Note
A Theorem on Permutations

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Out of the classical theory of Riemann surfaces, we extract the following combinatorial theorem: If \( \pi_1, \ldots, \pi_m \) are permutations of \( 1, 2, \ldots, n \), such that \( \pi_1 \cdots \pi_m = 1 \), then \( v(\pi_1) + \cdots + v(\pi_m) \geq 2(n - t) \), where \( v(\pi) = n - r \), \( r \) being the number of orbits of the cyclic group generated by the permutation \( \pi \) acting on the set \( \{1, \ldots, n\} \), and where \( t \) is the number of orbits of the group generated by \( \pi_1, \ldots, \pi_m \).

The direct proof of this theorem seems to be difficult.

Let \( \Omega \) be the finite set of \( n \) letters \( 1, 2, \ldots, n \). Any permutation \( \pi \) of \( \Omega \) can be written as a product of some disjoint cycles. If \( l_1, \ldots, l_r \) denote the length of these cycles, then \( l_1 + \cdots + l_r = n \). Define \( v(\pi) \) by

\[
v(\pi) = (l_1 - 1) + \cdots + (l_r - 1) = n - r.
\]

Thus \( v(\pi) = 1 \) if \( \pi \) is a transposition, and \( v(\pi) = n - 1 \) if \( \pi \) is a cycle of length \( n \).

**Theorem.** If \( \pi_1, \ldots, \pi_m \) are permutations of \( \Omega \) such that \( \pi_1 \cdots \pi_m = 1 \), the identity permutation, then

\[
v(\pi_1) + \cdots + v(\pi_m) \geq 2(n - t),
\]

where \( t \) is the number of transitivity components of \( \Omega \) under the group generated by \( \pi_1, \ldots, \pi_m \).

**Corollary 1.** If the permutations \( \pi_1, \ldots, \pi_m \) of \( \Omega \) generate a transitive permutation group of \( \Omega \), then

\[
v(\pi_1) + \cdots + v(\pi_m) \geq n - 1.
\]
Corollary 2. If the permutations $\pi_1, \ldots, \pi_m$ generate a transitive permutation group of $\Omega$ and

$$v(\pi_1) + \cdots + v(\pi_m) = n - 1,$$

then $\pi_1 \cdots \pi_m$ is a cycle of length $n$. (Notice that the ordering of $\pi_1, \ldots, \pi_m$ is arbitrary.)

The proof of the above theorem we give below is indirect, though essentially combinatorial. I have been unable to give a direct proof to the theorem, or even to Corollary 2 when all the $\pi_i$ are cycles of length 3.

Proof of Theorem. It can be seen easily that the special case $t = 1$ implies the general cases. Hence we shall assume that $t = 1$, i.e., that the group generated by $\pi_1, \ldots, \pi_m$ is transitive on $\Omega$.

Take $m + 1$ distinct points $z_1, z_2, \ldots, z_m, z^*$ on the Riemann sphere, assign the permutations $\pi_1, \ldots, \pi_m$ to $z_1, z_2, \ldots, z_m$, respectively, and construct a $n$-sheeted branched covering surface $R$ of the Riemann sphere as in [1, pp. 104-105]. Then the transitivity of the group generated by $\pi_1, \ldots, \pi_m$ implies that $R$ is a connected oriented closed surface. Now compute the genus $g$ of $R$ by Hurwitz' method [1, p. 125]. Then

$$v(\pi_1) + \cdots + v(\pi_m) - (2n - 2) = 2g \geq 0.$$

Remark. The above proof shows that $v(\pi_1) + \cdots + v(\pi_m)$ is even. This fact, however, follows directly from the fact that $v(\pi)$ is even if and only if $n$ is even.

Reference