Discretely weak $P$-sets

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**Abstract**

A discretely weak $P$-set is a nowhere dense closed set which is disjoint from the closure of any countable discrete subset of its complement. We show that the Stone–Čech remainder $\mathbb{N}^*$ of the discrete space $\mathbb{N}$ of natural numbers cannot be covered by discretely weak $P$-sets when the continuum hypothesis is assumed.

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1. Introduction

The general study of whether an infinite compact space can be covered by certain families of its closed nowhere dense subsets is one of the ongoing interests in topology. We are interested in the problem of covering the compact space $\mathbb{N}^*$ by its closed nowhere dense $P$-sets. A $P$-set in a topological space is one with the property that the intersection of countably many of its neighborhoods is again a neighborhood of it. A $P$-point is a point $x$ such that $\{x\}$ is a $P$-set. A compact space in which every point is a $P$-point must be finite, hence not every point of $\mathbb{N}^*$ is a $P$-point. Though not all points of $\mathbb{N}^*$ are $P$-points, one may wonder whether $\mathbb{N}^*$ may be covered with ‘small’ $P$-sets. We take small to mean closed and nowhere dense.

In 1980, Kunen, van Mill, and Mills [8] showed that no compact space of $\pi$-weight $\omega_1$ can be covered by nowhere dense closed $P$-sets. In particular, assuming the continuum hypothesis, $\mathbb{N}^*$ cannot be covered by nowhere dense closed $P$-sets. In addition they also constructed a compact space of weight $\omega_2$ which can be covered by nowhere dense closed $P$-sets. Dow and van Mill [3] also showed that no compact space can be covered by nowhere dense ccc $P$-sets, that is, $P$-sets satisfying the countable chain condition. These results left open the possibility that it might be possible to define a model in which the space $\mathbb{N}^*$ can be covered by nowhere dense closed $P$-sets.

In the same year, Balcar, Frankiewicz, and Mills [1] showed that it is consistent that $\mathbb{N}^*$ is covered by nowhere dense closed $P$-sets. They constructed their model by adding $\omega_1$ Cohen reals to a model of ZFC + MA + $2^{\omega} = \omega_2$. Similar results followed later using the NCF principle.
The principle NCF (Near Coherence of Filters) says that two ultrafilters on \( \mathbb{N}^* \) are nearly coherent, that is, if \( u, v \in \mathbb{N}^* \), then there is a finite-to-one map \( f: \mathbb{N} \to \mathbb{N} \) such that \( \beta(f(u)) = \beta(f(v)) \). Using NCF it can be shown that \( \mathbb{N}^* \) can be covered by nowhere dense closed \( P \)-sets. This is can be done as follows: NCF implies that for every \( u \in \mathbb{N}^* \), there is a finite-to-one map \( f: \mathbb{N} \to \mathbb{N} \) such that \( v = f(u) \) is a \( P \)-point. Then the set \( f^{-1}(v) \) is a closed nowhere dense \( P \)-set of \( \mathbb{N}^* \). In 1992, Zhu [11] improved this by showing that NCF implies that \( \mathbb{N}^* \) can be covered by an increasing sequence of nowhere dense closed \( P \)-sets. These results show that the statement \( \mathbb{N}^* \) can be covered by nowhere dense \( P \)-sets is independent of the axioms of ZFC.

In this paper we will extend the study of covering \( \mathbb{N}^* \) by closed nowhere dense sets by investigating whether \( \mathbb{N}^* \) can be covered by its discretely weak \( P \)-sets. The fact that every closed nowhere dense \( P \)-set is a discretely weak \( P \)-set makes our study an extension of covering \( \mathbb{N}^* \) by closed nowhere dense \( P \)-sets.

2. Weak and discretely weak \( P \)-sets

A subset \( K \) of a space \( X \) is called a weak \( P \)-set provided that \( K \cap F = \emptyset \) for each countable set \( F \subseteq X \setminus K \). A weak \( P \)-point is a point \( x \) such that \([x]\) is a weak \( P \)-set. The advantage of weak \( P \)-points over \( P \)-points is that their existence is provable in ZFC. As is well known, Kunen [7] proved the existence of a dense set of weak \( P \)-points in \( \mathbb{N}^* \). It is easily seen that every \( P \)-set is a weak \( P \)-set.

A subset \( K \subseteq X \) is called a discretely weak \( P \)-set if it is a nowhere dense closed set and \( K \cap F = \emptyset \) for each countable discrete set \( F \subseteq X \setminus K \). A point \( x \) in \( X \) is called a discretely weak \( P \)-point if \( x \notin F \) for every countable discrete set \( F \subseteq X \setminus [x] \) (these are called “discretely uncountable” in [10]). It follows from their definitions that every \( P \)-point is a weak \( P \)-point and every weak \( P \)-point is a discretely weak \( P \)-point. But the converses of these statements are not true. For example, in \( \mathbb{N}^* \) there is a discretely weak \( P \)-point which is not a weak \( P \)-point and there is a weak \( P \)-point which is not a \( P \)-point. These facts are first proved by Kunen [6] assuming CH and then by van Mill [9] in ZFC.

To our knowledge, no one has investigated coverings by discretely weak \( P \)-sets or weak \( P \)-sets for that matter, and we will study this problem in this paper. In particular we improve the results of Kunen, van Mill, and Mills [8] by showing that \( \mathbb{N}^* \) cannot be covered by discretely weak \( P \)-sets under CH. The techniques and methods used in showing every two-to-one continuous image of \( \mathbb{N}^* \) is homeomorphic to \( \mathbb{N}^* \) [4] motivated us to work on this interesting covering problem. In the study of two-to-one continuous functions defined on \( \mathbb{N}^* \) we deal with covering \( \mathbb{N}^* \) by certain types of sets and the question of covering by discretely weak \( P \)-sets arises naturally.

3. Inverse systems

Let \( \kappa \) be an ordinal. Suppose that for every \( \alpha \in \kappa \) corresponds a topological space \( X_\alpha \) and that for any \( \alpha, \beta \in \kappa \) satisfying \( \beta \leq \alpha \) a continuous mapping \( f_{\alpha \beta}: X_\alpha \to X_\beta \) is defined; suppose further that \( f_{\alpha \gamma} = f_{\beta \gamma} \circ f_{\alpha \beta} \) for any \( \alpha, \beta, \gamma \in \kappa \) satisfying \( \gamma \leq \beta \leq \alpha \) and that \( f_{\alpha \alpha} = \text{id} \) for every \( \alpha \in \kappa \). In this situation we say that the family \( \{X_\alpha, f_{\alpha \beta}, \kappa\} \) is called an inverse system. The inverse limit \( \lim\{X_\alpha, f_{\alpha \beta}, \kappa\} \) of the inverse system \( \{X_\alpha, f_{\alpha \beta}, \kappa\} \) is the subspace of the product space \( \prod_{\alpha < \kappa} X_\alpha \) consisting of all points \( x = (x_\alpha) \) such that \( f_{\alpha \beta}(x_\alpha) = x_\beta \), for \( \beta < \alpha \). An inverse system \( \{X_\alpha, f_{\alpha \beta}, \kappa\} \) is called continuous provided that \( X_\gamma = \lim\{X_\alpha, f_{\alpha \beta}, \gamma\} \) for each limit \( \gamma < \kappa \).

A \( \pi \)-base \( \mathcal{B} \) for a space \( X \) is a family of nonempty open subsets of \( X \) such that each nonempty open set in \( X \) contains some \( B \in \mathcal{B} \). The \( \pi \)-weight, \( \pi(X) \), of \( X \) is the minimum cardinal \( \kappa \) for which there is a \( \pi \)-base for \( X \) of cardinality \( \kappa \). The \( \pi \)-weight of \( \mathbb{N}^* \) is \( \omega \). We now state an important statement without proof that is useful in our analysis of covering \( \mathbb{N}^* \).

**Proposition 1.** ([8]) If \( X = \lim\{X_\alpha, f_{\alpha \beta}, \omega_1\} \), where \( \pi(X_\alpha) < \omega_1 \) for each \( \alpha < \omega_1 \) and \( (X_\alpha, f_{\alpha \beta}, \omega_1) \) is continuous, then for each closed subset \( A \subseteq X \) with empty interior there is some \( \alpha < \omega_1 \) such that \( f_{\omega_1 \alpha}(A) \) has empty interior.

**Lemma 2 (CH).** \( \mathbb{N}^* \) can be written as a continuous inverse limit of compact zero-dimensional metric spaces, \( \mathbb{N}^* = \lim\{X_\alpha, f_{\alpha \beta}, \omega_1\} \), so that for each ordinal \( \alpha < \omega_1 \), \( f_{\alpha + 1, \alpha}: X_{\alpha + 1} \to X_\alpha \) can be factored as \( f_{\alpha + 1, \alpha} = g \circ h \) where \( h: X_{\alpha + 1} \to X_{\alpha} \times 2^{\omega_0} \) is onto and \( g: X_{\alpha} \times 2^{\omega_0} \to X_\alpha \) is the projection map.

**Proof.** Suppose \( \mathbb{N}^* \) is a continuous inverse limit of compact metric spaces \( \{X_\alpha, f_{\alpha \beta}, \omega_1\} \), this is possible since \( w(\mathbb{N}^*) = \omega_1 \) under CH. For all \( \alpha < \omega_1 \), we have \( f_{\omega_1 \alpha}: \mathbb{N}^* \to X_\alpha \) and \( \pi_\alpha: X_\alpha \times 2^{\omega_0} \to X_\alpha \) where \( \pi_\alpha \) is the projection function. Then by using one of the properties of \( \mathbb{N}^* \) under CH [2] there is a function \( f: X_\alpha \times 2^{\omega_0} \to X_\alpha \) such that \( f_{\omega_1 \alpha} = f \circ \pi_\alpha \).

Now since the collection \( f^{-1}(\text{CO}(X_\alpha \times 2^{\omega_0})) \) is countable in \( \mathbb{N}^* \), there is a \( \gamma < \alpha \) such that \( f^{-1}(\text{CO}(X_\alpha \times 2^{\omega_0})) \subseteq f_{\omega_1 \gamma}^{-1}(\text{CO}(X_\gamma)) \).

Then we re-index \( \gamma \) as \( \alpha + 1 \) and we get \( f_{\gamma, \alpha}: X_\gamma \to X_\alpha \). Then \( f_{\gamma, \alpha} \) can be factored as \( f_{\gamma, \alpha} = g \circ h \) where \( g_\alpha: X_\alpha \times 2^{\omega_0} \to X_\alpha \) and \( h: X_\gamma \to X_\alpha \times 2^{\omega_0} \). \( \square \)
4. Covering by discretely weak P-sets

We now show, in the presence of CH, that \( \mathbb{N}^* \) cannot be covered by discretely weak P-sets. That is, there is a point \( p \in \mathbb{N}^* \) such that \( p \notin K \) for all discretely weak P-sets \( K \) of \( \mathbb{N}^* \).

**Theorem 3 (CH).** \( \mathbb{N}^* \) cannot be covered by discretely weak P-sets.

**Proof.** We first write \( \mathbb{N}^* \) as a continuous inverse limit of compact metric spaces, \( \mathbb{N}^* = \varprojlim \{X_\alpha, f_{\alpha\beta}, \omega_1 \} \) as per Lemma 2.

We construct, by transfinite induction, countable sets \( D^\alpha_\beta \) for \( \beta \leq \alpha \leq \omega_1 \) such that the following properties hold:

1. \( D^\alpha_\beta \) is dense in \( X_\alpha \) and disjoint from \( \bigcup_{\beta < \alpha} D^\alpha_\beta \);
2. \( D^\beta_{\beta+1} \) is discrete in \( X_{\beta+1} \);
3. if \( \beta < \gamma \leq \alpha \), then \( f_{\beta\gamma} \mid D^\alpha_\beta = D^\gamma_\beta \) and \( f_{\alpha\beta} \mid D^\alpha_\beta \) is one-to-one;
4. if \( \beta_n < \beta_{n-1} < \cdots < \beta_1 \leq \alpha \) and \( U_i \subseteq X_{\beta_i} \) is dense open, then

\[
D^\alpha_{\beta_n} \subseteq \text{cl} \left[ D^\alpha_{\beta_1} \cap \bigcap_{i<n} f^{-1}_{\alpha,\beta_i}(U_i) \right].
\]

The points of \( D^\alpha_\beta \subseteq X_{\alpha_1} = \mathbb{N}^* \) will be the points which are not covered by closed nowhere dense weak P-sets and discretely weak P-sets.

Assume \( \alpha \in \omega_1 \) and we have constructed countable sets \( D^\alpha_\beta \subseteq X_\alpha \) for all \( \beta \leq \lambda < \alpha \) satisfying conditions (1)–(4).

For limit \( \alpha, X_\alpha = \varprojlim \{X_\gamma, f_{\gamma\beta}, \alpha \} \). By the definition of inverse limit for all \( \beta < \alpha \) and for every \( d \in D^\beta_\alpha \), if \( \beta < \lambda < \alpha \), then there is a unique point \( x(d, \lambda) \in D^\beta_\lambda \) such that \( f_{\lambda\beta}(x(d, \lambda)) = d \) and if \( \lambda < \beta \) let \( x(d, \lambda) = f_{\beta\lambda}(d) \). Thus \( \langle x(d, \lambda) \rangle_{\lambda < \alpha} \) is in \( X_\alpha \) which is the point in \( D^\beta_\alpha \) mapping to \( d \). Let \( D^\alpha_\beta \) be any countable dense subset of \( X_\alpha \) which is disjoint from \( \bigcup_{\beta < \alpha} D^\beta_\alpha \).

We now check our inductive conditions for limit \( \alpha \). Suppose \( \beta_1 < \alpha \). Applying induction hypothesis for all \( \gamma \) such that \( \beta_1 < \gamma < \alpha \), that is,

\[
D^\gamma_{\beta_1} \subseteq \text{cl}_{X_\gamma} \left[ D^\gamma_{\beta_1} \cap \bigcap_{i<n} f^{-1}_{\gamma,\beta_i}(U_i) \right].
\]

we must show that

\[
D^\beta_{\beta_1} \subseteq \text{cl}_{X_\beta} \left[ D^\beta_{\beta_1} \cap \bigcap_{i<n} f^{-1}_{\beta,\beta_i}(U_i) \right].
\]

Let \( d \in D^\beta_{\beta_1} \) and \( d \in \mathcal{W} \) for some clopen set \( \mathcal{W} \) in \( X_\beta \). Then there is \( \gamma \) such that \( \beta_1 < \gamma < \alpha \), \( W = f_{\alpha\gamma}^{-1} \{ f_{\alpha\gamma}(\mathcal{W}) \} \) and \( f_{\alpha\gamma}(\mathcal{W}) \) is clopen in \( X_\gamma \) and \( f_{\alpha\gamma}(d) \in D^\gamma_{\beta_1} \). Since

\[
D^\gamma_{\beta_1} \subseteq \text{cl}_{X_\gamma} \left[ D^\gamma_{\beta_1} \cap \bigcap_{i<n} f^{-1}_{\gamma,\beta_i}(U_i) \right]
\]

and \( f_{\alpha\gamma}(\mathcal{W}) \) is clopen in \( X_\gamma \) containing \( f_{\alpha\gamma}(d) \) we get

\[
f_{\alpha\gamma}(\mathcal{W}) \cap \left[ D^\gamma_{\beta_1} \cap \bigcap_{i<n} f^{-1}_{\gamma,\beta_i}(U_i) \right] \neq \emptyset.
\]

Let \( e \in f_{\alpha\gamma}(\mathcal{W}) \cap \left[ D^\gamma_{\beta_1} \cap \bigcap_{i<n} f^{-1}_{\gamma,\beta_i}(U_i) \right] \). Then \( e' \in f_{\alpha\gamma}^{-1}(e) \cap D^\gamma_{\beta_1} \), thus \( e' \in \mathcal{W} \cap D^\alpha_{\beta_1} \cap \bigcap_{i<n} f^{-1}_{\alpha,\beta_i}(U_i) \).

If \( \beta_1 = \alpha \), we go down in the sequence to \( \beta_2 < \alpha \). We must show that

\[
D^\alpha_{\beta_2} \subseteq \text{cl} \left[ D^\alpha_{\beta_1} \cap \bigcap_{i<n} f^{-1}_{\alpha,\beta_i}(U_i) \right]
\]

and this is true since \( D^\alpha_{\beta_1} \) is dense and \( U_1 \) is open dense in \( X_\alpha \).

Now suppose \( \alpha = \lambda + 1 \). We construct countable sets \( D^{\lambda+1}_{\beta} \subseteq X_{\lambda+1} \) for each \( \beta \leq \lambda + 1 \). Let \( \{ r_n : n \in \omega \} \subset 2^\omega \) be a sequence converging to 0 in \( 2^\omega \). Let

\[
T = \left\{ \{ \beta_1, U_1 \}_{i < \omega} : n \in \omega, \beta_n < \beta_{n-1} < \cdots < \beta_1 \leq \lambda, U_i \subseteq X_{\beta_i} \text{ dense open} \right\}.
\]

Let \( \beta < \lambda + 1 = \beta_0 \) and \( \rho \in T \), say \( \rho = \{ (\beta_{i_0}, U_{i_0}), \ldots, (\beta_1, U_1) \} \) and fix maximal \( i_{\rho} \leq n_{\rho} \) such that \( \beta < \beta_{i_{\rho}} \).

By Lemma 2 \( f_{\lambda+1, \lambda} : X_{\lambda+1} \to X_\lambda \) can be factored as \( f_{\lambda+1, \lambda} = g \circ h \) where \( h : X_{\lambda+1} \to X_\lambda \times 2^\omega \) and \( g : X_\lambda \times 2^\omega \to X_\lambda \).

Let \( D^\lambda_{\beta_0} = \{ d_0^n : n \in \omega \} \), and for \( n \in \omega \), let \( D^\lambda_{\beta_0} = (d^\lambda_{\beta_0}, r_n) \). Then \( E_\lambda = \{ d^\lambda_{\beta_0} : n \in \omega \} \) is a discrete subset of \( X_\lambda \times 2^\omega \) and
projects one-to-one onto $D^\beta_\rho$. Since $D^\beta_\rho$ is dense in $X_\beta$, it follows that $g[\overline{X}_\beta] = X_\beta$; in fact $\overline{X}_\beta \supseteq X_\beta \times \{0\}$ and $g\upharpoonright_{X_\beta \times \{0\}}$ is a homeomorphism. For each $\beta < \lambda$, let $(d^\beta_\rho \lambda : \lambda \in \omega)$ be the one-to-one enumeration of $D^\beta_\rho \times \{0\}$, and observe that $f_\lambda \beta+1 \lambda (h^{-1}(d^\beta_\rho \lambda)) = g(d^\beta_\rho \lambda) \in D^\beta_\rho$ for all $\beta < \lambda$.

Then define $K(j, \beta, \rho)$ by induction on $i_\beta$, by

$$K(j, \beta, \rho) = h^{-1}(d^\beta_\rho j) \cap \bigcap_{i < i_\beta} \left[ \bigcup_K (l, \beta_i, \rho) \cap \bigcap_{i < m < i_\beta} f^{-1}_{\lambda+1, \beta_i} [U_m] \right].$$

**Lemma 4.** $K(j, \beta, \rho)$ is nonempty.

**Proof.** By induction on $i_\beta$, $K(l, \beta_i, \rho)$ is nonempty for each $l \in \omega$ and

$$K(l, \beta_i, \rho) \subseteq \bigcap_{m} \left[ \bigcup_K (m, \beta_i, \rho) \cap \bigcap_{i < m < i_\beta} f^{-1}_{\lambda+1, \beta_i} [U_m] \right].$$

We use induction assumption (4) for $i_\beta = 0$, and

$$f_\lambda \beta+1 \lambda \left( \bigcup_l K(l, \beta_i, \rho) \cap \bigcap_{i < i_\beta} f^{-1}_{\lambda+1, \beta_i} [U_l] \right) = D^\beta_\rho \cap \bigcap_{i < i_\beta} f^{-1}_{\lambda, \beta_i} [U_i]$$

and by induction assumption again

$$\bigcap_{i < i_\beta} \left[ \bigcup_K (\lambda, \beta_i, \rho) \cap \bigcap_{i < m < i_\beta} f^{-1}_{\lambda, \beta_i} [U_m] \right] \supseteq D^\beta_\rho.$$  

Therefore, for all $\beta < \beta_i$,

$$f_\lambda \beta+1 \lambda \left( \bigcap_{i < i_\beta} \left[ \bigcup_{l} K(l, \beta_i, \rho) \cap \bigcap_{i < m < i_\beta} f^{-1}_{\lambda+1, \beta_i} [U_l] \right] \right) \supseteq D^\beta_\rho.$$  

Thus, $K(j, \beta, \rho) \neq \emptyset$ for all $\rho \in T$ and $\beta$ such that $\beta_{i_\beta+1} \leq \beta < \beta_i$. $\square$

For $\rho \in T$, let $n_\rho$ and $\rho = \langle (\beta^{\rho_0}_{\rho}, U^{\rho_0}_{\rho}), \ldots, (\beta^1_{\rho}, U^1_{\rho}) \rangle$ with the first coordinates descending. We will omit the superscript $\rho$ when there is no danger of confusion. For $\rho, \sigma \in T$, define $\rho < \sigma$ if $\langle \beta^0_{\rho} \rangle_{i=1} \subseteq \langle \beta^0_{\rho} \rangle_{i=1}$ and $U^1_{\rho} \supseteq U^1_{\rho}$ whenever $\beta^0_{\rho} = \beta^0_{\rho}$. Notice that for each $\beta$, $i^0_\beta < i^1_\beta$ and $\langle \beta^1_{\beta} : i < i^1_\beta \rangle \subseteq \langle \beta^0_{\beta} : i < i^0_\beta \rangle$. Let $\beta = \lambda + 1$ and $\rho \in T$ such that $i^0_\beta \in n_\rho$ and $i^1_{\beta} + 1 < \beta < i^0_{\beta}$.  

**Lemma 5.** If $\rho < \sigma$ then $K(j, \beta, \rho) \supseteq K(j, \beta, \sigma)$.

**Proof.** Assume that $\rho < \sigma$. Then there is $i^0_\beta < n_\sigma$ such that

$$\beta_{i^0_\beta}^0 < \beta_{i^0_\beta}^0 < \beta_{i^1_\beta}^\sigma < \beta_{i^0_\beta}^\alpha.$$  

Then by the construction $i^0_\beta < n_\rho$,

$$\bigcup_l K(l, \beta^0_{i^0_\beta}, \rho) \cap \bigcap_{i < i^0_\beta} f^{-1}_{\lambda+1, \beta^0_{i^0_\beta}} [U^0_{i^0_\beta}] \supseteq \bigcup_l K(l, \beta^0_{i^0_\beta}, \sigma) \cap \bigcap_{i < i^0_\beta} f^{-1}_{\lambda+1, \beta^0_{i^0_\beta}} [U^0_{i^0_\beta}],$$

Hence, for $k \leq k'$ such that $\beta_{i^0_\beta}^0 = \beta_{i^0_\beta}^\alpha$

$$\bigcap_l K(l, \beta^0_{i^0_\beta}, \rho) \cap \bigcap_{i > i^0_\beta} f^{-1}_{\lambda+1, \beta^0_{i^0_\beta}} [U^0_{i^0_\beta}] \supseteq \bigcup_l K(l, \beta^0_{i^0_\beta}, \sigma) \cap \bigcap_{i > i^0_\beta} f^{-1}_{\lambda+1, \beta^0_{i^0_\beta}} [U^0_{i^0_\beta}],$$

Thus, for $\beta^0_{i^0_\beta} > \beta \geq \beta_{i^0_\beta}$ we have $K(j, \beta, \rho) \supseteq K(j, \beta, \sigma)$. $\square$

Therefore, since we showed that $K(j, \beta, \rho) \neq \emptyset$ for all $\rho$, by compactness we set

$$K(j, \beta) = \bigcap_{\rho \in T} K(j, \beta, \rho) \neq \emptyset.$$
Lemma 6. For each $\rho \in T$ and $\beta \in \lambda + 1$ such that $\beta_{\rho + 1} < \beta < \beta_{\rho}$

$$K(j, \beta) \subseteq cl \left[ \bigcup_{l} K(l, \beta_{l}) \cap \bigcap_{i \in \beta} f_{\lambda+1, \beta}^{-1}[U_{i}] \right].$$

**Proof.** Let $x \in K(j, \beta) = \bigcap_{\rho \in T} K(j, \beta, \rho)$. Then $x \in K(j, \beta, \rho)$ for all $\rho \in T$ and $f_{\lambda+1, \lambda}(x) = d_{1}^{\lambda, \beta}$. Suppose $W$ is a clopen set in $X_{\lambda+1}$ and $x \in W$. We must show that

$$W \cap \left[ \bigcup_{l} K(l, \beta_{l}) \cap \bigcap_{i \in \beta} f_{\lambda+1, \beta}^{-1}[U_{i}] \right] \neq \emptyset$$

that is, we have to show that

$$W \cap K(l, \beta_{l}) \cap \bigcap_{i \in \beta} f_{\lambda+1, \beta}^{-1}[U_{i}] \neq \emptyset$$

for at least one $l$, again this means,

$$W \cap \left[ \bigcap_{\sigma \in T} K(l, \beta_{\sigma}, \sigma) \cap \bigcap_{i \in \beta} f_{\lambda+1, \beta}^{-1}[U_{i}] \right] \neq \emptyset$$

for at least one $l$. Assume this is not the case. This implies

$$W \cap \left[ \bigcap_{\sigma \in T} K(l, \beta_{\sigma}, \sigma) \cap \bigcap_{i \in \beta} f_{\lambda+1, \beta}^{-1}[U_{i}] = \emptyset \right]$$

for all $l$. But $W \cap K(l, \beta_{l}, \sigma) \neq \emptyset$ for infinitely many $l$ since $x \in K(j, \beta) \subseteq K(j, \beta, \sigma)$ and $K(j, \beta, \sigma) \subseteq cl[\bigcup_{l} K(l, \beta_{l}, \sigma)]$ while $x \notin \bigcup_{l} K(l, \beta_{l}, \sigma)$. □

So now when we choose elements of $D^{\lambda+1}_{\beta}$, $\beta \leq \lambda$, we make sure to only pick points from $\bigcup_{l} K(l, \beta)$; that is, $d_{1}^{\lambda+1, \beta}$ will come from $K(l, \beta)$. Since this will ensure that $h[D^{\lambda+1}_{\beta}] = E_{\lambda}$ we will have that $D^{\lambda+1}_{\beta}$ will be discrete. Let $\{B(\beta, m) : m \in \omega\}$ enumerate a clopen base for $X_{\beta}$ for all $\beta \leq \lambda + 1$. Enumerate $\lambda + 1$ by $\{\beta_{0} = \lambda, \beta_{1}, \beta_{2}, \ldots\}$.

Here is how we choose elements of $D^{\lambda+1}_{\beta}$ by induction for each $\beta < \lambda + 1$. At stage $n$, we make finitely many choices. We pick $d_{j}^{\lambda+1, \beta_{l}} \in K(j, \beta_{l})$ for all $i, j < n$, if not already picked, and we make sure the following: For each $l_{0}, l_{1}, l_{2}, l_{3}, l_{4} < n$ such that $d_{j}^{\lambda+1, \beta_{i}} \in B(\lambda + 1, l_{2})$ and $B(\beta_{l_{4}}, l_{4}) \subseteq f_{\lambda+1, \beta_{l_{4}}}^{-1}B(\lambda + 1, l_{2})$ and

$$M = \{l : K(l, \beta_{l}) \cap B(\lambda + 1, l_{2}) \cap f_{\lambda+1, \beta_{l}}^{-1} B(\beta_{l_{4}}, l_{4}) \neq \emptyset\}$$

is infinite, we make sure that there is a $k \in M$ such that

$$d_{k}^{\lambda+1, \beta_{l}} \in K(k, \beta_{l}) \cap B(\lambda + 1, l_{2}) \cap f_{\lambda+1, \beta_{l}}^{-1} B(\beta_{l_{4}}, l_{4}).$$

Then, after the $\omega$ length induction, we now check if our construction satisfies the induction assumptions. If $\beta < \gamma \leq \lambda$ and $U_{l} \subseteq X_{\beta}$ is dense open for each $i < n$ with $\beta = \beta_{0} < \cdots < \beta_{1} = \gamma$, then we show that $D^{\gamma+1}_{\beta} \subseteq cl[D^{\lambda+1}_{\beta} \cap \bigcap_{\gamma \leq \lambda} f_{\lambda+1, \beta}^{-1}U_{i}]$. We consider two cases: when $\beta_{1} = \lambda + 1$ and $\beta_{1} < \lambda + 1$.

Suppose that $\beta_{1} = \lambda + 1$. Then $D^{\lambda+1}_{\lambda+1} \cap U_{1}$ is dense in $X_{\lambda+1}$ and so it is dense in $cl[\bigcap_{1 \leq i \leq n} f_{\lambda+1, \beta_{l}}^{-1}U_{i}]$. For all $\beta \leq \lambda$, $\lambda + 1$,

$$K(l, \beta) \subseteq cl \left[ \bigcap_{1 \leq i \leq n} f_{\lambda+1, \beta_{l}}^{-1}U_{i} \right]$$

and since all the elements of $D^{\lambda+1}_{\beta}$ came from $K(l, \beta)$ we have

$$D^{\lambda+1}_{\beta} \subseteq cl \left[ D^{\lambda+1}_{\lambda+1} \cap \bigcap_{1 \leq i \leq n} f_{\lambda+1, \beta_{l}}^{-1}U_{i} \right].$$

Suppose that $\beta_{1} < \lambda + 1$. Let $d \in D^{\lambda+1}_{\beta} = [d_{1}^{\lambda+1, \beta} : j \in \omega]$, say $d = d_{1}^{\lambda+1, \beta}$ in our enumeration. Then $d \in K(l_{0}, \beta)$ by construction. Let $W = B(\lambda + 1, l_{1})$ be a neighborhood of $d$ in $X_{\lambda+1}$. We must show that $W$ intersects $D^{\lambda+1}_{\beta} \cap \bigcap_{1 \leq i \leq n} f_{\lambda+1, \beta_{l}}^{-1}U_{i}$. Let

$$\hat{W} = W \cap \bigcup_{l} K(l, \gamma) \cap \bigcap_{1 \leq i \leq n} f_{\lambda+1, \beta_{l}}^{-1}U_{i}.$$
Assume that $f_{x+1,1}[W]$ is nowhere dense in $X$. Let $U$ be an open dense subset of $X$ such that $U \cap f_{x+1,1}[W] = \emptyset$. Then consider the set

$$\bigcup_{l} K(l, y) \cap f_{x+1,1,\beta}[U] \cap \bigcap_{i < n} f_{x+1,1,\beta}[U_i].$$

By Lemma 6

$$K(d_0, \beta) \subseteq U \left[ \bigcup_{l} K(l, y) \cap f_{x+1,1,\beta}[U] \cap \bigcap_{i < n} f_{x+1,1,\beta}[U_i] \right].$$

But

$$W \cap \left[ \bigcup_{l} K(l, y) \cap f_{x+1,1,\beta}[U] \cap \bigcap_{i < n} f_{x+1,1,\beta}[U_i] \right] = \emptyset.$$ 

This is a contradiction since $d \in K(d_0, \beta)$ and $W$ is a neighborhood of $d$. Therefore $f_{x+1,1}[W]$ has interior. So there exists $I_3$ such that $B(y, I_3)$ is contained in the closure of $f_{x+1,1}[W]$ which is a subset of $D^\alpha_x$. Thus $B(y, I_3) \subseteq f_{x+1,1}[W]$. Let $M_0 = \{l : f_{x+1,1}(K(l, y)) \in f_{x+1,1}[W] \cap B(y, I_3)\}$

which in effect is $f_{x+1,1}(W) \cap D^\alpha_x \cap B(y, I_3)$ and thus is infinite since it is dense in $B(y, I_3)$. We now show that $M_0 \subseteq M$. Since $f_{x+1,1}(K(l, y)) \in f_{x+1,1}[W]$ we have $K(l, y) \cap W \cap \bigcap_{i < n} f_{x+1,1,\beta}[U_i] \neq \emptyset$. $K(l, y) \subseteq f_{x+1,1,\beta}[B(y, I_3)]$, and $f_{x+1,1,\beta}[U_i] \subseteq f_{x+1,1,\beta}[B(y, I_3)]$. This shows that $M_0 \subseteq M$ and so $M$ was infinite for sufficiently large $\eta$. So we picked

$$d_3^{\lambda+1,1} \subseteq K(l, y) \cap B(\lambda + 1, I_3) \cap f_{x+1,1,\beta}[B(y, I_3)].$$

Since $B(y, I_3) \cap \bigcap_{i < n} f_{x+1,\beta}[U_i]$ contains $f_{x+1,1}[W] \cap B(y, I_3)$, $B(y, I_3) \cap \bigcap_{i < n} f_{x+1,\beta}[U_i]$ is a dense open subset of $B(y, I_3)$. By shrinking $B(y, I_3)$, possibly changing $I_3$, we can assume that $B(y, I_3) \subseteq \bigcap_{i < n} f_{x+1,\beta}[U_i]$. Thus we actually have $K(l, y) \subseteq f_{x+1,\beta}[B(y, I_3)] \cap \bigcap_{i < n} f_{x+1,\beta}[U_i]$ so that $d_3^{\lambda+1,1} \subseteq D^\alpha_x \cap W \cap \bigcap_{i < n} f_{x+1,\beta}[U_i]$ which verifies condition (4) of induction assumption. For each $\lambda < \omega_1$ define the countable set $D^\alpha_x \subseteq X_{\omega_1} = \mathbb{N}^+$ in exactly the same manner that $D^\alpha_x$ was defined in the case of limit ordinals $\alpha < (\omega, \omega_1)$. This completes our inductive construction.

We now show, as promised, that if $p \in D^\alpha_x$ then $p$ is not contained in any discretely weak $P$-set of $\mathbb{N}^+$ and this will complete our proof.

Let $L \subseteq \mathbb{N}^+$ be a discrete weak $P$-set. By Proposition 1, there is a $\lambda < \omega_1$ such that $f_{\omega_1,\lambda}K$ is nowhere dense in the space $X_{\lambda}$. If $U = X_{\lambda} \setminus f_{\omega_1,\lambda}[K]$, then $U$ is a dense open set in $X_{\lambda}$. Let $D = f_{\omega_1,\lambda}^{-1}[U] \cap D^\alpha_x$. Then $D$ is discrete in $X_{\omega_1} = \mathbb{N}^+$ and $D \cap K = \emptyset$. Since $K$ is a discrete weak $P$-set, $D \cap K = \emptyset$. But $p \in D^\alpha_x$ and by the same argument as in the case of limit $\alpha$, $D^\alpha_x \subseteq \bigcup_{i < n} f_{x+1,\beta}[U] \cap D^\alpha_x$. This shows that $p \notin K$. □

5. Open problems

This paper shows that it is independent whether $\mathbb{N}^+$ can be covered by discretely weak $P$-sets since in any model in which it is covered by nowhere dense closed $P$-sets it is also covered by discretely weak $P$-sets. The following questions arise naturally and are not answered.

1. Is it consistent with MA that $\mathbb{N}^+$ can be covered by discretely weak $P$-sets?
2. Is there a model in which $\mathbb{N}^+$ can be covered by discretely weak $P$-sets but not by nowhere dense closed $P$-sets?
3. Is there some quotable basis property of a point which implies it is not covered by a discretely weak $P$-set analogous to other special points (e.g. [7] or [5]).

References