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A new inversion free iteration for solving the equation $X + A^{\star}X^{-1}A = Q$

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Abstract

In this paper, we introduce a new inversion free variant of the basic fixed point iteration method for obtaining a maximal positive definite solution of the nonlinear matrix equation $X + A^{\star}X^{-1}A = Q$. It is more accurate than Zhan's algorithm (J. Sci. Comput. 17 (1996) 1167) and has less number of operations than the algorithm of Guo and Lancaster (Math. Comput. 68 (1999) 1589). We derive convergence conditions of the iteration and existence conditions of a solution to the problem. Finally, we give some numerical results to illustrate the behavior of the considered algorithm.

Keywords: Matrix equation; Inversion free variant of the basic fixed point iteration methods; Convergence rate

1. Introduction

We consider the nonlinear matrix equation

$$X + A^{\star}X^{-1}A = Q, \tag{1.1}$$

where $A,\ Q \in C^{n\times n}$ with Q positive definite matrix. It is easy to see that the matrix equation (1.1) can be reduced to

$$X + A^{\star}X^{-1}A = I, \tag{1.2}$$

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where I is the identity matrix, see [8,12]. This type of nonlinear matrix equations often arises in the analysis of ladder networks, dynamic programming, control theory, stochastic filtering, statistics and many applications, see [1,6] and the references therein. The equation can be viewed as a natural extension for the scalar equation $x + a^2/x = 1$. This scalar problem is equivalent to equation $\varphi(x) = a^2$, where $\varphi(x) = x(1 - x)$. This equation has a positive solution x so that $0 < x < 1$ if $a^2 \leqslant \max \varphi(x) = \varphi\left(\frac{1}{2}\right)$. The equation can also be viewed as a special case of a discrete-time algebraic Riccati equation

$$0 = Q + F^\star XF - X - (F^\star XB + A^\star)(R + B^\star XB)^{-1}(B^\star XF + A),$$

where Q is a positive definite matrix, see [6]. The discrete-time algebraic Riccati equation can be reduced to (1.2), by setting $F = 0$, $B = I$ and $R = 0$.

Eq. (1.2) has been studied recently by several authors [1–5,7–13]. Anderson et al. [1] discussed the existence of the positive solution to the matrix equation (1.2) with right-hand side an arbitrary matrix, while Engwerda et al. [2] established and proved theorems for the necessary and sufficient conditions of existence of a positive definite solution of same matrix equation as in [1]. They discussed both the real and complex case and established recursive algorithms to compute the largest and smallest solution of the equation. Engwerda [3] proved the existence of the positive definite solution of the real matrix equation (1.2) and also found an algorithm to calculate the solution. El-Sayed et al. [8,10–13] obtained necessary and sufficient conditions for existence of a positive definite solution of matrix equations with several forms instant of X^{-1} in (1.2). Zhan and Xie [13] were proposed several numerical algorithms for finding solutions for (1.2). In [12], Zhan was proposed an algorithm that avoids matrix inversion for every iteration called inversion free variant of the basic fixed point iteration.

$$\begin{aligned}
&\textit{Take}\quad X_0 = Y_0 = I,\\
&X_{n+1} = I - A^\star Y_n A,\\
&Y_{n+1} = Y_n(2I - X_n Y_n), \quad n = 0, 1, 2, \ldots\ .
\end{aligned} \tag{1.3}$$

Guo and Lancaster [4] modified Zhan's algorithm (1.3) to find the maximal positive definite solutions of Eq. (1.2) as the following:

$$\begin{aligned}
&\textit{Take}\quad X_0 = Y_0 = I,\\
&Y_{n+1} = Y_n(2I - X_n Y_n),\\
&X_{n+1} = I - A^\star Y_{n+1} A, \quad n = 0, 1, 2, \ldots\ .
\end{aligned} \tag{1.4}$$

He were gave more deep discussion of the convergence of the inversion free variant of the basic fixed point iteration method for Eq. (1.2) than the algorithm in [12].

The our goal of this paper is to discuss the matrix equation (1.2) with a new inversion free variant of the basic fixed point iteration method.

$$\begin{aligned}
&\textit{Take}\quad X_0 = Y_0 = I,\\
&Y_{n+1} = (I - X_n)Y_n + I,\\
&X_{n+1} = I - A^\star Y_{n+1} A, \quad n = 0, 1, 2, \ldots\ .
\end{aligned} \tag{1.5}$$

The suggested algorithm also avoid matrix inversion. Furthermore the algorithm requires only three matrix multiplications per step, whereas Zhan's algorithm (1.3) and Guo et al. algorithm (1.4) requires four matrix multiplications per step. We use the algorithm to obtain numerically the maximal solution of Eq. (1.2) under some additional conditions. We obtain the rate of convergence for the sequence generated by our suggested algorithm. Some numerical examples are given to show the behavior of the considered algorithm.

The paper is organized as follows. In Section 2, under some conditions on matrix A we obtain the rate of convergence of the iterative sequence of approximate solutions. Section 3 illustrates the performance of the method with some numerical examples. Conclusion drawn from the results obtained in this paper are in Section 4.

The following notations are used throughout the rest of the paper. The notation $A \geqslant 0$ $(A > 0)$ means that A is positive semidefinite (positive definite), $A^\star$ denotes the complex conjugate transpose of A, and I is the identity matrix. Moreover, $A \geqslant B$ $(A > B)$ is used as a different notation for $A - B \geqslant 0$ $(A - B > 0)$. We denote by ρ the largest eigenvalue of $A^\star A$. The norm used in this paper is the spectral norm of the matrix A, i.e., $\|A\| = \sqrt{\rho(AA^\star)}$ unless otherwise noted.

2. Conditions for existence of the solutions

In this section, we introduce an inversion free variant of the basic fixed point iteration method to avoid the computation of the matrix's inverse for every iteration. We will discuss some properties of Eq. (1.2) and obtain the conditions for existence of the solutions of Eq. (1.2).

We will prove that the sequence $\{X_n\}$ is monotone decreasing and converges to the maximal solution X_+.

Theorem 2.1. *If Eq.* (1.2) *has a positive definite solution and the two sequences* $\{X_n\}$ *and* $\{Y_n\}$ *are determined by the Algorithm* (1.5), *then* $\{X_n\}$ *is monotone decreasing and converges to the maximal solution* X_+ *. If matrix A is nonsingular and* $X_n > 0$ *for every n, then* (1.2) *has a positive definite solution.*

Proof. First, we will prove that $I = X_0 \geqslant X_1 \geqslant X_2 \geqslant \cdots \geqslant X_n \geqslant X_+$ and $I = Y_0 \leqslant Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_n \leqslant X_+^{-1}$.

Since X_+ is solution of (1.2), i.e.,

$$X_+ = I - A^\star X_+^{-1} A,$$

then $X_0 = I \geqslant X_+$. Also

$$X_1 = I - A^\star A \geqslant I - A^\star X_+^{-1} A = X_+,$$

i.e., $X_0 \geqslant X_1 \geqslant X_+$. For

$$X_2 = I - A^\star Y_2 A = I - A^\star A - A^{\star 2} A^2 = X_1 - A^{\star 2} A^2,$$

this implies to $X_2 \leqslant X_1$, i.e., $X_0 \geqslant X_1 \geqslant X_2$.

For the sequence $\{Y_n\}$ we have $Y_0 = Y_1 = I$ and since $X_+^{-1} \geqslant I$, then $Y_0 = Y_1 \leqslant X_+^{-1}$

$$Y_0 = Y_1 = I \leqslant Y_2 = (I - X_1)Y_1 + I = A^\star A + I,$$

i.e., $Y_0 = Y_1 \leqslant Y_2$.

We also have

$$Y_2 = (I - X_1)Y_1 + I \leqslant (I - X_+)X_+^{-1} + I = X_+^{-1},$$

i.e., $Y_1 \leqslant Y_2 \leqslant X_+^{-1}$. Concerning $\{X_2\}$, we get

$$X_2 = I - A^\star Y_2 A \geqslant I - A^\star X_+^{-1} A = X_+,$$

i.e., $X_0 \geqslant X_1 \geqslant X_2 \geqslant X_+$.

That means that the inequalities are true for $n = 0, 1, 2$. So, assume that the above inequalities are true for $n = k$, i.e.,

$$I = X_0 \geqslant X_1 \geqslant X_2 \geqslant \cdots \geqslant X_k \geqslant X_+$$

and

$$I = Y_0 \leqslant Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_k \leqslant X_+^{-1}.$$

Now we will prove inequalities at $n = k + 1$, then

$$Y_{k+1} = (I - X_k)Y_k + I \geqslant (I - X_{k-1})Y_{k-1} + I = Y_k.$$

We also have

$$Y_{k+1} = (I - X_k)Y_k + I \leqslant (I - X_+)X_+^{-1} + I = X_+^{-1},$$

i.e., $Y_k \leqslant Y_{k+1} \leqslant X_+^{-1}$. Concerning the sequence $\{X_n\}$, we have

$$X_k - X_{k+1} = A^\star (Y_{k+1} - Y_k)A,$$

since $Y_{k+1} \geqslant Y_k$, hence $X_k \geqslant X_{k+1}$. Therefore,

$$X_{k+1} = I - A^\star Y_{k+1} A \geqslant I - A^\star X_+^{-1} A = X_+,$$

i.e., $X_k \geqslant X_{k+1} \geqslant X_+$.

This completes the induction for $n = k + 1$. Therefore, $I = X_0 \geqslant X_1 \geqslant X_2 \geqslant \cdots \geqslant X_n \geqslant X_+$ and $I = Y_0 \leqslant Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_n \leqslant X_+^{-1}$ are true for all n, and $\lim_{n\to\infty} X_n$ and $\lim_{n\to\infty} Y_n$ exist. Taking limit in the Algorithm (1.5) leads to $Y = X^{-1}$ and $X = I - A^\star X^{-1} A$. Moreover, as each $X_n \geqslant X_+$ then $X = X_+$.

If matrix A is nonsingular and $X_n > 0$ for every n. Hence the above proof of the monotonicity of $\{Y_n\}$ remains valid (monotone increasing). It follows that sequence $\{X_n\}$ is monotone decreasing and bounded from below by the zero matrix. So, $\lim_{n\to\infty} X_n = X$ exists. Since A is nonsingular $Y_{n+1} = A^{-\star}(I - X_{n+1})A^{-1}$. Thus $\lim_{n\to\infty} Y_n = Y$ exist. As $Y_0 = I$ and $\{Y_n\}$ is monotone increasing, $Y \geqslant I$. Taking limit in the Algorithm (1.5) implies

$$\begin{aligned} Y &= (I - X)Y + I, \\ X &= I - A^\star Y A. \end{aligned} \tag{2.1}$$

Since $Y \geqslant I$, $X = Y^{-1} > 0$, and hence $X = I - A^\star X^{-1} A$. So Eq. (1.2) has a positive definite solution. □

Lemma 2.1. *Assume that Eq.* (1.2) *has a positive definite solution and* $\|A\| < \frac{1}{2}$, *then the sequence* $\{Y_n\}$ *satisfies* $\|Y_n A\| < 1$ *for every* $n = 0, 1, \ldots$.

Proof. Since $Y_0 = Y_1 = I$, it is clear that $\|Y_0 A\| = \|Y_1 A\| < \frac{1}{2} < 1$. For Y_2 we have $Y_2 = (I - X_1)Y_1 + I = A^\star A + I$, thus $\|Y_2 A\| = \|A^\star A^2 + A\| \leqslant \|A^\star A^2\| + \|A\| < \frac{5}{8} < 1$. That is means that the inequality holds for $n = 0, 1, 2$. So, assume that the inequality satisfies $n = k$, i.e., $\|Y_k A\| < 1$. Now we will prove inequality when $n = k + 1$.

$$\begin{aligned} Y_{k+1}A &= [(I - X_k)Y_k + I]A \\ &= [(I - (I - A^\star Y_k A))Y_k + I]A \\ &= A^\star Y_k A Y_k A + A. \end{aligned} \tag{2.2}$$

Then we get

$$\begin{aligned} \|Y_{k+1}A\| &\leqslant \|A^\star Y_k A Y_k A\| + \|A\| \\ &\leqslant \|A^\star\| \|Y_k A\|^2 + \|A\| \\ &\leqslant \|A^\star\| + \|A\| < 1. \end{aligned} \tag{2.3}$$

This completes the induction for $n = k + 1$ and the lemma. □

We now establish the following result to obtain the rate of convergence for the Algorithm (1.5).

Theorem 2.2. *If Eq.* (1.2) *has a positive definite solution and* $\|A\| < \frac{1}{2}$, *then the sequence* $\{X_n\}$ *satisfies*

$$\|Y_{n+1} - X_+^{-1}\| \leqslant \|A X_+^{-1}\| \|Y_n - X_+^{-1}\|, \tag{2.4}$$

and

$$\|X_{n+1} - X_+\| \leqslant \|A\|^2 \|Y_n - X_+^{-1}\|, \tag{2.5}$$

for all n large enough. If the matrix A is nonsingular, *we also have*

$$\|X_{n+1} - X_+\| \leqslant \|X_+^{-1} A\| \|X_n - X_+\|. \tag{2.6}$$

Proof.

$$\begin{aligned} Y_{n+1} &= (I - X_n)Y_n + I \\ &= A^\star Y_n A Y_n + I \\ &= A^\star (Y_n + X_+^{-1} - X_+^{-1}) A Y_n + I \\ &= A^\star (Y_n - X_+^{-1}) A Y_n + A^\star X_+^{-1} A Y_n + Y_n - Y_n + I \\ &= A^\star (Y_n - X_+^{-1}) A Y_n - (I - A^\star X_+^{-1} A) Y_n + Y_n + I \\ &= A^\star (Y_n - X_+^{-1}) A Y_n - X_+ Y_n + Y_n + I. \end{aligned} \tag{2.7}$$

Then we get

$$\begin{aligned} X_+^{-1} - Y_{n+1} &= X_+^{-1} + A^\star (X_+^{-1} - Y_n) A Y_n + X_+ Y_n - Y_n - I \\ &= (I - X_+)(X_+^{-1} - Y_n) + A^\star (Y_n - X_+^{-1}) A Y_n \\ &= A^\star X_+^{-1} A (X_+^{-1} - Y_n) + A^\star (Y_n - X_+^{-1}) A Y_n, \end{aligned} \tag{2.8}$$

i.e., we have

$$\begin{aligned}\|X_+^{-1} - Y_{n+1}\| &\leqslant \|A^\star X_+^{-1} A\| \|X_+^{-1} - Y_n\| + \|A^\star\| \|AY_n\| \|X_+^{-1} - Y_n\| \\ &\leqslant (\|X_+^{-1} A\| + \|AY_n\|) \|A^\star\| \|X_+^{-1} - Y_n\|. \end{aligned} \tag{2.9}$$

Since $\lim_{\to\infty} Y_n = X_+^{-1}$, then

$$\begin{aligned}\|Y_{n+1} - X_+^{-1}\| &\leqslant 2\|A^\star\| \|AX_+^{-1}\| \|Y_n - X_+^{-1}\| \\ &\leqslant \|AX_+^{-1}\| \|Y_n - X_+^{-1}\|. \end{aligned} \tag{2.10}$$

Then the inequality (2.4) is true. The second inequality (2.5) holds directly from the following equality.

$$X_{n+1} - X_+ = A^\star (X_+^{-1} - Y_{n+1}) A.$$

To prove the last inequality (2.6), we have from Eq. (2.8) the following:

$$\begin{aligned}X_+^{-1} - Y_{n+1} &= A^\star X_+^{-1} A (X_+^{-1} - Y_n) + A^\star (Y_k - X_+^{-1}) AY_n \\ &= A^\star X_+^{-1} AA^{-\star} (A^\star X_+^{-1} A - A^\star Y_n A) A^{-1} + (A^\star Y_k A - A^\star X_+^{-1} A) Y_n \\ &= A^\star X_+^{-1} AA^{-\star} (X_n - X_+) A^{-1} + (X_n - X_+) Y_n. \end{aligned} \tag{2.11}$$

Therefore,

$$\begin{aligned}X_{n+1} - X_+ &= A^\star (X_+^{-1} - Y_{n+1}) A \\ &= (A^\star)^2 X_+^{-1} AA^{-\star} (X_n - X_+) + A^\star (X_n - X_+) Y_n A. \end{aligned} \tag{2.12}$$

Taking norm for the above equation, we get

$$\begin{aligned}\|X_{n+1} - X_+\| &\leqslant \|A^\star\|^2 \|X_+^{-1} A\| \|A^{-\star}\| \|X_n - X_+\| + \|A^\star\| \|Y_n A\| \|X_n - X_+\| \\ &\leqslant (\|X_+^{-1} A\| + \|Y_n A\|) \|A^\star\| \|X_n - X_+\|. \end{aligned} \tag{2.13}$$

Since $\lim_{\to\infty} Y_n = X_+^{-1}$, then

$$\begin{aligned}\|X_{n+1} - X_+\| &\leqslant 2\|A^\star\| \|X_+^{-1} A\| \|X_n - X_+\| \\ &\leqslant \|X_+^{-1} A\| \|X_n - X_+^{-1}\|. \end{aligned} \tag{2.14}$$

Then the inequality (2.6) is fulfilled. $\square$

We note that from the Algorithm (1.5) $I - X_n Y_n = Y_{n+1} - Y_n \to 0$, as $n \to \infty$. Then one stopping criterion may be $\|I - X_n Y_n\| < \varepsilon$, for small $\varepsilon > 0$. The effect of the stopping criterion can be seen from the following Theorem.

Theorem 2.3. *If the Eq.* (1.2) *has a solution and after n iterative steps of the Algorithm* (1.5), *we have* $\|I - X_n Y_n\| < \varepsilon$, *thus*

$$\|X_n + A^\star X_n^{-1} A - I\| \leqslant \varepsilon \|A\|^2 \|X_+^{-1}\|.$$

Proof. Since,

$$\begin{aligned} X_n + A^\star X_n^{-1} A - I &= X_n - X_{n+1} + A^\star (X_n^{-1} - Y_{n+1}) A \\ &= A^\star (Y_{n+1} - Y_n) A + A^\star (X_n^{-1} - Y_{n+1}) A \\ &= A^\star (Y_{n+1} - X_n^{-1} + X_n^{-1} - Y_n) A + A^\star (X_n^{-1} - Y_{n+1}) A \\ &= A^\star X_n^{-1} (I - X_n Y_n) A. \end{aligned} \tag{2.15}$$

Take norm in both sides,

$$\begin{aligned} \|X_n + A^\star X_n^{-1} A - I\| &\leqslant \|A\|^2 \|X_+^{-1}\| \|I - X_n Y_n\| \\ &\leqslant \varepsilon \|A\|^2 \|X_+^{-1}\|. \end{aligned} \tag{2.16}$$

3. Numerical experiments

In this section the numerical experiments are given to display the flexibility of the new inversion free variant of the basic fixed point iteration methods. The maximal solution are computed for some different matrices A with different orders. We will compare the suggested Algorithm (1.5) with Algorithm (1.3) and Algorithm (1.4). The numerical experiments were carried out on an IBM-PC Pentium IV 2000 MHz computer. Double precision is used in the following calculations. The machine precision approximately $1.11022 \cdot 10^{-16}$. For the following examples, we use the practical stopping criterion $\|X + A^T X^{-1} A - I\| < 10^{-16}$.

Example 3.1. Consider Eq. (1.2) with normal matrix

$$A = \frac{1}{32} \begin{pmatrix} 0.2 & -0.1 & -0.5 & 0.1 \\ -0.1 & 0.6 & -0.5 & 0.7 \\ -0.5 & -0.5 & 0.1 & 0.8 \\ 0.1 & 0.7 & 0.8 & 0.5 \end{pmatrix}.$$

For this matrix the spectral norm is $\|A\| = 0.0412375$. The exact maximal solution can be found according to the formula

$$X_+ = \tfrac{1}{2} [I + (I - 4A^\star A)^{1/2}],$$

which is valid for any normal matrix A with $\|A\| \leqslant \frac{1}{2}$ (see [13]). Therefore the exact maximal solution is

$$X_+ = \begin{pmatrix} 0.999697 & -0.234558 \cdot 10^{-3} & 0.195301 \cdot 10^{-4} & 0.391194 \cdot 10^{-3} \\ -0.234558 \cdot 10^{-3} & 0.998915 & -0.254492 \cdot 10^{-3} & -0.352352 \cdot 10^{-3} \\ 0.195301 \cdot 10^{-4} & -0.254492 \cdot 10^{-3} & 0.998876 & -0.784171 \cdot 10^{-4} \\ 0.391194 \cdot 10^{-3} & -0.352352 \cdot 10^{-3} & -0.784171 \cdot 10^{-4} & 0.99864 \end{pmatrix}.$$

Algorithm (1.3) needs 9 iterations to find the above maximal solution, Algorithm (1.4) needs 5 iterations and the suggested algorithm needs 5 iterations as Algorithm (1.4) but the number of operations is less than Algorithm (1.4).

Example 3.2. We consider Eq. (1.2) with nonnormal matrix

$$A = \frac{1}{100}\begin{pmatrix} 0.2 & -0.1 & 0.4 \\ 0.7 & 0.6 & -0.5 \\ 0.4 & 0.8 & 0.6 \end{pmatrix}.$$

For this matrix the spectral norm is $\|A\| = 0.00796591$. We will obtain the maximal solution X_+ (with first fifteen digits) by any iterative algorithm. Therefore the maximal solution is

$$X_+ = \begin{pmatrix} 0.999931 & -0.72008 \cdot 10^{-4} & 0.299962 \cdot 10^{-5} \\ -0.72008 \cdot 10^{-4} & 0.999899 & -0.140023 \cdot 10^{-4} \\ 0.299962 \cdot 10^{-5} & -0.140023 \cdot 10^{-4} & 0.999923 \end{pmatrix}.$$

Algorithm (1.3) needs 5 iterations to find the maximal solution, Algorithm (1.4) needs 3 iterations and the suggested algorithm needs 3 iterations as Algorithm (1.4) but the number of operations is less than Algorithm (1.4).

4. Conclusions and remarks

In this paper we considered the nonlinear matrix equations than (1.2). We suggested a new inversion free variant of the basic fixed point iteration method. We achieved the conditions for the existence of a positive definite solution. We discussed an iterative algorithm from which a solution can always be calculated numerically whenever the equation is solvable. Moreover, two numerical examples are given to show the algorithm suggested is more accurate than Algorithm (1.3). We observe that our suggested algorithm also avoid matrix inversion and involves only matrix–matrix multiplication. Furthermore the algorithm requires only three matrix multiplications per step, whereas Algorithm (1.3) and Algorithm (1.4) requires four matrix multiplications per step.

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