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Asymptotic Stability of the Solutions of a Linear Singularly Perturbed System with Unbounded Delay

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Sufficient conditions for asymptotic stability of the solutions of a linear singularly perturbed system of differential equations with unbounded delay have been found. Under the same conditions it is proved that for a locally Lipschitz initial function the initial value problem for the system degenerates regularly. © 1991 Academic Press, Inc.

I. INTRODUCTION

Singularly perturbed systems of differential equations are often used in the applications. In the last few decades the theory of singularly perturbed ordinary differential equations develops intensively. One of the principal problems of this theory is the finding of sufficient conditions for regular degeneration of the system, i.e., conditions under which the solutions of the initial value problem tend as $\mu \to 0$ to the solutions of the initial value problem for the corresponding degenerate system. (Here μ is the perturbing parameter.)

In some mathematical models the history of the process described is taken account of. Thus the problem of finding sufficient conditions for regular degeneration of a singularly perturbed system of differential equations with retarding argument arises quite naturally. For systems with constant delay this problem was considered by A. Halanay [4], A. I. Klimushev [7, 8], K. L. Cooke [1], and K. L. Cooke and K. R. Meyer [2]. Linear singularly perturbed systems with variable bounded delay were

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considered by N. V. Stojanov and H. D. Voulov [16] and L. T. Magalhaes [11–13].

In the present paper sufficient conditions for asymptotic stability of the solutions of a linear singularly perturbed nonautonomous system with unbounded delay are found. Under the same conditions it is proved that for locally Lipschitz initial function the initial value problem for the system degenerates regularly.

II. NOTATIONS: PHASE SPACE AND DEFINITIONS

We shall denote the Euclidean norm by $|\cdot|$. If v is a scalar function of a scalar argument t, denote by $(d/dt)\,v$ its derivative, by $\dot{v}$ its right derivative and by D^+v, D_+v, D^-v, D_-v its Dini derivatives. Set $v'(t) = D^+v(t)$, $\mathscr{D}^+v(t) = \max\{|D^+v(t)|, |D_+v(t)|\}$, $\mathscr{D}^-v(t) = \max\{|D^-v(t)|, |D_-v(t)|\}$ and $\mathscr{D}v(t) = \max\{\mathscr{D}^+v(t), \mathscr{D}^-v(t)\}$. If x is a vector-valued function of a scalar argument, set $D^+x = \operatorname{col}(D^+x_1, ..., D^+x_n)$, $\mathscr{D}^+x = \operatorname{col}(\mathscr{D}^+x_1, \mathscr{D}^+x_2, ..., \mathscr{D}^+x_n)$. D_+x, D^-x, D_-x, $\mathscr{D}^-x$, and $\mathscr{D}x$ are defined in the same way. Denote by $\mathscr{I}[z_0, +\infty)$ the set of all functions $g \in C[z_0, \infty)$ such that $g(t) \leqslant t$ for $t \geqslant z_0$ and $g(t) \to +\infty$ as $t \to +\infty$. Set $x_t(s) = x(t+s)$ and $I = [t_0, +\infty)$, where t_0 is fixed. Let B be the linear space of the functions $\varphi\colon (-\infty, 0] \to R^p$ provided with the seminorm $\|\cdot\|$ and let B_τ be the space of the functions $\psi\colon (-\infty, 0] \to R^p$ such that ψ is continuous in $[-\tau, 0]$ and $\psi_{-\tau} \in B$, $\tau \geqslant 0$, where $\psi_t(s) = \psi(t+s)$ for $s \leqslant 0$. The space B is called admissible if for $\tau \geqslant 0$ and $\psi \in B_\tau$ we have

(B_1) $\psi_t \in B$ for $t \in [-\tau, 0]$

(B_2) ψ_t is continuous in t with respect to $\|\cdot\|$ for $t \in [-\tau, 0]$

(B_3) $M_0\,|\psi(0)| \leqslant \|\psi\| \leqslant K(\tau) \sup_{-\tau \leqslant s \leqslant 0} |\psi(s)| + M(\tau)\,\|\psi_{-\tau}\|$,

where $M_0 > 0$ is a constant and $K(s)$, $M(s)$ are continuous functions.

An admissible space B is said to have a fading memory if the functions $K(s)$ and $M(s)$ in (B_3) satisfy the condition:

(B_4) $K(s) = K = \text{const}$, $M(s) \to 0$ as $s \to +\infty$.

Assume, moreover, that

(B_5) for each $s \leqslant 0$ there exists a number $M^*(s)$ such that

$$|\varphi(s)| \leqslant M^*(s)\,\|\varphi\| \qquad \text{for} \quad \varphi \in B.$$

From [5, Lemma 2.4] it follows that under the conditions (B_1)–(B_3) the function $M^*(s)$ in (B_5) can be chosen continuous, positive, and monotone decreasing.

An important example of a phase space satisfying conditions (β_1)–(β_5) is the Banach space C_γ of the continuous functions $\varphi\colon(-\infty, 0] \to R^p$ for which there exists the limit $\lim_{s\to-\infty} e^{\gamma s}|\varphi(s)|$, provided with the norm $\|\varphi\| = \sup_{s\leqslant 0} e^{\gamma s}|\varphi(s)|$.

Consider the initial value problem

$$\dot{x}(t) = L(t, x_t), \qquad x_\sigma = \varphi \in B, \tag{1}$$

where $t \geqslant \sigma \in I$, $L(t, \varphi)$ is a functional which is defined and continuous for $(t, \varphi) \in I \times B$ and linear on φ. From [5, Theorem 2.1, Lemma 3.1; 15, Theorems 2.1, 2.2] it is known that for any $\sigma \geqslant t_0$, $\varphi \in B$ there exists a unique solution $x(t) = x(\sigma, \varphi)(t)$ of the initial value problem (1) defined for all $t \geqslant \sigma$.

DEFINITION. The trivial solution of (1) is called

(α_1) stable in R^p if for any $\varepsilon > 0$, $\sigma \in I$ there exists $\delta = \delta(\sigma, \varepsilon) > 0$ such that $|x(\sigma, \varphi)(t)| < \varepsilon$ for $t \geqslant \sigma$, $\|\varphi\| < \delta$;

(α_2) uniformly stable in R^p if in (α_1) δ does not depend on σ;

(α_3) equiasymptotically stable in R^p if it is stable and there exist functions $\delta_0 = \delta_0(\sigma)$ and $T = T(\sigma, \varepsilon)$ such that $|x(\sigma, \varphi)(t)| < \varepsilon$ for $t \geqslant T(\sigma, \varepsilon)$, $\|\varphi\| < \delta_0(\sigma)$;

(α_4) uniformly asymptotically stable in R^p if (α_3) is valid and $T(\sigma, \varepsilon) - \sigma$ does not depend on σ;

(α_5) exponentially stable in R^p if there exist positive constants α, M_1 such that

$$|x(\sigma, \varphi)(t)| \leqslant M_1 \|\varphi\| e^{-\alpha(t-\sigma)} \tag{2}$$

for $t \geqslant \sigma \in I$.

If in the above definition we replace $|x(\sigma, \varphi)(t)|$ and R^p respectively by $\|x_t(\sigma, \varphi)\|$ and B, we obtain the respective definitions of stability in B. Since B is an admissible space with a fading memory, the notions of uniform asymptotic stability in B and R^p are equivalent (see [5, Theorem 6.1]).

III. MAIN RESULTS

Let $\sigma \in I$. Consider for $t \geqslant \sigma$ the system

$$\begin{aligned} \dot{X}(t) &= L^{(1)}(t, X_t, \mu) + A(t, \mu)\, Y(t) \\ \mu \dot{Y}(t) &= L^{(2)}(t, X_t, \mu) + C(t, \mu)\, Y(t) \end{aligned} \tag{3}$$

with initial conditions $X_\sigma = \varphi \in B$, $Y(\sigma) = y_0 \in R^n$, where $\mu \in (c, \mu_0]$, $X(t) \in R^p$, $Y(t) \in R^n$, A and C are real matrices whose entries are functions of $(t, \mu) \in I \times [0, \mu_0]$, $L^{(\nu)}(t, \varphi, \mu)$, $\nu = 1, 2$, are real vectors whose components are continuous functionals defined for $(t, \varphi, \mu) \in I \times B \times [0, \mu_0]$ and linear on φ. In case that $\det C(t, 0) \neq 0$ for $t \in I$, the degenerate system corresponding to (3) (for $\mu = 0$) can be written in the more convenient form

$$\dot{x}(t) = [L^{(1)} - AC^{-1}L^{(2)}](t, x_t, 0) \tag{4}$$

$$y(t) = [-C^{-1}L^{(2)}](t, x_t, 0) \tag{5}$$

with initial condition $x_\sigma = \varphi$.

We shall say that conditions (H) are satisfied if the following conditions hold:

(H1) The components of $L^{(\nu)}(t, \varphi, \mu)$, $\nu = 1, 2$, $A(t, \mu)$, $C(t, \mu)$ are continuous for $(t, \varphi, \mu) \in I \times B \times [0, \mu_0]$, $C(t, 0) \in C^1(I)$, and there exists a function $g \in \mathscr{I}[t_0, +\infty)$ and a constant M_2 such that for $(t, \varphi, \mu) \in I \times B \times [0, \mu_0]$ the following inequalities hold

$$|A(t, \mu)| \leqslant M_2, \qquad |C(t, \mu)| \leqslant M_2, \qquad |D^+C(t, \mu)| \leqslant M_2$$
$$|L^{(\nu)}(t, \varphi, \mu)| \leqslant M_2 \sup_{g(t) - t \leqslant s \leqslant 0} |\varphi(s)|, \qquad \nu = 1, 2.$$

(H2) There exist functions $g \in \mathscr{I}[t_0, +\infty)$ and $\rho = \rho(\mu)$, the latter defined for $\mu \in [0, \mu_0]$, such that $\rho(\mu) \to 0$ as $\mu \to 0$ and the inequalities

$$|A(t, \mu) - A(t, 0)| \leqslant \rho(\mu), \qquad |C(t, \mu) - C(t, 0)| \leqslant \rho(\mu)$$
$$|L^{(\nu)}(t, \varphi, \mu) - L^{(\nu)}(t, \varphi, 0)| \leqslant \rho(\mu) \sup_{g(t) - t \leqslant s \leqslant 0} |\varphi(s)|, \qquad \nu = 1, 2$$

hold for any $(t, \varphi, \mu) \in I \times B \times [0, \mu_0]$.

(H3) There exists a function $g \in \mathscr{I}[t_0, +\infty)$ and a positive constant M_3 such that for any function $x: R^1 \to R^p$ such that $x_t \in B$ for $t \geqslant t_0$ the estimate

$$|D^+L^{(2)}(t, x_t, \mu)| \leqslant M_3 \sup_{g(t) \leqslant s \leqslant t} (|x(s)| + |\mathscr{D}x(s)|)$$

holds for $(t, \mu) \in I \times [0, \mu_0]$.

(H4) There exists a positive constant β such that all eigenvalues $\lambda_i(t)$, $i = 1, 2, \ldots, n$ of the matrix $C(t, 0)$ satisfy the condition

$$\operatorname{Re} \lambda_i(t) \leqslant -\beta \qquad \text{for} \quad t \in I.$$

(H5) The zero solution of the degenerate system (4) is uniformly asymptotically stable in B.

Since for $g_1, g_2 \in \mathscr{J}[t_0, +\infty)$ and $g(t) = \min\{g_1(t), g_2(t)\}$ we have $g \in \mathscr{J}[t_0, +\infty)$, we may assume that the function g in (H1), (H2), and (H3) is the same. From condition (H4) it follows that $|\det C(t, 0)| \geqslant \beta^n > 0$ and setting in (3) $\mu = 0$ we obtain the degenerate system (4), (5). Under the conditions (H4) and (H1) the right-hand sides of the linear systems (3) and (4) are continuous for $(t, \varphi) \in I \times B$. Then the initial value problems (3), $X_\sigma = \varphi$, $Y(\sigma) = y_0$ and (4), $x_\sigma = \varphi$ have unique solutions respectively $(X, Y)(t) = (X, Y)(\sigma, \varphi, y_0, \mu)(t)$ and $x(t) = x(\sigma, \varphi)(t)$, defined and continuous for $t \geqslant \sigma$.

The main result in the work is the following theorem.

THEOREM 1. *Let conditions* (H) *hold.*
Then the following assertions are valid:

(i) *There exists a positive number* $\mu_1 \leqslant \mu_0$ *such that for* $\mu \in (0, \mu_1)$ *the zero solution of* (3) *is equiasymptotically stable in* $B \times R^n$. *In the case of bounded delay*; *i.e., when the function* $t - g(t)$ *is bounded, the zero solution of* (3) *is uniformly asymptotically stable.*

(ii) *For any locally Lipschitz initial function* $\varphi \in B$ *and as* $\mu \to 0$ *the solution* $(X, Y)(t_0, \varphi, y_0, \mu)(t)$ *of* (3) *tends to the solution* $(x, y)(t_0, \varphi)(t)$ *of* (4), (5) *uniformly with respect to* $(t, y_0) \in [t_0 + \delta, +\infty) \times \{z \in R^n : |z| \leqslant Q\}$, *where* δ *and* Q *are arbitrary positive constants.*

The proof of Theorem 1 is given in Section IV.

In order to illustrate the role of condition (H) in Theorem 1, we shall consider several examples. If the estimate in condition (H4) is valid for each $t \in I$ but it is not uniform with respect to $t \in I$, i.e., β depends on t, then assertions (i) and (ii) are not valid which is seen from [6, Example E4]. The importance of condition (H2) is illustrated by [6, Examples E1, E2].

EXAMPLE 1. The condition $g(t) \to +\infty$ as $t \to +\infty$ in (H1) cannot be omitted. Consider the system

$$\begin{aligned} \dot{X}(t) &= -X(t) + X(g(t)) - Y(t), \qquad & X_\sigma &= \varphi \in C_\gamma \\ \mu \dot{Y}(t) &= X(g(t)) - Y(t), & Y(\sigma) &= y_0 \in R^1, \end{aligned} \tag{6}$$

where $g(t) \leqslant A = \text{const}$ for $t \geqslant t_0$. Choose a number $\sigma > \max\{A, t_0\}$ and set $\varphi(s) \equiv y_0$ for $s \leqslant 0$. Then $Y(t) \equiv y_0$ for $t \geqslant \sigma$, hence the trivial solution of (6) is not asymptotically stable.

EXAMPLE 2. In the case of bounded delay, i.e., when the function $g(t)$ in conditions (H) satisfies the inequality $g(t) \geqslant t - h$, $h = \text{const} > 0$, Theorem 1 shows that the zero solution of (3) is uniformly asymptotically stable for $t \in I$ and sufficiently small values of μ (see also [16]).

Consider system (6) for $g(t) = t/2$, $t_0 > 0$. It satisfies all conditions of Theorem 1, hence its zero solution is equiasymptotically stable for $t \geqslant t_0$ and sufficiently small values of μ. We shall prove that it is not uniformly asymptotically stable. Fix the number $\mu > 0$ and suppose that the last assertion is not true, i.e., that there exists a number $\delta > 0$ such that for any $\varepsilon > 0$ one can find a number $N(\varepsilon) > 0$ such that

$$|X(\sigma, \varphi, y_0, \mu)(t)| < \varepsilon, \ |Y(\sigma, \varphi, y_0, \mu)(t)| < \varepsilon$$

for $t \geqslant \sigma + N(\varepsilon)$, $\|\varphi\| + |y_0| < \delta$, $\sigma \geqslant t_0$. Set $\varepsilon = \delta/8$, $\sigma = t_0 + N(\varepsilon)$, $\varphi(s) \equiv y_0 = \delta/4$. Hence

$$|Y(\sigma, \varphi, y_0, \mu)(2\sigma)| < \delta/8.$$

But on the other hand for $t \in [\sigma, 2\sigma]$ the system is of the form

$$\begin{aligned} \dot{X}(t) &= -X(t) + \delta/4 - Y(t), \qquad & X(\sigma) &= \delta/4 \\ \mu \dot{Y}(t) &= \delta/4 - Y(t), & Y(\sigma) &= \delta/4 \end{aligned}$$

which implies that $Y(t) = \delta/4$ for $t \in [\sigma, 2\sigma]$.

This example shows that the condition of boundedness of the delay in [16, Theorem 1] is essential.

EXAMPLE 3. The convergence in assertion (ii) of Theorem 1 depends on the choice of the initial function φ; i.e., the rate of convergence changes under small changes of the initial function. Consider the system

$$\begin{aligned} \dot{X}(t) &= -X(t), \qquad & X_0 &= \varphi \in C[-1, 0] \\ \mu \dot{Y}(t) &= -Y(t) + X(t-1), & Y(0) &= 0. \end{aligned} \tag{7}$$

Let $\varepsilon > 0$ be fixed and let n be a positive integer. Set

$$\varphi(s) = \varphi_n(s) = \begin{cases} 0, & s \in [-1, -\varepsilon/n] \\ ns + \varepsilon, & s \in [-\varepsilon/n, 0]. \end{cases}$$

Integrate system (7) in the interval [0, 1]. For its solutions $X(\varphi_n, \mu)(t)$ and $Y(\varphi_n, \mu)(t)$ we have

$$X(\varphi_n, \mu)(t) = x(\varphi_n)(t) = \varepsilon e^{-t}$$

and for $t = 1$ the equality $Y(\varphi_n, \mu)(1) - y(\varphi_n)(1) = \mu n(e^{-\varepsilon/\mu n} - 1)$ holds where $x(\varphi_n)(t)$ and $y(\varphi_n)(t)$ are solutions of the respective degenerate system $\dot{x}(t) = -x(t)$, $y(t) = x(t-1)$, $x_0 = \varphi_n$. To complete the proof it suffices to note that $\|\varphi_n\| \leqslant \varepsilon$ for every n and $\mu n(e^{-\varepsilon/\mu n} - 1) \to 0$ nonuniformly with respect to n as $\mu \to 0$.

IV. Proof of Theorem 1

1. *Proof of Some Auxiliary Assertions*

LEMMA 1. *Let $v \in C[z_0, \tau)$, $\tau \leqslant +\infty$, $z_1 \geqslant z_0$, let δ and q be constants. Moreover, let $v'(t) \leqslant q$ for these values of $t \in [z_1, \tau)$ which satisfy the conditions $v(t) \geqslant \delta$ and $v(t) \geqslant v(s)$ for $s \in [z_0, t]$.*

Then for $t \in [z_1, \tau)$ the following inequality holds

$$v(t) \leqslant \max\{\delta, \sup_{s \in [z_0, z_1]} v(s)\} + (t - z_1) \max\{0, q\}. \tag{8}$$

Proof. (a) The case $\delta = 0$. Set $\bar{v}(t) = \max\{0, v(t)\}$ for $t \in [z_0, \tau)$. The function $\bar{v}$ is continuous and nonnegative in the interval $[z_0, \tau)$. It is easy to check that $\bar{v}'(t) \leqslant \max\{0, q\}$ if $t \in [z_1, \tau)$ and $\bar{v}(t) \geqslant \bar{v}(s)$ for $s \in [z_0, t)$. Then from [3, Lemma 1] it follows that for $t \in [z_1, \tau)$

$$v(t) \leqslant \max\{0, \sup_{s \in [z_0, z_1]} v(s)\} + (t - z_1) \max\{0, q\}.$$

(b) The general case. Set $w(t) = v(t) - \delta$ for $t \in [z_0, \tau)$. The function w satisfies the conditions of Lemma 1 in the particular case in which the lemma was proved above. Hence for $t \in [z_1, \tau)$ the inequality

$$v(t) - \delta \leqslant \max\{0, \sup_{s \in [z_0, z_1]} v(s) - \delta\} + (t - z_1) \max\{0, q\}$$

holds which is equivalent to (8). Lemma 1 is proved.

LEMMA 2. *Assume that $v \in C[z_2, \tau)$ where $\tau \leqslant +\infty$. Let $q > 0$ and δ be constants such that for all values of $t \in [z_2, \tau)$ satisfying the condition $v(t) \geqslant \delta$ the estimate $v'(t) \leqslant -q$ holds. Moreover, let $z_2 + T < \tau$, where*

$$T = q^{-1} \max\{0, v(z_2) - \delta\}.$$

Then $v(t) \leqslant \delta$ for $t \in [z_2 + T, \tau)$.

Proof. If $T = 0$, then the assertion of Lemma 2 follows from the definition of the number T and Lemma 1. Let $T > 0$. If $v(s) \leqslant \delta$ for some $s \in [z_2, z_2 + T]$, from Lemma 1 (for $z_0 = z_1 = s$) it follows that $v(t) \leqslant \delta$ for $t \in [s, \tau) \supset [z_2 + T, \tau)$. Suppose that this is not true, i.e., that $v(t) > \delta$ for all

$t \in [z_2, z_2+T]$. Then for the function $w(t) = v(t) + q(t - z_2)$ we have $w(z_2) = v(z_2)$ and $w'(t) \leqslant 0$ for $t \in [z_2, z_2+T]$. From Lemma 1 it follows that $w(z_2+T) \leqslant w(z_2)$, hence we obtain

$$\delta + qT < v(z_2+T) + qT = w(z_2+T) \leqslant w(z_2) = v(z_2)$$

which contradicts the definition of T. Lemma 2 is proved.

LEMMA 3. *Let $r \in \mathscr{J}[z_0, +\infty)$, $f \in C[0, +\infty)$ and $f(t) > t$ for $t > 0$. Let $c_0 > 0$ be a constant and let P be some subset of $C[z_0, +\infty)$ such that for $v \in P$ and $t \geqslant z_0$ the estimate $v(t) \leqslant c_0$ holds. Moreover, let for $\delta > 0$ the functions $\tau(\delta) = \tau(\delta, c_0, z_0) \geqslant z_0$ and $K(\delta) = K(\delta, c_0, z_0) > 0$ be defined so that for any function $v \in P$ and any value of $t \geqslant \tau(\delta)$ satisfying the conditions $v(t) \geqslant \delta$, $r(t) \geqslant z_0$ and $f(v(t)) \geqslant v(s)$ for $s \in [r(t), t]$ the estimate $v'(t) \leqslant -K(\delta)$ holds.*

Then there exists a function $\Gamma(\delta) = \Gamma(\delta, c_0, z_0)$ defined for $\delta > 0$ such that for $v \in P$ and $t \geqslant \Gamma(\delta)$ the estimate $v(t) \leqslant \delta$ holds.

Proof. Let δ be an arbitrary number from the interval $(0, c_0)$. By means of the numbers $K = K(\delta, c_0, z_0)$ and $\tau = \tau(\delta, c_0, z_0)$ we shall define $\Gamma(\delta)$. In virtue of the properties of the function f there exists a number $a = a(\delta, c_0)$ such that $f(s) - s > a$ for $s \in [\delta, c_0]$. Let $N = N(\delta, c_0)$ be a positive integer such that $\delta + Na \geqslant c_0$. From the properties of the function r it follows that a finite monotone increasing sequence of numbers $t_n = t_n(\delta, c_0, z_0)$ $(n = 0, 1, 2, ..., N)$ can be chosen so that $t_0 = \tau$ and $r(t) \geqslant t_{n-1}$ for $t \geqslant t_n - a/K$, $n = 1, 2, ..., N$.

Set $\Gamma(\delta) = t_N = t_N(\delta, c_0, z_0)$. We shall show that $v(t) \leqslant \delta$ for $t \geqslant t_N$ and $v \in P$. For this purpose it suffices to prove that for $n = 0, 1, 2, ..., N$, $v \in P$, and $t \geqslant t_n$ the inequality

$$v(t) \leqslant \delta + (N-n)a$$

holds. We shall prove the above assertion by induction on n. For $n = 0$ the assertion follows from the definition of N and the inequality $v(t) \leqslant c_0$ for $v \in P$.

Assume that the assertion holds for some $n < N$, i.e., $v(s) \leqslant \delta + (N-n)a$ for $s \geqslant t_n$, $v \in P$. Let $v \in P$, $t \geqslant t_{n+1} - a/K$, and $v(t) \geqslant \delta + (N-n-1)a$. Then the inequalities $t \geqslant r(t) \geqslant t_n \geqslant \tau$, $v(t) \geqslant \delta$, and $f(v(t)) > v(t) + a \geqslant \delta + (N-n)a \geqslant v(s)$ hold for $s \in [r(t), t]$, hence $v'(t) \leqslant -K$. From Lemma 2 it follows that $v(t) \leqslant \delta + a(N-n-1)$ for $t \geqslant t_{n+1} - a/K + T_1$, where

$$T_1 = K^{-1} \max\{0, v(t_{n+1} - a/K) - \delta - (N-n-1)a\}.$$

The inequalities $t_{n+1} - a/K \geqslant r(t_{n+1} - a/K) \geqslant t_n$ and $v(t_{n+1} - a/K) \leqslant \delta + (N-n)a$ show that $T_1 \leqslant a/K$. Hence $v(t) \leqslant \delta(N-n-1)a$ for $t \geqslant t_{n+1}$, $v \in P$.

This completes the proof of Lemma 3.

Remark 1. If $r(t) \geqslant t-h$ and the numbers $\tau(\delta, c_0, z_0)-z_0$ and $K(\delta, c_0, z_0)$ in the conditions of Lemma 3 do not depend on z_0, then the number $\Gamma(\delta, c_0, z_0)-z_0$ does not depend on z_0 too because in this case $t_n = t_{n-1} + h + a/K$, $n = 1, 2, 3, ..., N$ and $\Gamma(\delta, c_0, z_0) - z_0 = N(h + a/K)$.

We shall note that the proofs of Lemmas 2 and 3 have been inspired by [3, 9] with regard to certain ideas.

Next lemmas follow from the properties of lim sup and lim inf.

LEMMA 4. *For the scalar functions f and g we have*:

(a)
$$\limsup_{t\to s^+}(f(t)+g(t)) \leqslant \limsup_{t\to s^+} f(t) + \limsup_{t\to s^+} g(t),$$
$$\limsup_{t\to s^+} |f(t)\cdot g(t)| \leqslant \limsup_{t\to s^+} |f(t)| \cdot \limsup_{t\to s^+} |g(t)|.$$

(b) *If $\mathscr{D}^+ f(s) < +\infty$, then $f(t) \to f(s)$ as $t \to s^+$.*

(c)
$$\limsup_{h\to 0^+} |f(s+h)-f(s)|/h = \mathscr{D}^+ f(s).$$

(d) *If $\mathscr{D}^+ f(s) < +\infty$ and $\mathscr{D}^+ g(s) < +\infty$, then*

$$\mathscr{D}^+(fg)(s) \leqslant |f(s)|\, \mathscr{D}^+ g(s) + |g(s)|\, \mathscr{D}^+ f(s).$$

If $\mathscr{D}^+ f(s) < +\infty$, $f(s)=0$ and $g(t) \to g(s)$ as $t \to s^+$, then $\mathscr{D}^+(fg)(s) \leqslant |g(s)|\, \mathscr{D}^+ f(s)$.

Proof. The assertions (a) and (b) are trivial. Assertion (c) follows from the equality $\limsup |f(t)| = \max\{|\limsup f(t)|, |\liminf f(t)|\}$ and assertion (d) follows from (a), (b), (c), and the triangle inequality.

LEMMA 5. *Let f, g_1, and g_2 be continuous scalar functions such that $g_1(t) \leqslant D^+ f(t) \leqslant g_2(t)$ for $t \in (a, b)$.*

Then $D_+ f(t)$, $D^- f(t)$, $D_- f(t) \in [g_1(t), g_2(t)]$ for $t \in (a, b)$.

Proof. There exists a smooth function G_1 defined for $t \in (a, b)$ such that $\dot{G}_1(t) = g_1(t)$. The function $F_1 = f - G_1$ satisfies the estimate $D^+ F_1(t) \geqslant D^+ f(t) - g_1(t) \geqslant 0$ for $t \in (a, b)$. From [14, Appendix 1, Corollary 2.4] it follows that the derivatives $D_+ F_1(t)$, $D^- F_1(t)$, and $D_- F_1(t)$ are also nonnegative. Hence $D_+ f(t) \geqslant g_1(t)$ since

$$0 \leqslant D_+ F_1(t) \leqslant D_+ f(t) + D^+(-G_1)(t) = D_+ f(t) - g_1(t).$$

The inequalities $D_- f(t) \geqslant g_1(t)$, $D^- f(t) \geqslant g_1(t)$, $D_+ f(t) \leqslant g_2(t)$, $D^- f(t) \leqslant g_2(t)$, and $D_- f(t) \leqslant g_2(t)$ are proved in the same way.

COROLLARY 1. *Let the functions f and g be continuous for $t \in (a, b)$,*

$g(t) \in R$, $f(t) \in R^n$. Then the estimate $|D^+ f(t)| \leqslant g(t)$ for all $t \in (a, b)$ implies $|\mathscr{D} f(t)| \leqslant g(t)\sqrt{n}$.

LEMMA 6. *Let the numbers μ, c, d, r be positive and let a, b be non-negative numbers such that $a + b \geqslant r$.*

Then, if μ satisfies the condition

$$\mu \leqslant \mu_0 = \min\left\{\frac{2}{3c^2}, \frac{1}{6c}, \frac{(r/d)^2}{8c^2 + 4c(r/d)}\right\}$$

the inequality $a^2 + b^2\mu^{-1} \geqslant bc(a + b + d)$ holds.

If $r \geqslant 3d$, then $\mu_0 = \min\{2/3c^2, 1/6c\}$.

Proof. For $\mu \leqslant 2/(3c^2)$ the inequalities $a^2/2 + b^2/(3\mu) \geqslant a^2/2 + b^2c^2/2 \geqslant abc$ and for $\mu \leqslant 1/(6c)$ the inequality $b^2/(6\mu) \geqslant b^2 c$ is fulfilled. In order to complete the proof of Lemma 6, it suffices to show that for $\mu \leqslant (r/d)^2/(8c^2 + 4c(r/d))$ the estimate $a^2/2 + b^2/(2\mu) \geqslant bcd$ holds. For $a^2 \geqslant 2bcd$ the above inequality is obvious. On the other hand, for $a^2 < 2bcd$ the inequality $a + b \geqslant r$ implies the estimates

$$2r^2 \leqslant 4b^2 + 4a^2 \leqslant 4b^2 + 8bcd = (2b + 2cd)^2 - 4c^2d^2.$$

In such a case the inequalities

$$\frac{b}{2cd} \geqslant \frac{-2cd + \sqrt{4c^2d^2 + 2r^2}}{4cd} > \frac{(r/d)^2}{8c^2 + 4c(r/d)}$$

are fulfilled which imply $b^2 > 2bcd\mu$ for $\mu \leqslant (r/d)^2/(8c^2 + 4c(r/d))$.

This completes the proof of Lemma 6.

2. *Construction of a Functional of Lyapunov–Krasovskii Type*

From conditions (H1) and (H4) it follows that the functional $[L^{(1)} - AC^{-1}L^{(2)}](t, \varphi, 0)$ is continuous for $(t, \varphi) \in I \times B$ and linear on φ. Hence, from [15, condition E5, Theorem 3.2] it follows that the solution $x(\sigma, \varphi)(t)$ of the initial value problem (4), $x_\sigma = \varphi$ satisfies estimate (2) for some $M_1 > 0$ and $\alpha > 0$. By [15, Theorem 3.3] there exists a continuous functional

$$\bar{V}(t, \varphi) = \sup_{s \geqslant 0} \|x_{t+s}(t, \varphi)\|\, e^{\alpha s/2}$$

defined for $(t, \varphi) \in I \times B$ such that for $V(t, \varphi) = [\bar{V}(t, \varphi)]^2$ the following estimates are valid

$$\begin{gathered} \|\varphi\|^2 \leqslant V(t, \varphi) \leqslant b_1 \|\varphi\|^2 \\ |V(t, \varphi_1) - V(t, \varphi_2)| \leqslant b_1(\|\varphi_1\| + \|\varphi_2\|)\, \|\varphi_1 - \varphi_2\| \\ V'_{(4)}(t, \varphi) \leqslant -a_1 \|\varphi\|^2, \end{gathered} \tag{9}$$

where a_1 and b_1 are positive constants.

From conditions (H1), (H5), and [10, Lemma 2.4] it follows that

$$|e^{sC(t,0)}| \leqslant R_0 e^{-\beta s/2}, \qquad |e^{s\,{}^tC(t,0)}| \leqslant R_0 e^{-\beta s/2}, \tag{10}$$

where R_0 is a positive constant depending only on β and M_2 and the matrix ${}^tC$ is transposed to C.

Consider the linear autonomous system $\dot{Z}(s) = C(t, 0)\, Z(s)$ and denote its solution through $(0, \eta)$ by $Z(s, t, \eta)$. It is known that $Z(s, t, \eta) = e^{C(t,0)s}\eta$ and

$$\frac{\partial}{\partial t} Z(s, t, \eta) = e^{C(t,0)s} \cdot \int_0^s e^{-C(t,0)\tau} \frac{d}{dt} C(t, 0)\, e^{C(t,0)\tau}\, d\tau \cdot \eta.$$

Set $W(t, \eta) = \int_0^\infty |Z(s, t, \eta)|^2\, ds$. By means of (10) in a standard way it is proved that $W(t, \eta)\colon I \times R^n \to R$ is continuous, has partial derivatives with respect to all its arguments, and satisfies the conditions

$$\begin{gathered} a_2 |\eta|^2 \leqslant W(t, \eta) \leqslant b_2 |\eta|^2 \\ |W_t(t, \eta)| \leqslant b_2 |\eta|^2, \qquad |W_\eta(t, \eta)| \leqslant b_2 |\eta| \\ (W_\eta(t, \eta), C(t, 0)\eta) = -|\eta|^2, \end{gathered} \tag{11}$$

where a_2 and b_2 are positive constants depending only on β and M_2. For $(t, \varphi, \psi) \in I \times B \times R^n$ we set $U(t, \varphi, \psi) = V(t, \varphi) + W(t, \psi)$. The functional U is continuous and satisfies the estimates

$$a_3^2(\|\varphi\| + |\psi|)^2 \leqslant U(t, \varphi, \psi) \leqslant b_3^2(\|\varphi\| + |\psi|)^2. \tag{12}$$

3. *Proof of Assertion* (i) *of Theorem* 1

From conditions (H1), (H2), (H4), and Lemma 5 it follows that there exists a constant b_4 such that for $(t, \mu) \in I \times [0, \mu_0]$ the following inequalities hold

$$|C^{-1}(t, \mu)| \leqslant b_4, \qquad |\mathscr{D}^+ C^{-1}(t, \mu)| \leqslant b_4. \tag{13}$$

Set $L = \max\{1, M_1, M_2, M_3, M_0^{-1}, b_1, b_2, b_3, b_4\}$, $m = \min\{1, M_0, a_1, a_2, a_3\}$. There exists a number $\sigma_0 \geqslant t_0$ such that $g(t) \geqslant t_0$ for $t \geqslant \sigma_0$. For $t \geqslant \sigma_0$ set $G(t) = \inf_{g(t) \leqslant s \leqslant t} g(s)$. Since $G \in \mathscr{I}[\sigma_0, +\infty)$, then for each $\sigma \geqslant t_0$ there exists a number $\Sigma = \Sigma(\sigma)$ such that $G(t) > \sigma$ for $t \geqslant \Sigma$.

Let $\mu \in (0, \mu_0)$ and $\sigma \geqslant t_0$. The solution $(X, Y)(\sigma, \varphi, y_0, \mu)$ of the initial value problem (3), $X_\sigma = \varphi$, $Y(\sigma) = y_0$ we shall estimate in two steps.

For $t \in [\sigma, \Sigma]$, integrating system (3) over the interval $[\sigma, t]$, in view of conditions (B_1)–(B_5) and (H1) we obtain the following estimates of X and Y

$$|X(t)| \leqslant M_0^{-1} \|\varphi\| + L(C_0+1)\int_\sigma^t (\|X_\lambda\| + |Y(\lambda)|)\, d\lambda$$

$$|Y(t)| \leqslant |y_0| + L\mu^{-1}(C_0+1)\int_\sigma^t (\|X_\lambda\| + |Y(\lambda)|)\, d\lambda,$$

where $C_0 \geqslant \sup_{s\in[\sigma,\Sigma]} M^*(g(s)-s)$.

Then from condition (B_2) it follows that for $t\in[\sigma,\Sigma]$

$$\|X_t\| \leqslant [M_0^{-1}K(t-\sigma) + M(t-\sigma)]\,\|\varphi\|$$
$$+ K(t-\sigma)\, L(C_0+1)\int_\sigma^t (\|X_\lambda\| + |Y(\lambda)|)\, d\lambda.$$

Since the functions $\|X_t\|$, $|Y(t)|$, $K(t-\sigma)$, and $M(t-\sigma)$ are continuous for $t\in[\sigma,\Sigma]$, using Gronwall's inequality we obtain

$$\|X_t\| + |Y(t)| \leqslant C_1(\|\varphi\| + |y_0|) \tag{14}$$

for $t\in[\sigma,\Sigma]$, where $C_1 = C_1(\mu,\sigma) = (M_0^{-1}\hat{K} + \hat{M} + 1)\exp[L(C_0+1)(\mu^{-1}+\hat{K})(\Sigma-\sigma)]$, $\hat{K} = \max_{s\in[0,\Sigma-\sigma]} K(s)$, $\hat{M} = \max_{s\in[0,\Sigma-\sigma]} M(s)$.

Remark 2. Note that in the case of bounded delay, i.e., $g(t)\geqslant t-h$, where $h>0$ is a constant, the numbers C_0 and C_1 do not depend on σ since $\Sigma=\sigma+2h$ and $C_0 = M^*(-h)$ because the function M^* is monotone decreasing.

In order to estimate $\|X_t\|$ and $|Y(t)|$ for $t>\Sigma$, we shall use the functional $U(t,\varphi,\psi)$.

Set

$$\begin{aligned} \xi(t,\mu) &= X(t,\mu) - x(t) \\ \eta(t,\mu) &= Y(t,\mu) + C^{-1}(t,\mu)\, L^{(2)}(t, X_t, \mu), \end{aligned} \tag{15}$$

where $x(t) = x(\sigma,\varphi)(t)$ is the solution of the initial value problem (4), $x_\sigma = \varphi$. From equalities (15) and inequalities (B_5), (2) and (14) we deduce that

$$\|\xi_t\| + |\eta(t)| \leqslant C_2(\|\varphi\| + |y_0|) \tag{16}$$

for $t\in[\sigma,\Sigma]$, where C_2 depends on L, C_0, and C_1.

On the other hand, equalities (3), (4), and (15) show that the functions ξ and η satisfy the conditions

$$\begin{aligned} \dot{\xi}(t,\mu) &= [L^{(1)} - AC^{-1}L^{(2)}](t,\xi_t,\mu) + A(t,\mu)\,\eta(t,\mu) \\ &\quad + [L^{(1)} - AC^{-1}L^{(2)}](t,x_t,\mu) - [L^{(1)} - AC^{-1}L^{(2)}](t,x_t,0), \\ D^+\eta(t,\mu) &= C(t,\mu)\eta(t,\mu)\,\mu^{-1} + D^+[C^{-1}(t,\mu)\, L^{(2)}(t,X_t,\mu)], \\ D_+\eta(t,\mu) &= C(t,\mu)\,\eta(t,\mu)\,\mu^{-1} + D_+[C^{-1}(t,\mu)\, L^{(2)}(t,X_t,\mu)] \end{aligned} \tag{17}$$

for $t \geqslant \sigma$. Consider the functions

$$\begin{aligned} v_1(t) &= v_1(\sigma, \varphi, y_0, \mu)(t) = V(t, \xi_t), \\ v_2(t) &= v_2(\sigma, \varphi, y_0, \mu)(t) = W(t, \eta(t)), \\ v(t) &= v_1(t) + v_2(t). \end{aligned} \tag{18}$$

We shall estimate their right derivatives. Let $t > \sigma$. Denote by $Z(s)$ the solution of the initial value problem (4), $Z_t = \xi_t$. By (9) and (B_2) and Lemma 4 we obtain that for $t > \sigma$

$$\begin{aligned} v_1'(t) &\leqslant \limsup_{h \to 0^+} [V(t+h, \xi_{t+h}) - V(t+h, Z_{t+h})]/h \\ &\quad + \limsup_{h \to 0^+} [V(t+h, Z_{t+h}) - V(t, Z_t)]/h \\ &\leqslant 2L \, \|\xi_t\| \limsup_{h \to 0^+} \|\xi_{t+h} - Z_{t+h}\|/h + V'_{(4)}(t, Z_t) \\ &\leqslant 2LK(0) \, \|\xi_t\| \, |\dot{\xi}(t, \mu) - \dot{Z}(t)| - m \, \|\xi_t\|^2 \end{aligned}$$

since $\limsup_{h \to 0^+} \sup_{s \in (0, h]} |f(s)| = \limsup_{h \to 0^+} |f(h)|$ for any vector-valued function f defined for $h > 0$. Then from (4), (17), (13), and conditions (H1), (H2) it follows that

$$\begin{aligned} v_1'(t) \leqslant& -m \, \|\xi_t\|^2 + 2LK(0) \, \|\xi_t\| \{L \, |\eta(t)| \\ &+ (1 + 3L^2) \, \rho(\mu) (\sup_{s \in [g(t), t]} |\xi(s)| + \sup_{s \in [g(t), t]} |x(s)|) \}. \end{aligned} \tag{19}$$

Let $t \geqslant \sigma$ be a number such that $\sup_{s \in [g(t), t]} |\mathscr{D} X(\sigma, \varphi, y_0, \mu)(s)| < \infty$. Then from inequalities (13), condition (H3), and Lemmas 4 and 5 it follows that

$$|\mathscr{D}^+ [C^{-1}(t, \mu) \, L^{(2)}(t, X_t, \mu)]| \leqslant 2 \sqrt{n} \, L^2 \sup_{s \in [g(t), t]} (|X(s)| + |\mathscr{D} X(s)|) \tag{20}$$

and, by (17), $|\mathscr{D}^+ \eta(t)| < +\infty$. Hence

$$\begin{aligned} v_2'(t) &\leqslant W_t(t, \eta(t)) + \limsup_{h \to 0^+} [W(t, \eta(t+h)) - W(t, \eta(t))]/h \\ &\leqslant W_t(t, \eta(t)) + \limsup_{h \to 0^+} ([\eta(t+h) - \eta(t)]/h, \\ &\qquad W_\eta(t, \eta(t) + \theta(h)[\eta(t+h) - \eta(t)])) \\ &= W_t(t, \eta(t)) + \limsup_{h \to 0^+} ([\eta(t+h) - \eta(t)]/h, W_\eta(t, \eta(t))) \end{aligned} \tag{21}$$

and for $\lambda \geqslant t$ set $\zeta(\lambda) = \eta(\sigma, \varphi, y_0, \mu)(\lambda) - \lambda C(t, \mu) \, \eta(\sigma, \varphi, y_0, \mu)(t) \, \mu^{-1}$.

Then from (17) we obtain that

$$\mathscr{D}^+\zeta(t) = \mathscr{D}^+[C^{-1}(t, \mu)\, L^{(2)}(t, X_t, \mu)].$$

By Lemma 4 and inequality (20) we have

$$\begin{aligned}
&\limsup_{h \to 0^+}([\eta(t+h) - \eta(t)]/h - C(t, \mu)\, \eta(t)\, \mu^{-1}, W_\eta(t, \eta(t)))\\
&\quad = \limsup_{h \to 0^+}([\zeta(t+h) - \zeta(t)]/h, W_\eta(t, \eta(t)))\\
&\quad \leqslant |\mathscr{D}^+\zeta(t)| \cdot |W_\eta(t, \eta(t))|\\
&\quad \leqslant 2\sqrt{n}\, L^3\, |\eta(t)| \sup_{s \in [g(t), t]} (|X(s)| + |\mathscr{D}X(s)|).
\end{aligned} \tag{22}$$

From relations (11), in view of condition (H2), we deduce the inequality

$$(C(t, \mu)\, \eta(t)\, \mu^{-1}, W_\eta(t, \eta(t))) \leqslant -(m - L\rho(\mu))\, |\eta(t)|^2\, \mu^{-1}. \tag{23}$$

Combining estimates (21), (22), and (23) we see that for all values of $t \geqslant \sigma$ such that $\sup_{s \in [g(t), t]} |\mathscr{D}X(s)| < \infty$ the following inequality holds

$$\begin{aligned}
v_2'(t) \leqslant &-(m - L\rho(\mu))\, \mu^{-1}\, |\eta(t)|^2 + L\, |\eta(t)|^2\\
&+ 2\sqrt{n}\, L^3\, |\eta(t)| \sup_{s \in [g(t), t]} (|X(s)| + |\mathscr{D}X(s)|).
\end{aligned} \tag{24}$$

Now let $t \geqslant \Sigma$. Then $\sigma < G(t) \leqslant g(t)$ and from equalities (3), (15), condition (H1), and Lemma 5 it follows that

$$\sup_{s \in [g(t), t]} (|X(s)| + |\mathscr{D}X(s)|) \leqslant M_4 \sup_{s \in [G(t), t]} (|\xi(s)| + |x(s)| + |\eta(s)|),$$

where the number M_4 depends only on L.

Hence from estimates (19), (24), and (2) we obtain that for $t \geqslant \Sigma$ the following inequality holds

$$\begin{aligned}
v'(t) \leqslant &-m\, \|\xi_t\|^2 - (m - L\rho(\mu))\, \mu^{-1}\, |\eta(t)|^2\\
&+ M_5\, \|\xi_t\|\, \rho(\mu) \sup_{s \in [G(t), t]} \|\xi_s\| + M_5\, |\eta(t)| \{ \|\xi_t\|\\
&+ \sup_{s \in [G(t), t]} (\|\xi_s\| + |\eta(s)|) + \|\varphi\| \exp(-\alpha(G(t) - \sigma))\},
\end{aligned} \tag{25}$$

where the constant M_5 depends only on L, $K(0)$, p, and n. We shall note that there exists a positive number $\mu_1 \in (0, \mu_0)$ such that $L\rho(\mu) \leqslant m/2$ and $m^{-1}LM_5\sqrt{2}\, \rho(\mu) \leqslant m/2$ for $\mu \in (0, \mu_1)$. With every positive number δ we associate the set $P(\delta)$ consisting of all functions v defined by (18) such that $\|\varphi\| + |y_0| < \delta$. Let δ and a be positive numbers and $\mu \in (0, \mu_1)$. For any

$t \geqslant \Sigma$ and $v \in P(\delta)$ satisfying the condition $2v(t) \geqslant v(s)$ for $s \in [G(t), t]$, from (25) and (12) there follows the estimate

$$v'(t) \leqslant -m \|\xi_t\|^2/2 - m\mu^{-1} |\eta(t)|^2/2$$
$$+ M_6 |\eta(t)| \{ \|\xi_t\| + |\eta(t)| + \|\varphi\| \exp[-\alpha(G(t)-\sigma)] \}, \quad (26)$$

where the number M_6 depends only on M_5, L, and m.

Let, moreover, the condition $v(t) \geqslant a$ be fulfilled. Then from inequalities (12) and (26) it follows, by Lemma 6, that for

$$0 < \mu \leqslant \mu_2 = \min \left\{ 1, \mu_1, \frac{m^2}{24M_6^2}, \frac{m}{24M_6} \right\}$$

and $\delta \exp[-\alpha(G(t)-\sigma)] \leqslant \sqrt{a}/(3L)$ the following inequalities hold

$$v'(t) \leqslant -\frac{m}{4} \|\xi_t\|^2 - \frac{m}{4} |\eta(t)|^2 \leqslant \frac{-ma}{8L^2}. \quad (27)$$

Now fix the number $\mu \in (0, \mu_2)$. We shall prove that the zero solution of (3) is equiasymptotically stable. First we shall show that it is stable. Let $\sigma \in I$ and $\varepsilon > 0$. Set $\varepsilon_1 = \varepsilon_1(\varepsilon) = 3m\varepsilon/(4L^2+7)$ and $\delta = \delta(\varepsilon, \sigma) = \min\{\varepsilon_1/(3L), \varepsilon_1/(LC_2), \varepsilon/C_1\}$, where the numbers C_1 and C_2 are the same as in (14) and (16). Let $\|\varphi\| + |y_0| \leqslant \delta$. We shall verify that for any $t \geqslant \sigma$ the inequality $\|X_t\| + |Y(t)| \leqslant \varepsilon$ holds. For $t \in [\sigma, \Sigma]$ from (12) and (16) it follows that $v(t) \leqslant L^2 C_2^2 \delta^2 \leqslant \varepsilon_1^2$. Combining this estimate and inequalities (27) for $a = \varepsilon_1^2$, $\mu \in (0, \mu_2)$, $\delta \leqslant \varepsilon_1/(3L)$, we obtain, by Lemma 1, the inequality

$$m^2(\|\xi_t\| + |\eta(t)|)^2 \leqslant v(t) \leqslant \varepsilon_1^2 \qquad \text{for} \quad t \geqslant \sigma$$

which, together with (15) and (2), yields

$$\|X_t\| \leqslant \|\xi_t\| + \|x_t\| \leqslant \varepsilon_1/m + L\delta \leqslant 4\varepsilon_1/(3m) \qquad \text{for} \quad t \geqslant \sigma$$

and

$$|Y(t)| \leqslant |\eta(t)| + L^2 \sup_{s \in [g(t), t]} \|X_s\| \leqslant \varepsilon_1/m + 4\varepsilon_1 L^2/(3m)$$

for $t \geqslant \Sigma$. Hence $\|X_t\| + |Y(t)| \leqslant (4L^2+7)\varepsilon_1/(3m) = \varepsilon$ for $t \geqslant \Sigma$, which, together with (14), yields $\|X_t\| + |Y(t)| \leqslant \varepsilon$ for $t \geqslant \sigma$, hence the zero solution of (3) is stable. In the case of bounded delay, i.e., when $g(t) \geqslant t - h$, $h = \text{const}$, the stability is uniform on $\sigma \in I$, because in this case, by Remark 2, the numbers C_1 and C_2 do not depend on σ.

To complete the proof of assertion (i) we shall apply Lemma 3. Let

$\sigma \geqslant t_0$. Set $\delta_0 = \min\{1/(3L), 1/(LC_2)\}$. For $v \in P(\delta_0)$ and $t \geqslant \sigma$ we have $v(t) \leqslant 1$. Since $G(t) \to +\infty$ as $t \to +\infty$, for any $a > 0$ there exists a number $\lambda(a) = \lambda(a, \sigma) \geqslant \Sigma$ such that $G(t) \geqslant \sigma + \alpha^{-1} \ln(3L\delta_0/\sqrt{a})$ for $t \geqslant \lambda(a)$. Then from estimate (27) it follows that Lemma 3 can be applied to the set $P = P(\delta_0)$ for $z_0 = \sigma$, $r(t) = G(t)$, $f(t) = 2t$, $c_0 = 1$, $\tau(a) = \lambda(a)$ and $K(a) = ma/(8L^2)$. By Lemma 3 there exists a function $\Gamma(a) = \Gamma(a, 1, \sigma)$ defined for $a > 0$ such that for $v \in P(\delta_0)$ and $t \geqslant \Gamma(a)$ the inequality $v(t) \leqslant a$ holds.

Let ε be an arbitrary positive number. Set $\varepsilon_2 = \varepsilon/(2L^3 + 2)$, $Q_1 = Q_1(\varepsilon, \sigma) = \Gamma(m^2\varepsilon_2^2, 1, \sigma)$ and $Q_2 = \sigma + \alpha^{-1} \ln(L\delta_0/\varepsilon_2)$. For $t \geqslant Q_1$ and $\|\varphi\| + |y_0| \leqslant \delta_0$ from (12) and (18) it follows that $\|\xi_t\| + |\eta(t)| \leqslant \varepsilon_2$. On the other hand, inequality (2) shows that for $t \geqslant Q_2$ and $\|\varphi\| \leqslant \delta_0$ the estimate $\|x_t\| \leqslant L\, \|\varphi\|\, e^{-\alpha(t-\sigma)} \leqslant \varepsilon_2$ holds. Moreover, there exists a number $Q_3 = Q_3(\varepsilon, \sigma)$ such that $t \geqslant g(t) \geqslant \max\{Q_1, Q_2\}$ for $t \geqslant Q_3$. Hence for $\|\varphi\| + |y_0| < \delta_0$ and $t \geqslant Q_3(\varepsilon, \sigma)$ the following inequalities are satisfied

$$L^{-1}\, |X(t)| \leqslant \|X_t\| \leqslant \|\xi_t\| + \|x_t\| \leqslant 2\varepsilon_2$$

$$|Y(t)| \leqslant |\eta(t)| + L^2 \sup_{s \in [g(t), t]} |X(s)| \leqslant (2L^3 + 1)\, \varepsilon_2$$

which imply the inequality $\|X_t\| + |Y(t)| \leqslant \varepsilon$; i.e., the zero solution of (3) is equiasymptotically stable. In the case of bounded delay when $g(t) \geqslant t - h$, $h = \text{const}$, we have $\Sigma - \sigma = 2h$,

$$\lambda(a) - \sigma = 2h + \alpha^{-1} \ln(3L\delta_0/\sqrt{a}),$$

where δ_0 does not depend on σ. By Remark 1 the numbers $\Gamma(a, 1, \sigma) - \sigma$ and $Q_1(\varepsilon, \sigma) - \sigma$ do not depend on σ. In such a case the zero solution of (3) is uniformly asymptotically stable since $Q_3(\varepsilon, \sigma) - \sigma = h + \max\{Q_1 - \sigma, Q_2 - \sigma\}$ does not depend on σ.

This completes the proof of assertion (i).

4. *Proof of Assertion* (ii) *of Theorem* 1

There exists a number τ_0 such that $g(t) \geqslant t_0$ for $t \geqslant \tau_0$. Choose the number b in such a way that $t_0 - b \leqslant g(t)$ for $t \in [t_0, \tau_0]$. Then for $g(t) \leqslant t_0 \leqslant t$ we obtain $g(t) - t_0 \in [-b, 0]$. Since the function φ satisfies the Lipschitz condition in the interval $[-b, 0]$, there exists a number $L_1 = L_1(\varphi)$ such that

$$L^2\, \|\varphi\| \leqslant L_1 \qquad \text{and} \qquad |\varphi(s)| \leqslant L_1, \qquad |\mathscr{D}\varphi(s)| \leqslant L_1 \tag{28}$$

for $s \in [-b, 0]$. By means of equalities (15) we again introduce the functions $\xi(t) = \xi(y_0, \mu)(t)$ and $\eta(t) = \eta(y_0, \mu)(t)$, where the number $\mu \in (0, \mu_0)$ and the vector $y_0 \in R^n$ are parameters. Note that in (15) the dependence on y_0 and φ is omitted. Let Q, h_0 and ε be arbitrary positive numbers. We shall prove that there exists a number $\mu_3=\mu_3(\varphi, Q, h_0, \varepsilon)\in(0,\mu_0)$ such that for $|y_0|\leqslant Q$, $\mu\in(0,\mu_3)$ the following inequalities are valid

$$\begin{aligned} |\xi(y_0,\mu)(t)| &\leqslant \varepsilon \quad \text{for} \quad t\geqslant t_0 \\ |\eta(y_0,\mu)(t)| &\leqslant \varepsilon \quad \text{for} \quad t\geqslant t_0+h_0. \end{aligned} \tag{29}$$

Then assertion (ii) of Theorem 1 follows immediately from (29) in view of equalities (15), (5), condition (H2) and estimates (2) and (13).

First note that $\xi(s)=0$ for $s\leqslant t_0$ and $L\rho(\mu)\leqslant m/2$ for $\mu\in(0,\mu_1)$. Then from inequalities (19) and (24), in view of equalities (3) and (15), condition (H1), Lemma 5, and estimates (2) and (28), we obtain that for $\mu\in(0,\mu_1)$ and $t\geqslant t_0$ the following inequalities are valid

$$v_2'(t)\leqslant -\frac{m}{2\mu}|\eta(t)|^2+M_7|\eta(t)|\{1+\sup_{s\in[t_0,t]}(|\xi(s)|+|\eta(s)|)\} \tag{30}$$

$$v'(t)\leqslant -\frac{m}{2}\|\xi_t\|^2-\frac{m}{2\mu}|\eta(t)|^2+M_8|\eta(t)|\{1+\sup_{s\in[t_0,t]}(|\xi(s)|+|\eta(s)|)\}, \tag{31}$$

where M_7 and M_8 are positive constants depending only on L and L_1.

Since $\xi_{t_0}=0$, from estimates (9), (12), and (28) it follows that

$$v(y_0,\mu)(t_0)=v_2(y_0,\mu)(t_0)\leqslant L^2|\eta(t_0)|^2\leqslant L^2(|y_0|+L^2L_1)^2\leqslant q^2 \tag{32}$$

for $|y_0|\leqslant Q$, where $q=L(Q+L^2L_1)$.

On the other hand, for any $t\geqslant t_0$ satisfying the conditions $v(y_0,\mu)(t)\geqslant q^2$ and $v(y_0,\mu)(t)\geqslant v(y_0,\mu)(s)$ for $s\in[t_0,t]$, from inequalities (31) and (12) it follows that

$$v'(y_0,\mu)(t)\leqslant -\frac{m}{2}\|\xi_t\|^2-\frac{m}{2\mu}|\eta(t)|^2+\frac{L^2M_8}{m}|\eta(t)|\{1+\|\xi_t\|+|\eta(t)|\}$$

and by Lemma 6 there exists $\gamma_1=\gamma_1(q)\leqslant\mu_1$ such that for $\mu\leqslant\gamma_1$ we have $v'(y_0,\mu)(t)\leqslant 0$. Then from Lemma 1 it follows that for $t\geqslant t_0$, $\mu\leqslant\gamma_1$, $|y_0|\leqslant Q$ we have $v(y_0,\mu)(t)\leqslant q^2$ and

$$|\xi(y_0,\mu)(t)|+|\eta(y_0,\mu)(t)|\leqslant Lq/m. \tag{33}$$

Since $\xi(y_0,\mu)(s)=0$ for $s\leqslant t_0$, from equality (17), estimates (2), (13), (33), and condition (H1), by Lemma 5 we obtain that $|(d/dt)(|\xi(y_0,\mu)(t)|^2)|\leqslant F_1$ for $t>t_0$, where the constant F_1 depends only on L, L_1, m, and Q.

Set $h_1 = \min\{h_0, m^2\varepsilon^2/(4F_1 L^2)$. From the finite increment theorem it follows that for $t \in [t_0, t_0 + h_1]$, $\mu \leqslant \gamma_1$, $|y_0| \leqslant Q$ we have

$$|\xi(y_0, \mu)(t)| \leqslant m\varepsilon/(2L). \tag{34}$$

On the other hand, inequalities (30) and (33) show that

$$v_2'(t) \leqslant -m\,|\eta(t)|^2/(2\mu) + F_2 \tag{35}$$

for $t \geqslant t_0$, $\mu \leqslant \gamma_1$, $|y_0| \leqslant Q$, where the numbers γ_1 and F_2 depend only on L, m, and Q.

Set $N_1 = \max\{0, 4L^2q^2 - m^4\varepsilon^2\}/(4L^2h_1)$ and $\gamma_2 = m^5\varepsilon^2/[8L^4(N_1 + F_2)]$. For $\mu \leqslant \min\{\gamma_1, \gamma_2\}$ and $|y_0| \leqslant Q$ from inequalities (35) and the definition of γ_2 it follows that for each $t \geqslant t_0$ which satisfies the condition $v_2(t) \geqslant m^4\varepsilon^2/(4L^2)$ the estimate $v_2'(t) \leqslant -N_1$ holds. Then, by Lemma 2 and the definition of the number N_1, we obtain $v_2(t_0 + h_1) \leqslant m^4\varepsilon^2/(4L^2)$, whence it follows that

$$|\eta(y_0, \mu)(t_0 + h_1)| \leqslant m\varepsilon/(2L) \tag{36}$$

for $\mu \leqslant \min\{\gamma_1, \gamma_2\}$ and $|y_0| \leqslant Q$. From (34), (36), and (12) we deduce the inequality

$$v(y_0, \mu)(t_0 + h_1) \leqslant m^2\varepsilon^2. \tag{37}$$

For any $t \geqslant t_0 + h_1$ such that $v(y_0, \mu)(t) \geqslant m^2\varepsilon^2$ from inequalities (12), (31), and (33) it follows that $v'(y_0, \mu)(t) < 0$ for $\mu \leqslant \gamma_3$, where by Lemma 6 the number $\gamma_3 > 0$ depends only on m, L, L_1, Q, and ε. Then from Lemma 1 and inequality (37) for $t \geqslant t_0 + h_1$, $|y_0| \leqslant Q$, $\mu \leqslant \mu_3 = \min\{\gamma_1, \gamma_2, \gamma_3\}$ it follows that $v(y_0, \mu)(t) \leqslant m^2\varepsilon^2$ and in view of (12) and (36) we obtain the inequality (29) since $h_1 \leqslant h_0$. Theorem 1 is proved.

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