}_p^*(n, \alpha)$.

Proof. From (2.1), we have $h(z) \in \mathcal{A}(n)$. Also

$$\begin{aligned} |h'(z) - 1| &= \left|\int_o^z h''(t)dt\right| \\ &\leq \int_o^{|z|} |h''(\rho e^{i\theta})| d\rho \\ &\leq \frac{n+1}{\sqrt{(n+1)^2+1}}|z| \quad \text{(by the given condition (2.7))} \\ &< \frac{n+1}{\sqrt{(n+1)^2+1}} \quad (n \in \mathbb{N}\, z \in \mathbb{U}). \end{aligned}$$

This shows that $h(z)$ satisfies the condition of lemma. Thus $h(z) \in \mathcal{S}^*(n, 0)$, which leads to $f(z) \in \mathcal{S}_p^*(n, \alpha)$. □

Setting $\alpha = 1/2$ in Theorems 1 and 2, we obtain the following results.

Corollary 1. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left|\left(\frac{f(z)}{z}\right)^{2/(2p-1)} \left(z^{\frac{1}{2p-1}}\frac{f'(z)}{f(z)} - \frac{1}{2}z^{\frac{2(1-p)}{2p-1}}\right) - p + \frac{1}{2}\right| < \frac{(n+1)(2p-1)}{2\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U}, p \in \mathbb{N}), \tag{2.8}$$

then $f(z) \in \mathcal{S}_p^*(n, 1/2)$.

Corollary 2. *Let $f(z) \in \mathcal{A}_p(n)$ and a function $h(z)$ is defined by*

$$h(z) = \left(\frac{f(z)}{z^{1/2}}\right)^{2/(2p-1)} \quad (z \in \mathbb{U}, p \in \mathbb{N}). \tag{2.9}$$

If $h(z)$ satisfies

$$\left|h''(z)\right| < \frac{(n+1)}{\sqrt{(n+1)^2+1}},$$

then $f(z) \in \mathcal{S}_p^*(n, 1/2)$.

Remark. Putting $p = n = 1$ in Theorems 1 and 2, we get Theorems 2.1 and 2.2 established recently by Uyanik et al. [3]. Further for $\alpha = 0$, Theorem 1 gives a Lemma 2.1 by Mocanu [4].

3. Conditions for convexity of order α

In this section we obtain conditions for $f(z) \in \mathcal{A}_p(n)$ to be convex of order α in $\mathbb{U}$.

Theorem 3. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left|\left\{\frac{(f'(z))^{\alpha+1-p}}{pz^{p-1}}\right\}^{\frac{1}{p-\alpha}} \left[zf''(z) + (1-\alpha)f'(z)\right] - p + \alpha\right| < \frac{(n+1)(p-\alpha)}{\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U}) \tag{3.1}$$

for some real $\alpha (0 \leq \alpha < p)$, then $f(z) \in \mathcal{C}_p(n, \alpha)$.

Proof. Let us define a function $h(z)$ by

$$h(z) = \int_0^z \left(\frac{f'(t)}{pt^{p-1}}\right)^{\frac{1}{p-\alpha}} dt = z + \frac{p+n}{(n+1)p(p-\alpha)} a_{p+n} z^{n+1} + \cdots. \tag{3.2}$$

Further, let

$$g(z) = zh'(z) = z\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}} = z + \frac{p+n}{p(p-\alpha)} a_{p+n} z^{n+1} + \cdots. \tag{3.3}$$

Obviously $h(z)$ and $g(z) \in \mathcal{A}(n)$.

Now

$$g(z) = z\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}}.$$

Differentiating logarithmically, we find after some computation that

$$g'(z) = \left\{\frac{(f'(z))^{\alpha+1-p}}{pz^{p-1}}\right\}^{\frac{1}{p-\alpha}} \frac{1}{p-\alpha}\left[zf''(z) + (1-\alpha)f'(z)\right],$$

or

$$\left|g'(z) - 1\right| = \frac{1}{p-\alpha}\left|\left\{\frac{(f'(z))^{\alpha+1-p}}{pz^{p-1}}\right\}^{\frac{1}{p-\alpha}}\left[zf''(z) + (1-\alpha)f'(z)\right] - p + \alpha\right| < \frac{n+1}{\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U}).$$

Therefore, application of the lemma gives us that

$$g(z) = zh'(z) \in \mathcal{S}^*(n, 0) \Rightarrow h(z) \in \mathcal{C}(n, 0).$$

Since

$$\frac{zh''(z)}{h'(z)} = \frac{1}{p-\alpha}\left\{\frac{zf''(z)}{f'(z)} - (p-1)\right\}, \tag{3.4}$$

therefore

$$\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) = \mathrm{Re}\left[\frac{1}{p-\alpha}\left(1 - \alpha + \frac{zf''(z)}{f'(z)}\right)\right] \quad (z \in \mathbb{U},\ 0 \le \alpha < p)$$

which imply that

$$\frac{1}{p-\alpha}\mathrm{Re}\left(1 - \alpha + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (\text{as } h(z) \in \mathcal{C}(n, 0))$$

or

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha. \tag{3.5}$$

It follows from above that $f(z) \in \mathcal{C}_p(n, \alpha)$. This completes the proof of Theorem 3. □

Theorem 4. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left|f''(z)\left(\frac{(f'(z))^{1+\alpha-p}}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}} - \frac{(p-1)}{z}\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}}\right| \le \frac{(p-\alpha)(n+1)}{2\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U}) \tag{3.6}$$

for some real $\alpha (0 \le \alpha < p)$, then $f(z) \in C_p(n, \alpha)$.

Proof. Let

$$h(z) = \int_0^z \left(\frac{f'(t)}{pt^{p-1}}\right)^{\frac{1}{p-\alpha}} dt. \tag{3.7}$$

Then

$$zh'(z) = z\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}}. \tag{3.8}$$

Further, suppose that $g(z) = zh'(z)$. Then we obtain

$$g(z) = z + \frac{p+n}{p(p-\alpha)}a_{p+n}z^{n+1} + \cdots \in \mathcal{A}(n) \tag{3.9}$$

and

$$\begin{aligned} |g'(z) - 1| &= |h'(z) + zh''(z) - 1| \qquad (3.10)\\ &\le |h'(z) - 1| + |zh''(z)| \\ &= \left|\int_0^z h''(t)dt\right| + |zh''(z)| \\ &\le \int_0^{|z|}\left|\frac{1}{p-\alpha}\left[f''(t)\left(\frac{f'(t)^{1+\alpha-p}}{pt^{p-1}}\right)^{\frac{1}{p-\alpha}} - \frac{(p-1)}{t}\left(\frac{f'(t)}{pt^{p-1}}\right)^{\frac{1}{p-\alpha}}\right]dt\right| \\ &\quad + \left|\frac{1}{p-\alpha}z\left[f''(z)\left(\frac{f'(z)^{1+\alpha-p}}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}} - \frac{(p-1)}{z}\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}}\right]\right| \\ &\le \frac{(n+1)}{\sqrt{(n+1)^2+1}}|z| \\ &< \frac{(n+1)}{\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U}). \qquad (3.11) \end{aligned}$$

Thus, using the lemma, we obtain that $g(z) \in \mathcal{S}^*(n, 0)$ that is

$zh'(z) \in \mathcal{S}^*(n, 0)$.

This means that $h(z) \in \mathcal{C}(n, 0)$. Consequently, we infer that

$f(z) \in \mathcal{C}_p(n, \alpha)$. □

Setting $\alpha = 1/2$ in Theorems 3 and 4, we get the following corollary.

Corollary 3. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left|\left\{\frac{(f'(z))^{(3/2)-p}}{pz^{p-1}}\right\}^{\frac{2}{2p-1}}[2zf''(z) + f'(z)] - 2p + 1\right| < \frac{(n+1)(2p-1)}{\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U}, p \in \mathbb{N}), \tag{3.12}$$

then $f(z) \in \mathcal{C}_p(n, 1/2)$.

Corollary 4. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left|f''(z)\left(\frac{(f'(z))^{(3/2)-p}}{pz^{p-1}}\right)^{\frac{2}{2p-1}} - \frac{(p-1)}{z}\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{2}{2p-1}}\right| \le \frac{(n+1)(2p-1)}{4\sqrt{((n+1)^2+1)}} \quad (z \in \mathbb{U}), \tag{3.13}$$

then $f(z) \in \mathcal{C}_p(n, 1/2)$.

Putting $p = 1$ and $\alpha = 0$ in Theorem 4, we have the following corollary.

Corollary 5. *If $f(z) \in \mathcal{A}(n)$ satisfies*

$$\left|f''(z)\right| \le \frac{(n+1)}{2\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U}), \tag{3.14}$$

then $f(z) \in C(n, 0)$.

Remark. Setting $n = p = 1$ in Theorems 3 and 4, we get Theorems 3.1 and 3.2 obtained recently by Uyanik et al. [3]. Further for $\alpha = 0$, we get a result by Nunokawa et al. [5].

4. Generalized Alexander integral operator

For $f(z) \in \mathcal{A}_p(n)$, define

$$g(z) = \int_0^z \left(\frac{f(t)}{t^p}\right)^{\gamma} dt = z + \frac{\gamma}{n+1} a_{p+n} z^{n+1} + \cdots. \tag{4.1}$$

Here note that $g(z) \in \mathcal{A}(n)$, and for $p = 1$ and $\gamma = 1$ we obtain the well-known Alexander integral operator [6].
Our next theorem provides us the sufficient conditions for starlikeness for the generalized Alexander operator.

Theorem 5. *If $\gamma \geq \frac{1}{p}$ and $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\gamma \left| \frac{(f(z))^{\gamma}}{z^{p\gamma+1}} \left[\frac{zf'(z)}{f(z)} - p\right]\right| < \frac{n+1}{2\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U}) \tag{4.2}$$

then $f(z) \in \mathcal{S}_p^*(n, 0)$.

Proof. From (4.1), we get

$$g'(z) = \left(\frac{f(z)}{z^p}\right)^{\gamma}. \tag{4.3}$$

Now, differentiating (4.3) logarithmically and multiplied by 'z', we get

$$\frac{zg''(z)}{g'(z)} = \gamma \left[\frac{zf'(z)}{f(z)} - p\right]. \tag{4.4}$$

Therefore,

$$\left|g''(z)\right| = \gamma \left| \frac{(f(z))^{\gamma}}{z^{p\gamma+1}} \left[\frac{zf'(z)}{f(z)} - p\right]\right| \leq \frac{n+1}{2\sqrt{(n+1)^2+1}} \quad (z \in \mathbb{U}). \tag{4.5}$$

Since $g(z) \in \mathcal{A}(n)$, therefore by Corollary 5 we find that $g(z) \in \mathcal{C}(n, 0)$
From (4.4), we obtain

$$\begin{aligned} \mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) &= \gamma \mathrm{Re}\left(\frac{zf'(z)}{f(z)} - p\right) + 1 \\ &\Rightarrow \gamma \mathrm{Re}\left(\frac{zf'(z)}{f(z)} - p\right) + 1 > 0, \quad (\because\ g(z) \in \mathcal{C}(n, 0)) \\ &\Rightarrow \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > p - \frac{1}{\gamma} > 0, \quad (z \in \mathbb{U}) \end{aligned}$$

which proves that $f(z) \in \mathcal{S}_p^*(n, 0)$.
For $p = n = \gamma = 1$ in Theorem 5, we get a result recently obtained by Uyanik et al. [3, Theorem 4.1]. □

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