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Subharmonic solutions for nonautonomous sublinear second order Hamiltonian systems ☆

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Abstract

Some existence theorems are obtained for subharmonic solutions of nonautonomous second order Hamiltonian systems by the minimax methods in critical point theory.

Keywords: Subharmonic solution; Second order Hamiltonian system; Saddle point theorem; (PS) condition; Sobolev's inequality; Wirtinger's inequality

1. Introduction and main results

Consider the second order Hamiltonian systems

$$\ddot{u}(t) + \nabla F\big(t, u(t)\big) = 0 \quad \text{a.e. } t \in R, \tag{1}$$

where $F : R \times R^N \to R$ is T-periodic ($T > 0$) in t for all $x \in R^N$, that is,

$$F(t + T, x) = F(t, x) \tag{2}$$

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for all $x \in R^N$ and a.e. $t \in R$, and satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for each $x \in R^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(R^+, R^+)$, $b \in L^1(0,T; R^+)$ such that

$$\left|F(t,x)\right| + \left|\nabla F(t,x)\right| \leqslant a\left(|x|\right)b(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$.

Under the conditions that there exists $h \in L^1(0,T; R^+)$ such that

$$\left|\nabla F(t,x)\right| \leqslant h(t) \tag{3}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, and that

$$\int_0^T F(t,x)\,dt \to +\infty \tag{4}$$

as $|x| \to +\infty$, the existence of T-periodic solutions is proved in [11]. Meanwhile, [7] proves that problem (1) has infinitely distinct subharmonic solutions (kT-periodic solution for some positive integer k is called to be subharmonic) under (3) and the condition that

$$F(t,x) \to +\infty \tag{5}$$

as $|x| \to +\infty$ uniformly for a.e. $t \in [0,T]$. Motivated by the results of [7,11], a natural question is whether problem (1) has infinitely distinct subharmonic solutions under (3) and (4). In [6] a positive answer was given if in addition $F(t,x)$ is convex in x for every $t \in [0,T]$. In this paper we shall consider the nonconvex case and prove that problem (1) has infinitely distinct subharmonic solutions under (3) and a condition weaker than (5) but stronger than (4) (see Theorem 1 below).

It has been proved that problem (1) has infinitely distinct subharmonic solutions under suitable conditions (see [1–13,16–18]). After [12] consider the superquadratic second order Hamiltonian systems, [1,4] consider the superquadratic second order Hamiltonian systems with a changing sign potential. The convex potentials (see [3,6,18]), the even potentials (see [16,17]), the periodic potential (see [13]), the subquadratic potential (see [8–10,12]) and bounded nonlinearity (see [2,5,7]) were also considered, where [2,5,8,9] only consider the special systems

$$\ddot{u}(t) + \nabla G\left(u(t)\right) = e(t) \quad \text{a.e. } t \in R.$$

Recently Chun-Lei Tang [14] generalizes the existence result of T-periodic solutions in [11] mentioned above to the sublinear case. The existence of T-periodic solutions is proved in [14] under the conditions that there exist $g, h \in L^1(0,T; R^+)$ and $\alpha \in [0,1)$ such that

$$\left|\nabla F(t,x)\right| \leqslant g(t)|x|^\alpha + h(t) \tag{6}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, and that

$$|x|^{-2\alpha} \int_0^T F(t,x)\,dt \to +\infty$$

as $|x| \to +\infty$. In this paper, we also consider the existence of infinitely distinct subharmonic solutions for problem (1) in the case that $\nabla F(t,x)$ is sublinear in x (see Theorem 2 below). Some existence theorems are obtained for infinitely distinct subharmonic solutions of problem (1), which generalizes the corresponding result in [7] even if $\nabla F(t,x)$ is bounded in x. The following main results are obtained by the minimax methods.

Theorem 1. *Suppose that F satisfies assumption* (A), (2) *and* (3). *Assume that there exists $\gamma \in L^1(0,T)$ such that*

$$F(t,x) \geqslant \gamma(t) \tag{7}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, and that there exists a subset E of $[0,T]$ *with* $\mathrm{meas}(E) > 0$ *such that*

$$F(t,x) \to +\infty \quad \text{as } |x| \to \infty$$

for a.e. $t \in E$. Then problem (1) *has kT-periodic solution $u_k \in H^1_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to \infty$ as $k \to \infty$, where*

$$H^1_{kT} = \bigl\{u : [0,kT] \to R^N \mid u \text{ is absolutely continuous}, \\ u(0) = u(kT) \text{ and } \dot{u} \in L^2(0,kT;R^N)\bigr\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \left(\int_0^{kT} \bigl|u(t)\bigr|^2\,dt + \int_0^{kT} \bigl|\dot{u}(t)\bigr|^2\,dt\right)^{1/2}$$

and

$$\|u\|_\infty = \max_{0 \leqslant t \leqslant kT} \bigl|u(t)\bigr|$$

for $u \in H^1_{kT}$.

Remark 1. Theorem 1 extends Theorem 4.1 in [7]. There are functions F satisfying our Theorem 1 and not satisfying the results in [1–13,16–18]. For example, let

$$F(t,x) = |\sin \omega t| \ln\bigl(1 + |x|^2\bigr)$$

for all $x \in R^N$ and $t \in R$. Then F satisfies our Theorem 1. But F does not satisfy the results in [1–13,16–18], because that $F(t,x)$ is neither superquadratic in x, nor subquadratic in x, nor convex in x, nor periodic in x, nor uniformly coercive in x for a.e. t, nor belongs to the special case $G(x) + (e(t),x)$.

Theorem 2. *Suppose that $F(t,x)$ satisfies assumption* (A), (2) *and* (6). *Assume that*

$$|x|^{-2\alpha} F(t,x) \to +\infty \tag{8}$$

as $|x| \to +\infty$ uniformly for a.e. $t \in [0,T]$, where α is the same as in (6). *Then problem* (1) *has kT-periodic solution $u_k \in H^1_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to \infty$ as $k \to \infty$.*

Remark 2. Theorem 2 also generalizes Theorem 4.1 in [7] which is the special case of our Theorem 2 corresponding to $\alpha = 0$. There are functions F satisfying our Theorem 2 and not satisfying the results in [1–13,16–18]. For example, let

$$F(t,x) = g(t)|x|^{1+\alpha},$$

where $0 < \alpha < 1$ and $g: R \to R$ is T-periodic, $g \in L^1[0,T]$ and $\inf_{t\in[0,T]} g(t) > 0$. Then F satisfies our Theorem 2. But F does not satisfy the results in [1–13,16–18], because that $F(t,x)$ is neither superquadratic in x, nor subquadratic in x, nor convex in x, nor periodic in x, nor with bounded $\nabla F(t,x)$, nor belong to C^2-class, nor belong to the special case $G(x) + (e(t), x)$.

We shall prove more general results than Theorems 1 and 2.

Theorem 3. *Suppose that F satisfies assumption* (A), (2), (6) *and* (7). *Assume that there exists a subset E of $[0,T]$ with* $\operatorname{meas}(E) > 0$ *such that*

$$|x|^{-2\alpha} F(t,x) \to +\infty \quad \text{as } |x| \to \infty \tag{9}$$

for a.e. $t \in E$. Then problem (1) *has kT-periodic solution $u_k \in H^1_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to \infty$ as $k \to \infty$.*

Remark 3. Without loss of generality, we may assume that functions b in assumption (A), g, h in (6) and γ in (7) are T-periodic and assumption (A), (6) and (7) hold for all $t \in R$ by the T-periodicity of $F(t,x)$ in the first variable.

2. Proof of Theorem 3

Let k be a positive integer. For $u \in H^1_{kT}$, let

$$\bar{u} = (kT)^{-1} \int_0^{kT} u(t)\,dt \quad \text{and} \quad \tilde{u}(t) = u(t) - \bar{u}.$$

Then one has

$$\|\tilde{u}\|_\infty^2 \leqslant \frac{kT}{12} \int_0^{kT} |\dot{u}(t)|^2\,dt \quad \text{(Sobolev's inequality)} \tag{10}$$

and

$$\int_0^{kT} |\tilde{u}(t)|^2\,dt \leqslant \frac{k^2T^2}{4\pi^2} \int_0^{kT} |\dot{u}(t)|^2\,dt \quad \text{(Wirtinger's inequality).} \tag{11}$$

It follows from assumption (A) that the functional φ_k on H^1_{kT} given by

$$\varphi_k(u) = \frac{1}{2} \int_0^{kT} |\dot{u}(t)|^2\,dt - \int_0^{kT} F\big(t, u(t)\big)\,dt$$

is continuously differentiable on H^1_{kT} (see [11]). Moreover one has

$$\langle \varphi_k'(u), v\rangle = \int_0^{kT} \big[\big(\dot u(t), \dot v(t)\big) - \big(\nabla F\big(t, u(t)\big), v(t)\big)\big]\,dt$$

for all $u, v \in H^1_{kT}$. It is well known that the kT-periodic solutions of problem (1) correspond to the critical points of the functional φ_k.

For convenience to quote we state an analog of Egorov's theorem (see Lemma 2 in [15]).

Lemma 1 [15]. *Suppose that F satisfies assumption* (A) *and E is a measurable subset of* $[0, T]$. *Assume that*

$$F(t, x) \to +\infty \quad \text{as } |x| \to \infty$$

for a.e. $t \in E$. Then for every $\delta > 0$ there exists a subset E_δ of E with $\operatorname{meas}(E \setminus E_\delta) < \delta$ *such that*

$$F(t, x) \to +\infty \quad \text{as } |x| \to \infty$$

uniformly for all $t \in E_\delta$.

Lemma 2. *Assume that F satisfies assumption* (A), (2), (6), (7) *and* (9). *Then φ_k satisfies the* (*PS*) *condition, that is, u_n has a convergent subsequence whenever it satisfies $\varphi_k'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi_k(u_n)\}$ is bounded.*

Proof. By Wirtinger's inequality, we have

$$\left(\int_0^{kT} \big|\dot u_n(t)\big|^2\,dt\right)^{1/2} \leqslant \|\tilde u_n\| \leqslant \left(\frac{k^2T^2}{4\pi^2} + 1\right)^{1/2} \left(\int_0^{kT} \big|\dot u_n(t)\big|^2\,dt\right)^{1/2} \tag{12}$$

for all n.

It follows from (6) and Sobolev's inequality that

$$\begin{aligned}
\left|\int_0^{kT} \big(\nabla F\big(t, u(t)\big), \tilde u(t)\big)\,dt\right| &\leqslant \int_0^{kT} g(t)\big|\bar u + \tilde u(t)\big|^\alpha \big|\tilde u(t)\big|\,dt + \int_0^{kT} h(t)\big|\tilde u(t)\big|\,dt \\
&\leqslant \int_0^{kT} 2g(t)\big(|\bar u|^\alpha + \big|\tilde u(t)\big|^\alpha\big)\big|\tilde u(t)\big|\,dt + \int_0^{kT} h(t)\big|\tilde u(t)\big|\,dt \\
&\leqslant 2\big(|\bar u|^\alpha + \|\tilde u\|_\infty^\alpha\big)\|\tilde u\|_\infty \int_0^{kT} g(t)\,dt + \|\tilde u\|_\infty \int_0^{kT} h(t)\,dt \\
&\leqslant \frac{3}{kT}\|\tilde u\|_\infty^2 + \frac{kT}{3}|\bar u|^{2\alpha}\left(\int_0^{kT} g(t)\,dt\right)^2 + 2\|\tilde u\|_\infty^{\alpha+1}\int_0^{kT} g(t)\,dt + \|\tilde u\|_\infty \int_0^{kT} h(t)\,dt
\end{aligned}$$

$$\leqslant \frac{1}{4}\int_0^{kT}\left|\dot u(t)\right|^2 dt + C_1|\bar u|^{2\alpha} + C_2\left(\int_0^{kT}\left|\dot u(t)\right|^2 dt\right)^{(\alpha+1)/2} + C_3\left(\int_0^{kT}\left|\dot u(t)\right|^2 dt\right)^{1/2}$$

for all $u \in H^1_{kT}$ and some positive constants C_1, C_2 and C_3.

Hence one has

$$\begin{aligned}\|\tilde u_n\| &\geqslant \left|\langle \varphi_k'(u_n), \tilde u_n\rangle\right| = \left|\int_0^{kT}\left|\dot u_n(t)\right|^2 dt - \int_0^{kT}\big(\nabla F\big(t, u_n(t)\big), \tilde u_n(t)\big)\,dt\right| \\ &\geqslant \frac{3}{4}\int_0^{kT}\left|\dot u_n(t)\right|^2 dt - C_1|\bar u_n|^{2\alpha} \\ &\quad - C_2\left(\int_0^{kT}\left|\dot u_n(t)\right|^2 dt\right)^{(\alpha+1)/2} - C_3\left(\int_0^{kT}\left|\dot u_n(t)\right|^2 dt\right)^{1/2}\end{aligned}$$

for large n. By (12) and the above inequality we have

$$C|\bar u_n|^{\alpha} \geqslant \left(\int_0^{kT}\left|\dot u_n(t)\right|^2 dt\right)^{1/2} - C_4 \tag{13}$$

for some constants $C > 0$, $C_4 > 0$ and all large n, which implies that

$$\|\tilde u_n\|_\infty \leqslant C_5\big(|\bar u_n|^{\alpha} + 1\big)$$

for all large n and some positive constant C_5 by Sobolev's inequality. Then one has

$$\left|u_n(t)\right| \geqslant |\bar u_n| - \left|\tilde u_n(t)\right| \geqslant |\bar u_n| - \|\tilde u_n\|_\infty \geqslant |\bar u_n| - C_5\big(|\bar u_n|^{\alpha} + 1\big)$$

for all large n and every $t \in [0, kT]$, which implies that

$$\left|u_n(t)\right| \geqslant \frac{1}{2}|\bar u_n| \tag{14}$$

for all large n and every $t \in [0, kT]$.

If $(|\bar u_n|)$ is unbounded, we may assume that, going to a subsequence if necessary,

$$|\bar u_n| \to \infty \quad \text{as } n \to \infty. \tag{15}$$

Set $\delta = \operatorname{meas} E/2$. It follows from (9) and Lemma 1 that there exists a subset E_δ of E with $\operatorname{meas}(E \setminus E_\delta) < \delta$ such that

$$|x|^{-2\alpha} F(t, x) \to +\infty \quad \text{as } |x| \to \infty$$

uniformly for all $t \in E_\delta$, which implies that

$$\operatorname{meas} E_\delta = \operatorname{meas} E - \operatorname{meas}(E \setminus E_\delta) > \delta > 0 \tag{16}$$

and for every $\beta > 0$, there exists $M \geqslant 1$ such that

$$|x|^{-2\alpha} F(t, x) \geqslant \beta \tag{17}$$

for all $|x| \geqslant M$ and all $t \in E_\delta$. By (14) and (15), one has

$$\big|u_n(t)\big| \geqslant M \tag{18}$$

for all large n and every $t \in [0, kT]$. It follows from (13), (7), (18), (17), (14) and (16) that

$$\begin{aligned}
\varphi_k(u_n) &\leqslant \big(C|\bar{u}_n|^\alpha + C_4\big)^2 - \int\limits_{[0,kT]\setminus E_\delta} \gamma(t)\,dt - \int\limits_{E_\delta} \beta\big|u_n(t)\big|^{2\alpha}\,dt \\
&\leqslant \big(C|\bar{u}_n|^\alpha + C_4\big)^2 - \int\limits_{[0,kT]\setminus E_\delta} \gamma(t)\,dt - 2^{-2\alpha}|\bar{u}_n|^{2\alpha}\delta\beta
\end{aligned}$$

for all large n. Hence we have

$$\limsup_{n\to\infty} |\bar{u}_n|^{-2\alpha}\varphi_k(u_n) \leqslant C^2 - 2^{-2\alpha}\delta\beta.$$

By the arbitrariness of $\beta > 0$, one has

$$\limsup_{n\to\infty} |\bar{u}_n|^{-2\alpha}\varphi_k(u_n) = -\infty,$$

which contradicts the boundedness of $\varphi_k(u_n)$. Hence $(|\bar{u}_n|)$ is bounded. Furthermore, (u_n) is bounded by (13) and (12). Arguing then as in Proposition 4.1 in [11], we conclude that the (PS) condition is satisfied. □

Proof of Theorem 3. It follows from Lemma 2 that φ_k satisfies the (PS) condition. We now prove that φ_k satisfies the other conditions of the saddle point theorem. Set

$$e_k(t) = k(\cos k^{-1}\omega t)x_0$$

for all $t \in R$ and some $x_0 \in R^N$ with $|x_0| = 1$, where $\omega = 2\pi/T$. Then we have

$$\dot{e}_k(t) = -\omega(\sin k^{-1}\omega t)x_0$$

for all $t \in R$, which implies that

$$\|\dot{e}_k\|^2_{L^2(0,kT;R^N)} = \frac{1}{2}kT\omega^2.$$

Hence one has

$$\varphi_k(x + e_k) = \frac{1}{4}kT\omega^2 - \int_0^{kT} F\big(t, x + k(\cos k^{-1}\omega t)x_0\big)\,dt$$

for all $x \in R^N$. It follows from (17) that

$$\begin{aligned}
\varphi_k(x + e_k) &\leqslant \frac{1}{4}kT\omega^2 - \int\limits_{[0,kT]\setminus E_\delta} \gamma(t)\,dt - \beta\int\limits_{E_\delta} \big|x + k(\cos k^{-1}\omega t)x_0\big|^{2\alpha}\,dt \\
&\leqslant \frac{1}{4}kT\omega^2 - \int\limits_{[0,kT]\setminus E_\delta} \gamma(t)\,dt - \beta M^{2\alpha}\,\mathrm{meas}\,E_\delta \\
&\leqslant \frac{1}{4}kT\omega^2 - \int\limits_{[0,kT]\setminus E_\delta} \gamma(t)\,dt - \beta\,\mathrm{meas}\,E_\delta
\end{aligned}$$

for all $|x| \geqslant M + k$, which implies that

$$\varphi_k(x + e_k) \to -\infty \quad \text{as } |x| \to \infty \tag{19}$$

by the arbitrariness of β.

Let $\tilde{H}^1_{kT}$ be the subspace of H^1_{kT} given by

$$\tilde{H}^1_{kT} = \{u \in H^1_{kT} \mid \bar{u} = 0\}.$$

Then one has

$$\varphi_k(u) \to +\infty \tag{20}$$

as $\|u\| \to \infty$ in $\tilde{H}^1_{kT}$. In fact, it follows from Sobolev's inequality that

$$\begin{aligned}
\left|\int_0^{kT} \big[F\big(t, u(t)\big) - F(t, 0)\big]\, dt\right| &= \left|\int_0^{kT}\int_0^1 \big(\nabla F\big(t, su(t)\big), u(t)\big)\, ds\, dt\right| \\
&\leqslant \int_0^{kT}\int_0^1 g(t)\big|su(t)\big|^\alpha \big|u(t)\big|\, ds\, dt + \int_0^{kT}\int_0^1 h(t)\big|u(t)\big|\, ds\, dt \\
&\leqslant \int_0^{kT} g(t)\big|u(t)\big|^\alpha \big|u(t)\big|\, dt + \int_0^{kT} h(t)\big|u(t)\big|\, dt \\
&\leqslant \|u\|_\infty^{\alpha+1} \int_0^{kT} g(t)\, dt + \|u\|_\infty \int_0^{kT} h(t)\, dt \\
&\leqslant C_5\left(\int_0^{kT} \big|\dot{u}(t)\big|^2\, dt\right)^{(\alpha+1)/2} + C_6\left(\int_0^{kT} \big|\dot{u}(t)\big|^2\, dt\right)^{1/2}
\end{aligned}$$

for all $u \in \tilde{H}^1_{kT}$ and some positive constants C_5 and C_6.

Hence one has

$$\begin{aligned}
\varphi_k(u) &= \frac{1}{2}\int_0^{kT} \big|\dot{u}(t)\big|^2\, dt - \int_0^{kT} \big[F\big(t, u(t)\big) - F(t, 0)\big]\, dt - \int_0^{kT} F(t, 0)\, dt \\
&\geqslant \frac{1}{2}\int_0^{kT} \big|\dot{u}(t)\big|^2\, dt - C_5\left(\int_0^{kT} \big|\dot{u}(t)\big|^2\, dt\right)^{(\alpha+1)/2} \\
&\quad - C_6\left(\int_0^{kT} \big|\dot{u}(t)\big|^2\, dt\right)^{1/2} - \int_0^{kT} F(t, 0)\, dt
\end{aligned}$$

for all $u \in \tilde{H}^1_{kT}$, which implies (20) by (12).

By (19), (20) and the saddle point theorem (see Theorem 4.6 in [11]), there exists a critical point $u_k \in H^1_{kT}$ for φ_k such that

$$-\infty < \inf_{\tilde{H}^1_{kT}} \varphi_k \leqslant \varphi_k(u_k) \leqslant \sup_{R^N+e_k} \varphi_k.$$

For fixed $x \in R^N$, set

$$A_k = \left\{t \in [0, kT] \mid \left|x + k(\cos k^{-1}\omega t)x_0\right| \leqslant M\right\}.$$

Then we have

$$\text{meas}\, A_k \leqslant k\delta/2 \tag{21}$$

for all large k. In fact, if $\text{meas}\, A_k > k\delta/2$, there exists $t_1 \in A_k$ such that

$$\frac{1}{8}k\delta \leqslant t_1 \leqslant \frac{1}{2}kT - \frac{1}{8}k\delta \tag{22}$$

or

$$\frac{1}{2}kT + \frac{1}{8}k\delta \leqslant t_1 \leqslant kT - \frac{1}{8}k\delta. \tag{23}$$

Moreover, there exists $t_2 \in A_k$ such that

$$|t_2 - t_1| \geqslant \frac{1}{8}k\delta \tag{24}$$

and

$$\left|t_2 - (kT - t_1)\right| \geqslant \frac{1}{8}k\delta. \tag{25}$$

It follows from (25) that

$$\left|\frac{1}{2}(k^{-1}t_1 + k^{-1}t_2) - \frac{1}{2}T\right| \geqslant \frac{1}{16}\delta. \tag{26}$$

By (22) and (23), one has

$$\frac{1}{16}\delta \leqslant \frac{1}{2}(k^{-1}t_1 + k^{-1}t_2) \leqslant T - \frac{1}{16}\delta. \tag{27}$$

From (26) and (27) we obtain

$$\left|\sin\left(\frac{1}{2}(k^{-1}t_1 + k^{-1}t_2)\omega\right)\right| \geqslant \sin\left(\frac{1}{16}\omega\delta\right).$$

Furthermore, by (24) we have

$$\begin{aligned}
&\left|\cos(k^{-1}\omega t_1) - \cos(k^{-1}\omega t_2)\right| \\
&\quad = 2\left|\sin\left(\frac{1}{2}(k^{-1}t_1 + k^{-1}t_2)\omega\right)\right|\left|\sin\left(\frac{1}{2}(k^{-1}t_1 - k^{-1}t_2)\omega\right)\right| \geqslant 2\sin^2\left(\frac{1}{16}\omega\delta\right).
\end{aligned}$$

But due to $t_1, t_2 \in A_k$, one has

$$\begin{aligned}
&\left|\cos(k^{-1}\omega t_1) - \cos(k^{-1}\omega t_2)\right| \\
&\quad = \frac{1}{k}\left|x + k(\cos k^{-1}\omega t_1)x_0 - \left(x + k(\cos k^{-1}\omega t_2)x_0\right)\right| \leqslant \frac{2M}{k}
\end{aligned}$$

which is a contradiction for large k. Hence (21) holds. Let

$$E_k = \bigcup_{j=0}^{k-1} (jT + E_\delta).$$

Then it follows from (21) that

$$\operatorname{meas}(E_k \setminus A_k) \geqslant \frac{1}{2} k\delta$$

for large k. By (17) we have

$$\begin{aligned} k^{-1}\varphi_k(x+e_k) &= \frac{1}{4}T\omega^2 - k^{-1}\int_0^{kT} F\big(t, x + k(\cos k^{-1}\omega t)x_0\big)\,dt \\ &\leqslant \frac{1}{4}T\omega^2 - k^{-1}\int_{[0,kT]\setminus(E_k\setminus A_k)} \gamma(t)\,dt - k^{-1}\beta \operatorname{meas}(E_k \setminus A_k) \\ &\leqslant \frac{1}{4}T\omega^2 + \int_0^T \big|\gamma(t)\big|\,dt - \frac{1}{2}\delta\beta \end{aligned}$$

for every $x \in R^N$ and all large k. Hence one has

$$\sup_{x\in R^N} k^{-1}\varphi_k(x+e_k) \leqslant \frac{1}{4}T\omega^2 + \int_0^T \big|\gamma(t)\big|\,dt - \frac{1}{2}\delta\beta$$

for all large k, which implies that

$$\limsup_{k\to\infty} \sup_{x\in R^N} k^{-1}\varphi_k(x+e_k) \leqslant \frac{1}{4}T\omega^2 + \int_0^T \big|\gamma(t)\big|\,dt - \frac{1}{2}\delta\beta.$$

By the arbitrariness of β, we obtain

$$\limsup_{k\to\infty} \sup_{x\in R^N} k^{-1}\varphi_k(x+e_k) = -\infty,$$

which follows that

$$\limsup_{k\to\infty} k^{-1}\varphi_k(u_k) = -\infty. \tag{28}$$

Now we prove that $\|u_k\|_\infty \to \infty$ as $k \to \infty$. If not, going to a subsequence if necessary, we may assume that

$$\|u_k\|_\infty \leqslant C_7$$

for all $k \in N$ and some positive constant C_7. Hence we have

$$k^{-1}\varphi_k(u_k) \geqslant -k^{-1}\int_0^{kT} F\big(t, u_k(t)\big)\,dt \geqslant -k^{-1}\max_{0\leqslant s\leqslant C_7} a(s)\int_0^{kT} b(t)\,dt$$
$$= -\max_{0\leqslant s\leqslant C_7} a(s)\int_0^{T} b(t)\,dt.$$

It follows that

$$\liminf_{k\to\infty} k^{-1}\varphi_k(u_k) > -\infty,$$

which contradicts (28). Therefore we complete our proof. □

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