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$\mathcal{W}$-Gorenstein modules

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ABSTRACT

Let $\mathcal{W}$ be a self-orthogonal class of left R-modules. We introduce and study $\mathcal{W}$-Gorenstein modules as a common generalization of some known modules such as Gorenstein projective (injective) modules (Enochs and Jenda, 1995 [7]) and V-Gorenstein projective (injective) modules (Enochs et al., 2005 [12]). Special attention is paid to $\mathcal{W}_P$-Gorenstein and $\mathcal{W}_I$-Gorenstein modules, where $\mathcal{W}_P = \{C \otimes_R P \mid P$ is a projective left R-module$\}$ and $\mathcal{W}_I = \{\mathrm{Hom}_S(C, E) \mid E$ is an injective left S-module$\}$ with ${}_SC_R$ a faithfully semidualizing bimodule.

1. Introduction and preliminaries

Auslander and Bridger [2] introduced the G-dimension for finitely generated modules. Enochs and Jenda [7] defined Gorenstein projective modules whether the modules are finitely generated or not. Also, they defined the Gorenstein projective dimension for arbitrary (non-finitely generated) modules. It is well known that for finitely generated modules over a commutative Noetherian ring, the Gorenstein projective dimension agrees with the G-dimension. Along the same lines, Gorenstein injective modules were introduced in [7]. Since then, various generalizations of these modules are given over specific rings (see, e.g., [9,10,12,22]).

In Section 2 of this paper, we define and study $\mathcal{W}$-Gorenstein modules for a self-orthogonal class $\mathcal{W}$ of left R-modules. A left R-module M is said to be $\mathcal{W}$-*Gorenstein* if there exists an exact sequence $W_\bullet = \cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots$ of modules in $\mathcal{W}$ such that $M = \ker(W^0 \to W^1)$ and

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$W_\bullet$ is both $\mathrm{Hom}_R(\mathcal{W},-)$ and $\mathrm{Hom}_R(-,\mathcal{W})$ exact. For different choices of $\mathcal{W}$, the class $\mathcal{G}_w$ of $\mathcal{W}$-Gorenstein modules encompasses all of the aforementioned modules, and some results existing in the literature for the modules above can be obtained as particular instances of the results on $\mathcal{W}$-Gorenstein modules.

Section 3 is devoted to investigating $\mathcal{W}_P$-Gorenstein and $\mathcal{W}_I$-Gorenstein modules for a faithfully semidualizing bimodule ${}_SC_R$ over associative rings R and S, where $\mathcal{W}_P = \{C \otimes_R P \mid P$ is a projective left R-module$\}$ and $\mathcal{W}_I = \{\mathrm{Hom}_S(C, E) \mid E$ is an injective left S-module$\}$, and we simply call them C-Gorenstein projective and C-Gorenstein injective modules respectively. We prove that $\mathcal{W}_P = \mathrm{Add}\,{}_SC$ and $\mathcal{W}_I = \mathrm{Prod}\,C^+$, where $C^+ = \mathrm{Hom}_S(C, Q)$ with ${}_SQ$ an injective cogenerator. We also prove that the subcategories of C-Gorenstein injective left R-modules (Gorenstein projective left R-modules in the Auslander class $\mathcal{A}_C(R)$) and Gorenstein injective left S-modules in the Bass class $\mathcal{B}_C(S)$ (C-Gorenstein projective left S-modules) are equivalent under Foxby equivalence. For a commutative Noetherian ring R and a semidualizing R-module C, it is shown that a finitely generated R-module M is $\mathrm{add}\,C$-Gorenstein if and only if it is $\mathrm{Add}\,C$-Gorenstein. This result generalizes [5, Theorem 4.2.6].

Next we shall recall some notions and definitions which we need in the later sections.

Let $\mathcal{C}$ be a class of left R-modules. We define

$$^{\perp}\mathcal{C} = \bigcap_{i=1}^{\infty} {}^{\perp_i}\mathcal{C}, \quad \text{where } {}^{\perp_i}\mathcal{C} = \big\{X \mid \mathrm{Ext}^i(X, C) = 0 \text{ for all } C \in \mathcal{C}\big\},\ i \geqslant 1,$$

$$\mathcal{C}^{\perp} = \bigcap_{i=1}^{\infty} \mathcal{C}^{\perp_i}, \quad \text{where } \mathcal{C}^{\perp_i} = \big\{X \mid \mathrm{Ext}^i(C, X) = 0 \text{ for all } C \in \mathcal{C}\big\},\ i \geqslant 1.$$

A $\mathcal{C}$ *resolution* of a left R-module M is an exact sequence $C_\bullet = \cdots \to C_1 \to C_0 \to M \to 0$ with $C_i \in \mathcal{C}$ for all $i \geqslant 0$; moreover, if the sequence $\mathrm{Hom}(C, C_\bullet)$ is exact for every $C \in \mathcal{C}$, then we say that $C_\bullet$ is *proper*. The $\mathcal{C}$ *resolution dimension* $\mathit{resdim}_{\mathcal{C}}(M)$ of M is the minimal nonnegative integer n such that M has a $\mathcal{C}$ resolution of length n. Dually we have the definitions of a (coproper) $\mathcal{C}$ coresolution and the $\mathcal{C}$ coresolution dimension $\mathit{coresdim}_{\mathcal{C}}(M)$ of M. We say that $\mathit{resdim}_{\mathcal{C}}(M) < \infty$ ($\mathit{coresdim}_{\mathcal{C}}(M) < \infty$) if $\mathit{resdim}_{\mathcal{C}}(M) = n$ ($\mathit{coresdim}_{\mathcal{C}}(M) = n$) for some nonnegative integer n.

Let R and S be rings. Following [16], an (S, R)-bimodule $C = {}_SC_R$ is *semidualizing* if:

(1) ${}_SC$ admits a degreewise finite S-projective resolution.
(2) C_R admits a degreewise finite R-projective resolution.
(3) The homothety map ${}_SS_S \xrightarrow{{}_S\gamma} \mathrm{Hom}_R(C, C)$ is an isomorphism.
(4) The homothety map ${}_RR_R \xrightarrow{\gamma_R} \mathrm{Hom}_S(C, C)$ is an isomorphism.
(5) $\mathrm{Ext}_S^{\geqslant 1}(C, C) = 0$.
(6) $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$.

A semidualizing bimodule $C = {}_SC_R$ is *faithfully semidualizing* if it satisfies the following conditions for all modules ${}_SN$ and M_R.

(1) If $\mathrm{Hom}_S(C, N) = 0$, then $N = 0$.
(2) If $\mathrm{Hom}_R(C, M) = 0$, then $M = 0$.

Let $C = {}_SC_R$ be a semidualizing bimodule. The *Auslander class* $\mathcal{A}_C(R)$ with respect to C consists of all left R-modules M satisfying

(1) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0 = \mathrm{Ext}_S^{\geqslant 1}(C, C \otimes_R M) = 0$, and
(2) the natural evaluation homomorphism $\mu_M : M \to \mathrm{Hom}_S(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* $\mathcal{B}_C(S)$ with respect to C consists of all left S-modules N satisfying

(1) $\mathrm{Ext}_S^{\geqslant 1}(C,N)=0=\mathrm{Tor}^R_{\geqslant 1}(C,\mathrm{Hom}_S(C,N))=0$, and
(2) the natural evaluation homomorphism $\nu_N: C\otimes_R \mathrm{Hom}_S(C,N)\to N$ is an isomorphism.

The class $\mathcal{A}_C(R)$ contains the flat left R-modules and the class $\mathcal{B}_C(S)$ contains the injective left S-modules [16, Lemma 4.1].

Throughout this paper, all rings are associative with identities and all modules are unitary. ${}_RM$ (M_R) denotes a left (right) R-module. M^I ($M^{(I)}$) is the direct product (sum) of copies of a module M indexed by a set I. As usual, $\mathrm{pd}(M)$ ($\mathrm{id}(M)$) denotes the projective (injective) dimension of an R-module M, and $\mathrm{Add}_R M$ ($\mathrm{add}_R M$) stands for the category consisting of all modules isomorphic to direct summands of (finite) direct sums of copies of M and $\mathrm{Prod}_R M$ the category consisting of all modules isomorphic to direct summands of direct products of copies of M.

2. $\mathcal{W}$-Gorenstein modules

We start with the following

Definition 2.1. Let $\mathcal{W}$ be a class of left R-modules. $\mathcal{W}$ is called self-orthogonal if it satisfies the following condition:

$$\mathrm{Ext}^i(W,W')=0 \quad \text{for all } W,W'\in\mathcal{W} \text{ and all } i\geqslant 1.$$

In what follows, $\mathcal{W}$ always denotes a self-orthogonal class of left R-modules which is closed under finite direct sums and direct summands.

Definition 2.2. A left R-module M is said to be $\mathcal{W}$*-Gorenstein* if there exists an exact sequence

$$W_\bullet=\cdots\to W_1\to W_0\to W^0\to W^1\to\cdots$$

of modules in $\mathcal{W}$ such that $M=\ker(W^0\to W^1)$ and $W_\bullet$ is $\mathrm{Hom}_R(\mathcal{W},-)$ and $\mathrm{Hom}_R(-,\mathcal{W})$ exact.

In the following, we denote by $\mathcal{G}_w$ the class of $\mathcal{W}$-Gorenstein left R-modules.

Remark 2.3. (1) It is clear that each module in $\mathcal{W}$ is $\mathcal{W}$-Gorenstein. If $W_\bullet=\cdots\to W_1\to W_0\to W^0\to W^1\to\cdots$ is a $\mathrm{Hom}_R(\mathcal{W},-)$ and $\mathrm{Hom}_R(-,\mathcal{W})$ exact exact sequence of modules in $\mathcal{W}$, then by symmetry, all the images, the kernels and the cokernels of $W_\bullet$ are $\mathcal{W}$-Gorenstein.

(2) If R is commutative and Noetherian and $\mathcal{W}=\mathrm{add}_R R$, then $\mathcal{W}$-Gorenstein modules are exactly modules with G-dimension zero [2] which coincide with finitely generated Gorenstein projective modules. If $\mathcal{W}=\mathrm{Add}_R R$ ($\mathrm{Prod}_R E$ with E an injective cogenerator), then $\mathcal{W}$-Gorenstein modules are exactly Gorenstein projective (injective) modules [7].

(3) If $\mathcal{W}=\mathrm{add}({}_SC)$ for a semidualizing bimodule ${}_SC_R$, then $\mathcal{W}$-Gorenstein modules are just ω-Gorenstein modules ($\omega=C$) [22] by noting that faithfully balanced self-orthogonal modules in [22] are precisely semidualizing modules in [16].

(4) Let ${}_SC_R$ be a semidualizing bimodule, and let $\mathcal{W}_P=\{C\otimes_R P \mid P \text{ is a projective left } R\text{-module}\}$ and $\mathcal{W}_I=\{\mathrm{Hom}_S(C,E)\mid E \text{ is an injective left } S\text{-module}\}$. Then $\mathcal{W}_P$ and $\mathcal{W}_I$ are self-orthogonal and closed under finite direct sums and direct summands by Corollary 3.2 and Theorem 3.1 below. If R and S are right and left Noetherian rings respectively and ${}_SC_R$ is a dualizing bimodule, then $\mathcal{W}_P$-Gorenstein ($\mathcal{W}_I$-Gorenstein) modules are just V-Gorenstein projectives (injectives) ($V=C$) [12,13]; if $R=S$ is a local Cohen–Macaulay ring admitting a dualizing module C, then $\mathcal{W}_P$-Gorenstein ($\mathcal{W}_I$-Gorenstein) modules coincide with Ω-Gorenstein projective (injective) modules ($\Omega=C$) [10,11].

(5) We note that the class $\mathcal{G}_w$ of $\mathcal{W}$-Gorenstein left R-modules is just the class $\mathcal{G}(\mathcal{W})$ in [19] when the abelian category $\mathcal{A}$ is taken to be the category of left R-modules. So $\mathcal{G}_w=\mathcal{G}(\mathcal{G}_w)$ by [19, Corollary 4.10].

The following proposition is immediate by definition.

Proposition 2.4. *A left R-module M is $\mathcal{W}$-Gorenstein if and only if $M \in {}^{\perp}\mathcal{W} \cap \mathcal{W}^{\perp}$ and M has a proper $\mathcal{W}$ resolution and a coproper $\mathcal{W}$ coresolution.*

Remark 2.5. If M is a left R-module with $resdim_{\mathcal{W}} M < \infty$, then $M \in \mathcal{G}_w^{\perp}$. In fact, let $resdim_{\mathcal{W}} M = n < \infty$, then there is an exact sequence $0 \to W_n \to \cdots \to W_1 \to W_0 \to M \to 0$ with $W_i \in \mathcal{W}$ for $0 \leqslant i \leqslant n$. So $\mathrm{Ext}_R^j(G, M) \cong \mathrm{Ext}_R^{j+n}(G, W_n) = 0$ for all $j \geqslant 1$ and all $G \in \mathcal{G}_w$ by Proposition 2.4. Dually, if M is a left R-module with $coresdim_{\mathcal{W}} M < \infty$, then $M \in {}^{\perp}\mathcal{G}_w$. Similarly, if $resdim_{\mathcal{G}_w} N < \infty$ ($coresdim_{\mathcal{G}_w} N < \infty$), then $N \in \mathcal{W}^{\perp}$ ($N \in {}^{\perp}\mathcal{W}$).

Corollary 2.6. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of left R-modules with $M \in \mathcal{G}_w$.*

(1) *If $M' \in \mathcal{G}_w$ and $M'' \in {}^{\perp_1}\mathcal{W}$, then $M'' \in \mathcal{G}_w$.*
(2) *If $M'' \in \mathcal{G}_w$ and $M' \in \mathcal{W}^{\perp_1}$, then $M' \in \mathcal{G}_w$.*

Proof. (1) Since $M' \in \mathcal{G}_w$, there is an exact sequence $0 \to M' \to W^0 \to L \to 0$ with $W^0 \in \mathcal{W}$ and $L \in \mathcal{G}_w$. Consider the following pushout diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & W^0 & \longrightarrow & D & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

From the middle column, we get that D is $\mathcal{W}$-Gorenstein by Remark 2.3(5) and [19, Corollary 4.5]. Note that the middle row splits since $\mathrm{Ext}_R^1(M'', W^0) = 0$ by hypothesis. So M'' is $\mathcal{W}$-Gorenstein by Remark 2.3(5) and [19, Corollary 4.11].

(2) The proof is dual to that of (1). □

Recall that a class of modules is called *resolving* (*coresolving*) if it is closed under extensions and kernels of surjections (cokernels of injections), and it contains all projective (injective) modules. By Corollary 2.6, we get that the class of Gorenstein projective (injective) modules is resolving (coresolving).

Proposition 2.7. *Let $W_\bullet = \cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots$ be an exact sequence of modules in $\mathcal{W}$ such that $M = \ker(W^0 \to W^1)$ is $\mathcal{W}$-Gorenstein. Then $W_\bullet$ is $\mathrm{Hom}_R(\mathcal{W}, -)$ and $\mathrm{Hom}_R(-, \mathcal{W})$ exact if and only if every kernel and cokernel is $\mathcal{W}$-Gorenstein.*

Proof. Let $K_i = \mathrm{coker}(W_{i+1} \to W_i)$ and $L^i = \ker(W^i \to W^{i+1})$ for $i \geqslant 1$. Then the left half $\cdots \to W_1 \to W_0 \to M \to 0$ is $\mathrm{Hom}_R(-, \mathcal{W})$ exact since $M \in \mathcal{W}^{\perp}$ by Proposition 2.4, and it is $\mathrm{Hom}_R(\mathcal{W}, -)$ exact if and only if $\mathrm{Ext}_R^1(\mathcal{W}, K_i) = 0$ for all $i \geqslant 1$ if and only if each K_i is $\mathcal{W}$-Gorenstein by Corollary 2.6. Similarly, the right half $0 \to M \to W^0 \to W^1 \to \cdots$ is $\mathrm{Hom}_R(\mathcal{W}, -)$ and $\mathrm{Hom}_R(-, \mathcal{W})$ exact if and only if each L^i is $\mathcal{W}$-Gorenstein for $i \geqslant 1$. This completes the proof. □

Proposition 2.8. *Let G be a $\mathcal{W}$-Gorenstein left R-module.*

(1) *If* $\mathrm{pd}(G) < \infty$ *or* $\mathrm{id}(G) < \infty$, *then* $G \in \mathcal{W}$.
(2) *If* $resdim_{\mathcal{W}} G < \infty$ *or* $coresdim_{\mathcal{W}} G < \infty$, *then* $G \in \mathcal{W}$.

Proof. (1) Since G is $\mathcal{W}$-Gorenstein, there exists an exact sequence $\cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots$ in $\mathcal{W}$ such that $G = \ker(W^0 \to W^1)$. Suppose that $\mathrm{pd}(G) = n < \infty$ or $\mathrm{id}(G) = n < \infty$. Let $K_i = \mathrm{coker}(W_{i+1} \to W_i)$ and $L^i = \ker(W^i \to W^{i+1})$ for $i \geqslant 1$, then $\mathrm{Ext}_R^1(G, K_1) \cong \mathrm{Ext}_R^{n+1}(G, K_{n+1}) = 0$ or $\mathrm{Ext}_R^1(L^1, G) \cong \mathrm{Ext}_R^{n+1}(L^{n+1}, G) = 0$ since $G \in {}^{\perp}\mathcal{W} \cap \mathcal{W}^{\perp}$ by Proposition 2.4. It follows that the sequence $0 \to K_1 \to W_0 \to G \to 0$ or $0 \to G \to W^0 \to L^1 \to 0$ splits. So $G \in \mathcal{W}$.

(2) If $resdim_{\mathcal{W}} G < \infty$, then there is an exact sequence $0 \to K \to W \to G \to 0$ with $W \in \mathcal{W}$ and $resdim_{\mathcal{W}} K < \infty$. So this sequence splits by Remark 2.5, as desired.

A dual argument gives the result for $coresdim_{\mathcal{W}} G < \infty$. □

Corollary 2.9. *Every Gorenstein projective (injective) R-module with finite projective dimension or finite injective dimension is projective (injective).*

Proposition 2.10. *Let M be a left R-module. Then M has a $\mathcal{W}$ resolution if and only if M has a $\mathcal{G}_w$ resolution.*

Proof. It is enough to show the "if" part. Let $0 \to N \to G_0 \to M \to 0$ be an exact sequence with $G_0 \in \mathcal{G}_w$ and N having a $\mathcal{G}_w$ resolution. Then we have the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G' & = & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & H & \to & W_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & N & \to & G_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with $W_0 \in \mathcal{W}$ and $G' \in \mathcal{G}_w$. Consider the following pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & G' & \to & L & \to & G_1 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & G' & \to & H & \to & N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

where K has a $\mathcal{G}_W$ resolution and $G_1 \in \mathcal{G}_W$. So $L \in \mathcal{G}_W$ by Remark 2.3(5) and [19, Corollary 4.5], and then H has a $\mathcal{G}_W$ resolution. Note that $0 \to H \to W_0 \to M \to 0$ is exact. By repeating the preceding process, we have that M has a $\mathcal{W}$ resolution. □

Remark 2.11. Let M be a left R-module with $resdim_{\mathcal{G}_w} M = n \geqslant 1$, and let $0 \to N \to G_0 \to M \to 0$ be an exact sequence with $G_0 \in \mathcal{G}_W$ and $resdim_{\mathcal{G}_w} N = n-1$. By the proof of the proposition above, we have an exact sequence $0 \to H \to W_0 \to M \to 0$ such that $W_0 \in \mathcal{W}$ and $resdim_{\mathcal{G}_w} H = resdim_{\mathcal{G}_w} N$.

Proposition 2.12. *Let M be a left R-module with a finite $\mathcal{G}_W$ resolution and n a nonnegative integer. Then the following are equivalent*:

(1) *$resdim_{\mathcal{G}_w} M \leqslant n$.*
(2) *There is an exact sequence $0 \to G \to W_{n-1} \to \cdots \to W_1 \to W_0 \to M \to 0$ with $W_i \in \mathcal{W}$ for $0 \leqslant i \leqslant n-1$ and $G \in \mathcal{G}_W$.*
(3) *M has a proper $\mathcal{G}_W$-resolution of length n.*
(4) *There is an exact sequence $0 \to W_n \to W_{n-1} \to \cdots \to W_1 \to G \to M \to 0$ with $W_i \in \mathcal{W}$ for $1 \leqslant i \leqslant n$ and $G \in \mathcal{G}_W$.*
(5) *There is an exact sequence $0 \to W_n \to \cdots \to W_{i+1} \to G \to W_{i-1} \cdots \to W_0 \to M \to 0$ with $W_j \in \mathcal{W}$ for $1 \leqslant j \leqslant n$, $j \neq i$, $0 \leqslant i \leqslant n$ and $G \in \mathcal{G}_W$.*
(6) *$\mathrm{Ext}_R^{n+j}(M, W) = 0$ for all $j \geqslant 1$ and all $W \in \mathcal{W}$.*
(7) *$\mathrm{Ext}_R^{n+j}(M, N) = 0$ for all $j \geqslant 1$ and all left R-modules N with finite $\mathcal{W}$ resolutions.*
(8) *$\mathrm{Ext}_R^{n+1}(M, N) = 0$ for all left R-modules N with finite $\mathcal{W}$ resolutions.*

Furthermore, we have that

$$\begin{aligned} resdim_{\mathcal{G}_w} M &= \sup\{n \in \mathbb{N} \mid \mathrm{Ext}_R^n(M, W) \neq 0 \text{ for some } W \in \mathcal{W}\} \\ &= \sup\{n \in \mathbb{N} \mid \mathrm{Ext}_R^n(M, N) \neq 0 \text{ for some } N \text{ with } resdim_{\mathcal{W}} N < \infty\}. \end{aligned}$$

Proof. We first prove the equivalences of (1) through (5). The case $n = 0$ is trivial. We may assume $n \geqslant 1$.

(1) ⇒ (2): By (1), there exists an exact sequence $0 \to N \to G_0 \to M \to 0$ with $G_0 \in \mathcal{G}_W$ and $resdim_{\mathcal{G}_w} N \leqslant n-1$. By Remark 2.11, we have an exact sequence $0 \to H \to W_0 \to M \to 0$ such that $resdim_{\mathcal{G}_w} H = resdim_{\mathcal{G}_w} N$. By repeating this process, we have an exact sequence $0 \to G_n \to W_{n-1} \to \cdots \to W_1 \to W_0 \to M \to 0$ with $W_i \in \mathcal{W}$ for all $0 \leqslant i \leqslant n-1$ and $G_n \in \mathcal{G}_W$.

(2) ⇒ (3): Suppose M satisfies (2). Since G is $\mathcal{W}$-Gorenstein by (2), there is a $\mathrm{Hom}_R(-, \mathcal{W})$ exact exact sequence $0 \to G \to W^0 \to \cdots \to W^{n-1} \to G' \to 0$ with each $W^i \in \mathcal{W}$ and $G' \in \mathcal{G}_W$. So the diagram

$$\begin{array}{ccccccccccccc} 0 & \to & G & \to & W^0 & \to & \cdots & \to & W^{n-1} & \to & G' & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & G & \to & W_{n-1} & \to & \cdots & \to & W_0 & \to & M & \to & 0 \end{array}$$

can be completed to a commutative diagram. Then we have a mapping cone $0 \to G \to G \oplus W^0 \to W_{n-1} \oplus W^1 \to \cdots \to W_1 \oplus W^{n-1} \to W_0 \oplus G' \to M \to 0$ which gives an exact sequence $W_\bullet = 0 \to W^0 \to W_{n-1} \oplus W^1 \to \cdots \to W_1 \oplus W^{n-1} \to W_0 \oplus G' \to M \to 0$. Note that each cokernel of $W_\bullet$ except M has a finite $\mathcal{W}$-resolution. So $W_\bullet$ is $\mathrm{Hom}_R(\mathcal{G}_W, -)$ exact by Remark 2.5. It follows that $W_\bullet$ is just a proper $\mathcal{G}_W$ resolution of M of length n.

(2) ⇒ (4): Note that $W_\bullet$ in the proof of (2) ⇒ (3) is just the desired exact sequence.

(3) ⇒ (1), (4) ⇒ (1) and (5) ⇒ (1) are obvious.

(1) ⇒ (5) is immediate by Remark 2.11 and the equivalence of (1) and (4).

Next we show the equivalences of (1), (6), (7) and (8).

(1) ⇒ (6): By assumption, there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ with $G_i \in \mathcal{G}_w$ for $0 \leqslant i \leqslant n$. So $\operatorname{Ext}_R^{n+j}(M, W) \cong \operatorname{Ext}_R^j(G_n, W) = 0$ for all $j \geqslant 1$ and all $W \in \mathcal{W}$ by Proposition 2.4.

(6) ⇒ (7) follows from the usual dimension shifting argument.

(7) ⇒ (8) is clear.

(8) ⇒ (1): By hypothesis, let $resdim_{\mathcal{G}_w} M = s < \infty$. If $s \leqslant n$, there is nothing to prove. So we assume $s > n$. Then there is an exact sequence $0 \to W_s \to \cdots \to W_1 \to G \to M \to 0$ with $W_i \in \mathcal{W}$ for $1 \leqslant i \leqslant s$ and $G \in \mathcal{G}_w$ by the equivalence of (1) and (4). Let $K_i = \operatorname{coker}(W_{i+1} \to W_i)$ for $1 \leqslant i \leqslant s-1$.

If $n = 0$, then $\operatorname{Ext}_R^{n+1}(M, K_1) = 0$ by (8) since $resdim_{\mathcal{W}} K_1 < \infty$. Thus the exact sequence $0 \to K_1 \to G \to M \to 0$ is split, and so $M \in \mathcal{G}_w$, as desired.

Let $n \geqslant 1$. Since $resdim_{\mathcal{W}} K_{n+1} < \infty$, we have that $\operatorname{Ext}_R^1(K_n, K_{n+1}) \cong \operatorname{Ext}_R^{n+1}(M, K_{n+1}) = 0$ by Remark 2.5 and (8). So the exact sequence $0 \to K_{n+1} \to W_n \to K_n \to 0$ splits. Thus $K_n \in \mathcal{W}$, and so (1) follows.

The last claim is an immediate consequence of the equivalences of (1), (6) and (7). □

Remark 2.13. By an argument similar to the proof of the equivalence of (1) and (6) in Proposition 2.12, we have that if $resdim_{\mathcal{W}} M < \infty$ then

$$resdim_{\mathcal{W}} M = \sup\{n \in \mathbb{N} \mid \operatorname{Ext}_R^n(M, W) \neq 0 \text{ for some } W \in \mathcal{W}\}.$$

So let M be a left R-module, if $resdim_{\mathcal{W}} M < \infty$, then $resdim_{\mathcal{G}_w} M = resdim_{\mathcal{W}} M$.

Remark 2.14. We note that all the foregoing results on resolutions and resolution dimensions (from Proposition 2.10 to Remark 2.13) have the dual versions on coresolutions and coresolution dimensions.

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [6], we say that a homomorphism $\phi : X \to M$ is a *$\mathcal{C}$-precover of M* if $X \in \mathcal{C}$ and the abelian group homomorphism $\operatorname{Hom}(X', \phi) : \operatorname{Hom}(X', X) \to \operatorname{Hom}(X', M)$ is surjective for every $X' \in \mathcal{C}$.

Let $C = {}_SC_R$ be a semidualizing bimodule over associative rings R and S, and let $(-)^* = \operatorname{Hom}_R(-, C)$ (or $\operatorname{Hom}_S(-, C)$). A finitely generated right R-module M is said to have *generalized Gorenstein dimension zero* (with respect to ${}_SC_R$) [3] if the following conditions are satisfied: (1) $M \cong M^{**}$; (2) $\operatorname{Ext}_R^i(M, C) = 0 = \operatorname{Ext}_S^i(M^*, C)$ for all $i \geqslant 1$.

We conclude this section with the following theorem.

Theorem 2.15. *Let R be a commutative Noetherian ring and C a semidualizing R-module, and let $\mathcal{W} = \operatorname{add} C$. Then the following are equivalent for a finitely generated R-module M:*

(1) $M \in \mathcal{G}_w$.
(2) *M has generalized Gorenstein dimension zero with respect to C and $M \in \mathcal{B}_C(R)$.*
(3) *There exists an exact sequence $W_\bullet = \cdots \to C^{l_1} \to C^{l_0} \to C^{n_0} \to C^{n_1} \to \cdots$ such that $M = \ker(C^{n_0} \to C^{n_1})$ and $W_\bullet$ is $\operatorname{Hom}_R(C, -)$ and $\operatorname{Hom}_R(-, C)$ exact, where l_i and n_j are positive integers for all $i, j \geqslant 0$.*

Proof. (1) ⇒ (2) is immediate by Proposition 2.4, Remark 2.3(3) and [22, Proposition 2.2].

(3) ⇒ (1) is clear.

(2) ⇒ (3): Since M has generalized Gorenstein dimension zero with respect to C by (2), there exists an exact sequence

$$0 \to M \to C^{n_0} \to C^{n_1} \to \cdots \qquad (\flat)$$

which is $\operatorname{Hom}_R(-, C)$ exact by [17, Theorem 1] and $\operatorname{Hom}_R(C, -)$ exact since $\operatorname{Ext}_R^i(C, M) = 0$ for all $i \geqslant 1$, where n_j ($j \geqslant 0$) are positive integers.

Next, we prove that there exists an exact sequence

$$\cdots \to C^{l_1} \to C^{l_0} \to M \to 0 \qquad (\flat\flat)$$

which is $\mathrm{Hom}_R(C,-)$ exact, where l_j $(j \geqslant 0)$ are positive integers.

Let $f_1,\dots,f_{l_0}$ be a system of generators of the R-module $\mathrm{Hom}_R(C,M)$. Taking the direct sum of $f_1,\dots,f_{l_0}$, we construct a homomorphism $f: C^{l_0} \to M$. It is easily seen that f is an $\mathrm{add}\,C$-precover of M. Let $K=\ker f$. Then we have the exact sequence $0 \to \mathrm{Hom}_R(C,K) \to \mathrm{Hom}_R(C,C^{l_0}) \to \mathrm{Hom}_R(C,M) \to 0$. Since $\mathrm{Tor}_i^R(C,\mathrm{Hom}_R(C,M))=0$ for all $i \geqslant 1$ (for $M \in \mathcal{B}_C(R)$) and $\mathrm{Tor}_i^R(C,\mathrm{Hom}_R(C,C^{l_0}))=0$ for all $i \geqslant 1$, $\mathrm{Tor}_i^R(C,\mathrm{Hom}_R(C,K))=0$ for all $i \geqslant 1$. Furthermore we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C\otimes_R \mathrm{Hom}_R(C,K) & \longrightarrow & C\otimes_R \mathrm{Hom}_R(C,C^{l_0}) & \longrightarrow & C\otimes_R \mathrm{Hom}_R(C,N) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \nu_K} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & K & \longrightarrow & C^{l_0} & \xrightarrow{f} & M. & &
\end{array}$$

Thus f is epic, and hence $C\otimes_R \mathrm{Hom}_R(C,K)\cong K$ and $\mathrm{Ext}_R^i(C,K)=0$ for $i \geqslant 1$. Therefore K is finitely generated (for R is Noetherian) and $K \in \mathcal{B}_C(R)$. Repeating the foregoing process, we have the desired exact sequence.

Since $\mathrm{Ext}_R^i(M,C)=0$ for all $i \geqslant 1$, the sequence $(\flat\flat)$ is also $\mathrm{Hom}_R(-,C)$ exact. By pasting the sequences $(\flat)$ and $(\flat\flat)$ above we get the desired complex $W_\bullet$. $\square$

Remark 2.16. We note that [5, Theorem 4.1.4] is a particular case of Theorem 2.15 where $C=R$.

3. $\mathcal{W}_P$-Gorenstein and $\mathcal{W}_I$-Gorenstein modules

Let $C={}_SC_R$ be a semidualizing bimodule, $\mathcal{W}_P=\{C\otimes_R P \mid P \text{ is a projective left } R\text{-module}\}$ and $\mathcal{W}_I=\{\mathrm{Hom}_S(C,E) \mid E \text{ is an injective left } S\text{-module}\}$. In this section, we shall particularly investigate $\mathcal{W}_P$-Gorenstein and $\mathcal{W}_I$-Gorenstein modules which will be called *C-Gorenstein projective* and *C-Gorenstein injective* modules respectively. Accordingly, the $\mathcal{W}_P$-Gorenstein (Gorenstein projective) resolution dimension is called *C-Gorenstein projective* (*Gorenstein projective*) dimension and the $\mathcal{W}_I$-Gorenstein (Gorenstein injective) coresolution dimension is called *C-Gorenstein injective* (*Gorenstein injective*) dimension simply.

We start with the following descriptions of the classes $\mathcal{W}_P$ and $\mathcal{W}_I$ which may be of independent interest.

Theorem 3.1. *Let ${}_SC_R$ be a semidualizing bimodule. Then*:

(1) $\mathcal{W}_P=\mathrm{Add}\,{}_SC$.
(2) $\mathcal{W}_I=\mathrm{Prod}\,C^+$, *where* $C^+=\mathrm{Hom}_S(C,Q)$ *with* ${}_SQ$ *an injective cogenerator.*

Proof. (1) It is clear that $\mathcal{W}_P\subseteq \mathrm{Add}\,{}_SC$. Conversely, for any left S-module X, the evaluation homomorphism $\nu_X: C\otimes_R \mathrm{Hom}_S(C,X)\to X$ is defined by $\nu_X(c\otimes f)=f(c)$ for $c\in C$ and $f\in \mathrm{Hom}_S(C,X)$. We claim that $\nu_{C^{(K)}}$ is an isomorphism for any index set K.

Since ${}_SC$ is finitely generated, there is an isomorphism $\mathrm{Hom}_S(C,C^{(K)})\to \mathrm{Hom}_S(C,C)^{(K)}$ defined by $f\mapsto(\pi_k f)$, where $\pi_k: C^{(K)}\to C$ is the kth projection for $k\in K$. Thus we have an isomorphism

$$\alpha_1: C\otimes_R \mathrm{Hom}_S\big(C,C^{(K)}\big)\to C\otimes_R \mathrm{Hom}_S(C,C)^{(K)}$$

given by $c\otimes f\mapsto c\otimes(\pi_k f)$ for $f\in \mathrm{Hom}_S(C,C^{(K)})$.

Note that $C \otimes_R -$ commutes with direct sums, so there is an isomorphism

$$\alpha_2 : C \otimes_R \mathrm{Hom}_S(C, C)^{(K)} \to \big(C \otimes_R \mathrm{Hom}_S(C, C)\big)^{(K)}$$

given by $c \otimes (g_k) \mapsto (c \otimes g_k)$ for $c \in C$ and $(g_k) \in \mathrm{Hom}_S(C, C)^{(K)}$.

Since ${}_SC_R$ is semidualizing, the homothety map $\gamma_R : R \to \mathrm{Hom}_S(C, C)$ is an isomorphism. Hence there is an isomorphism

$$\alpha_3 : \big(C \otimes_R \mathrm{Hom}_S(C, C)\big)^{(K)} \to (C \otimes_R R)^{(K)}$$

defined by $(c_k \otimes g_k) \mapsto (c_k \otimes \gamma_R^{-1}(g_k))$, where $c_k \in C$ and $g_k \in \mathrm{Hom}_S(C, C)$ for $k \in K$.

Finally, the natural isomorphism $C \otimes_R R \to C$ induces an isomorphism

$$\alpha_4 : (C \otimes_R R)^{(K)} \to C^{(K)}$$

given by $(c_k \otimes r_k) \mapsto (c_k r_k)$, where $c_k \in C$ and $r_k \in R$ for $k \in K$.

Let $c \in C$ and $f \in \mathrm{Hom}_S(C, C^{(K)})$. Then $\alpha_2\alpha_1(c \otimes f) = \alpha_2(c \otimes (\pi_k f)) = (c \otimes \pi_k f)$. By [1, Proposition 4.10], the homothety map $\gamma_R : R \to \mathrm{Hom}_S(C, C)$ is defined by $\gamma_R(r)(c) = cr$ for $r \in R$ and $c \in C$. Let $r_k = \gamma_R^{-1}(\pi_k f)$, then $\pi_k f = \gamma_R(r_k)$, and so $\pi_k f(c) = \gamma_R(r_k)(c) = cr_k$ for $c \in C$ and $k \in K$. Thus $\alpha_4\alpha_3\alpha_2\alpha_1(c \otimes f) = \alpha_4\alpha_3((c \otimes \pi_k f)) = \alpha_4((c \otimes r_k)) = (cr_k) = (\pi_k f(c)) = f(c) = \nu_{C^{(K)}}(c \otimes f)$. This shows that $\nu_{C^{(K)}} = \alpha_4\alpha_3\alpha_2\alpha_1$ is an isomorphism.

Suppose ${}_SM \in \mathrm{Add}\,{}_SC$ and $M \oplus N = C^{(K)}$ for some left S-module N and some index set K. Then there is a split exact sequence $0 \to M \xrightarrow{\lambda} C^{(K)} \xrightarrow{p} N \to 0$ which induces the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C \otimes_R \mathrm{Hom}_R(C, M) & \xrightarrow{1\otimes\lambda_*} & C \otimes_R \mathrm{Hom}_R(C, C^{(K)}) & \xrightarrow{1\otimes p_*} & C \otimes_R \mathrm{Hom}_R(C, N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nu_M} & & \downarrow{\scriptstyle \nu_{C^{(K)}}} & & \downarrow{\scriptstyle \nu_N} & & \\ 0 & \longrightarrow & M & \xrightarrow{\lambda} & C^{(K)} & \xrightarrow{p} & N & \longrightarrow & 0. \end{array}$$

The Five Lemma shows that ν_M is monic. Thus ν_N is also monic, and so ν_M is an isomorphism by the Five Lemma again.

Note that $\mathrm{Hom}_S(C, M)$ is a projective left R-module since $\mathrm{Hom}_S(C, M) \oplus \mathrm{Hom}_S(C, N) \cong \mathrm{Hom}_S(C, C^{(K)}) \cong R^{(K)}$. So ${}_SM \cong C \otimes_R \mathrm{Hom}_S(C, M) \in \mathcal{W}_P$.

(2) It is clear that $\mathcal{W}_I \subseteq \mathrm{Prod}\, C^+$. Conversely, for any left R-module X, the evaluation homomorphism $\mu_X : X \to \mathrm{Hom}_S(C, C \otimes_R X)$ is defined by $\mu_X(x)(c) = c \otimes x$ for $x \in X$ and $c \in C$. We claim that $\mu_{(C^+)^J} : (C^+)^J \to \mathrm{Hom}_S(C, C \otimes_R (C^+)^J)$ is an isomorphism for any index set J.

Since C_R is finitely presented, there is an isomorphism $\alpha : C \otimes_R (C^+)^J \to (C \otimes_R C^+)^J$ defined by $c \otimes (f_j) \mapsto (c \otimes f_j)$ for $c \in C$ and $(f_j) \in (C^+)^J$. Thus we have an isomorphism

$$\beta_1 = \alpha_* : \mathrm{Hom}_S\big(C, C \otimes_R \big(C^+\big)^J\big) \to \mathrm{Hom}_S\big(C, \big(C \otimes_R C^+\big)^J\big).$$

Note that $\mathrm{Hom}_S(C, -)$ commutes with direct products, so there is an isomorphism

$$\beta_2 : \mathrm{Hom}_S\big(C, \big(C \otimes_R C^+\big)^J\big) \to \big(\mathrm{Hom}_S\big(C, C \otimes_R C^+\big)\big)^J$$

given by $f \mapsto (p_j f)$, where $p_j : (C \otimes_R C^+)^J \to C \otimes_R C^+$ is the jth projection for $j \in J$.

Since ${}_SC_R$ is semidualizing, the homothety map ${}_S\gamma : S \to \mathrm{Hom}_R(C, C)$ is an isomorphism and ${}_S\gamma$ is defined by ${}_S\gamma(s)(c) = sc$ for $s \in S$ and $c \in C$ by [1, Proposition 4.10]. Note that C_R is finitely presented and ${}_SQ$ is injective, and so, by [18, Lemma 3.60], the evaluation map $\nu_Q : C \otimes_R C^+ \to Q$ given by $\nu_Q(c \otimes f) = f(c)$ for $c \in C$ and $f \in C^+$ is an isomorphism. Hence we have an isomorphism

$$\beta_3 = \big((\nu_Q)_*\big)^J : \big(\mathrm{Hom}\big(C, C \otimes_R C^+\big)\big)^J \to \big(C^+\big)^J.$$

It is easy to verify that $\beta_3\beta_2\beta_1\mu_{(C^+)^K} = \mathrm{id}_{(C^+)^J}$, and so $\mu_{(C^+)^J}$ is an isomorphism.

Suppose ${}_RM \in \mathrm{Prod}\, C^+$ and $M \oplus N = (C^+)^J$ for some left R-module N and some index set J. Then there is a split exact sequence $0 \to M \xrightarrow{\lambda} (C^+)^J \xrightarrow{p} N \to 0$ which induces the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\lambda} & (C^+)^J & \xrightarrow{p} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_M} & & \downarrow{\scriptstyle \mu_{(C^+)^J}} & & \downarrow{\scriptstyle \mu_N} & & \\ 0 & \longrightarrow & \mathrm{Hom}_S(C, C \otimes_R M) & \xrightarrow{(1\otimes\lambda)_*} & \mathrm{Hom}_S(C, C \otimes_R (C^+)^J) & \xrightarrow{(1\otimes p)_*} & \mathrm{Hom}_S(C, C \otimes_R N) & \longrightarrow & 0. \end{array}$$

By the corresponding proof in (1), μ_M is an isomorphism. Note that $C \otimes_R M$ is an injective left S-module since $C \otimes_R M \oplus C \otimes_R N = C \otimes_R (C^+)^J \cong (C \otimes_R C^+)^J \cong Q^J$. So ${}_RM \cong \mathrm{Hom}_S(C, C \otimes_R M) \in \mathcal{W}_I$. □

Corollary 3.2. *Let ${}_SC_R$ be a semidualizing bimodule. Then*:

(1) $\mathcal{A}_C(R) \subseteq {}^\perp\mathcal{W}_I$.
(2) $\mathcal{B}_C(S) \subseteq \mathcal{W}_P^\perp$.
(3) $\mathcal{W}_I \subseteq \mathcal{A}_C(R) \cap (\mathcal{A}_C(R))^\perp$.
(4) $\mathcal{W}_P \subseteq \mathcal{B}_C(S) \cap {}^\perp(\mathcal{B}_C(S))$.

Proof. (1) For any $A \in \mathcal{A}_C(R)$ and any index set J, by [18, Theorem 7.14] and [8, Theorem 3.2.1], we have

$$\mathrm{Ext}_R^{\geqslant 1}\big(A, \big(C^+\big)^J\big) \cong \big(\mathrm{Ext}_R^{\geqslant 1}\big(A, C^+\big)\big)^J \cong \big(\big(\mathrm{Tor}_{\geqslant 1}^R(C, A)\big)^+\big)^J = 0.$$

So $\mathcal{A}_C(R) \subseteq {}^\perp\mathcal{W}_I$ by Theorem 3.1(2).

(2) For any $B \in \mathcal{B}_C(S)$ and any index set K, by [18, Theorem 7.13], we have

$$\mathrm{Ext}_S^{\geqslant 1}\big(C^{(K)}, B\big) \cong \big(\mathrm{Ext}_S^{\geqslant 1}(C, B)\big)^K = 0.$$

So $\mathcal{B}_C(S) \subseteq \mathcal{W}_P^\perp$ by Theorem 3.1(1).

(3) and (4) follow from (1), (2) and [16, Lemma 4.1 and Proposition 4.1]. □

In what follows, we assume that $C = {}_SC_R$ is a faithfully semidualizing bimodule. Note that all semidualizing modules are faithfully semidualizing over a commutative ring [16, Proposition 3.1].

The following lemma is needed frequently in the sequel.

Lemma 3.3. *(See [16, Corollary 6.3].) The classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$ have the property that if two of three modules in a short exact sequence are in the class then so is the third.*

Proposition 3.4. *Let $W_\bullet = \cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots$ be an exact sequence of modules in $\mathcal{W}_I$ and $M = \ker(W^0 \to W^1)$. Then the sequence $W_\bullet$ is $\mathrm{Hom}_R(-, \mathcal{W}_I)$ exact if and only if $M \in \mathcal{A}_C(R)$.*

Proof. Suppose $M \in \mathcal{A}_C(R)$. By Corollary 3.2(3) and Lemma 3.3, every kernel and cokernel of $W_\bullet$ is in $\mathcal{A}_C(R)$, and so $W_\bullet$ is $\mathrm{Hom}_R(-,\mathcal{W}_I)$ exact by Corollary 3.2(1).

Conversely, if $W_\bullet$ is $\mathrm{Hom}_R(-,\mathcal{W}_I)$ exact then $C \otimes_R W_\bullet$ is still exact by the adjoint isomorphism. Since $\mathrm{Tor}_i^R(C,W)=0$ for all $W \in \mathcal{W}_I$ and all $i \geqslant 1$, $\mathrm{Tor}_i^R(C,M)=0$ for all $i \geqslant 1$. So using a projective resolution of M we have an exact sequence $X_\bullet = \cdots \to P_1 \to P_0 \to W^0 \to W^1 \to \cdots$ of left R-modules with P_i projective for $i \geqslant 1$ such that $C \otimes_R X_\bullet$ is exact. Hence $M \in \mathcal{A}_C(R)$ by [16, Theorem 2]. □

A dual argument of the proof of Proposition 3.4 gives the following proposition.

Proposition 3.5. *Let $U_\bullet = \cdots \to U_1 \to U_0 \to U^0 \to U^1 \to \cdots$ be an exact sequence of modules in $\mathcal{W}_P$ and $N = \ker(U^0 \to U^1)$. Then the sequence $U_\bullet$ is $\mathrm{Hom}_R(\mathcal{W}_P,-)$ exact if and only if $N \in \mathcal{B}_C(S)$.*

The next proposition shows that C-Gorenstein projectives and C-Gorenstein injectives defined here are different from those defined in [23] ([15]) when $S=R$ is a commutative (Noetherian) ring.

Proposition 3.6. *An R-module M is C-Gorenstein projective (C-Gorenstein injective) defined here if and only if M is C-Gorenstein projective (C-Gorenstein injective) defined in [23] ([15]) and in $\mathcal{B}_C(R)$ ($\mathcal{A}_C(R)$) when $S=R$ is a commutative (Noetherian) ring.*

Proof. We only prove the case for C-Gorenstein projectives.

"⇒" Let M be C-Gorenstein projective defined here. Then M is C-Gorenstein projective given in [23] ([15]) and $M \in \mathcal{B}_C(R)$ by Propositions 2.4 and 3.5.

"⇐" Let M be a C-Gorenstein projective modules given in [23] ([15]) and $M \in \mathcal{B}_C(R)$. Then $M \in {}^\perp\mathcal{W}_P$ and M has a coproper $\mathcal{W}_P$ coresolution by [23, Proposition 2.2] (or [15, Definition 2.7]), and $M \in \mathcal{W}_P^\perp$ by Corollary 3.2. On the other hand, there exists an exact sequence

$$U_\bullet = \cdots \to U_1 \to U_0 \to M \to 0$$

with each $U_i \in \mathcal{W}_P$ such that $U_\bullet$ is $\mathrm{Hom}_R(C,-)$ exact by [16, Theorem 6.1]. Thus $U_\bullet$ is $\mathrm{Hom}_R(\mathcal{W}_P,-)$ exact by Theorem 3.1(1), and so M has a proper $\mathcal{W}_P$ resolution. Therefore M is C-Gorenstein projective defined here by Proposition 2.4. □

Let M be a module over a commutative ring R admitting a semidualizing module C. The *$\mathcal{W}_P$-projective dimension* and *$\mathcal{W}_I$-injective dimension* of M, denoted by $\mathcal{W}_P$-pd(M) and $\mathcal{W}_I$-id(M), are defined in [21]. By [21, Corollary 2.10], $\mathcal{W}_P$-pd$(M) < \infty$ ($\mathcal{W}_I$-id$(M) < \infty$) if and only if $\mathit{resdim}_{\mathcal{W}_P} M < \infty$ ($\mathit{resdim}_{\mathcal{W}_I} M < \infty$). So the following corollary generalizes [21, Corollary 2.9] to noncommutative rings.

Corollary 3.7. *Let M be a left R-module.*

(1) *If M has finite C-Gorenstein injective dimension, then $M \in \mathcal{A}_C(R)$.*
(2) *If M has finite C-Gorenstein projective dimension, then $M \in \mathcal{B}_C(S)$.*

Proof. By Propositions 3.4 and 3.5 respectively, all C-Gorenstein injective modules are in $\mathcal{A}_C(R)$ and all C-Gorenstein projective modules are in $\mathcal{B}_C(S)$. So the results are immediate by Lemma 3.3. □

Remark 3.8. If a left R-module M has a finite C-Gorenstein injective resolution, then M is C-Gorenstein injective. In fact, let $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ be an exact sequence with G_i C-Gorenstein injective for $0 \leqslant i \leqslant n$, and let $L_i = \mathrm{coker}(G_{i+1} \to G_i)$ for $0 \leqslant i \leqslant n-1$, where $L_0 = M$. Then each $L_i \in \mathcal{A}_C(R)$ by Proposition 3.4 and Lemma 3.3. So each L_i is C-Gorenstein injective by Corollaries 3.2(1) and 2.6(1) for $0 \leqslant i \leqslant n-1$. Thus M is C-Gorenstein injective. Similarly, if a left S-module N has a finite C-Gorenstein projective coresolution, then N is C-Gorenstein projective.

The following lemma is stated in [21] for a commutative ring, but the proof there also works in the present context.

Lemma 3.9. *(See [21, Theorem 2.8].)*

(1) *Let M be a left R-module, then $M \in \mathcal{A}_C(R)$ if and only if $C \otimes_R M \in \mathcal{B}_C(S)$.*
(2) *Let N be a left S-module, then $N \in \mathcal{B}_C(S)$ if and only if $\mathrm{Hom}_S(C, N) \in \mathcal{A}_C(R)$.*

Remark 3.10. (1) In what follows, we denote by G_C-*Proj* (G_C-*Inj*) the class of C-Gorenstein projective left S-modules (C-Gorenstein injective left R-modules). *G-Proj* (*G-Inj*) stands for the class of Gorenstein projective left R-modules (Gorenstein injective left S-modules).

(2) Let R and S be right and left Noetherian rings respectively admitting a dualizing bimodule (see [13, Definition 3.1]). Then all Gorenstein projective left R-modules are in $\mathcal{A}(R)$ [13, Proposition 3.9] and all Gorenstein injective left S-modules are in $\mathcal{B}(S)$ [13, Proposition 3.8]. Furthermore, if each flat left R-module has finite projective dimension, then a left S-module $N \in \mathcal{B}(S)$ only if N has finite Gorenstein injective dimension by the proof of [13, Lemma 3.15], and hence $N \in \mathcal{B}(S)$ if and only if N has finite Gorenstein injective dimension by [13, Proposition 3.13]; dually, we have that a left R-module $M \in \mathcal{A}(R)$ if and only if M has finite Gorenstein projective dimension. The following theorem is the counterpart of [13, Theorem 4.5] in the present context.

Theorem 3.11. *There are equivalences of categories:*

$$G_C - Inj \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} G - Inj \cap \mathcal{B}_C(S),$$

$$G - Proj \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} G_C - Proj.$$

Proof. It suffices to prove the first assertion. The second has a dual argument. We first show that the functor $C \otimes_R -$ maps G_C-*Inj* to *G-Inj* $\cap\, \mathcal{B}_C(S)$. Let $M \in G_C$-*Inj*, then there exists an exact sequence

$$W_\bullet = \cdots \to \mathrm{Hom}_S(C, E_1) \to \mathrm{Hom}_S(C, E_0) \to \mathrm{Hom}_S\big(C, E^0\big) \to \mathrm{Hom}_S\big(C, E^1\big) \to \cdots$$

with E_i, E^j injective for $i, j \geqslant 0$ and $M = \ker(\mathrm{Hom}_S(C, E^0) \to \mathrm{Hom}_S(C, E^1))$ such that $W_\bullet$ is $\mathrm{Hom}_R(\mathcal{W}_I, -)$ and $\mathrm{Hom}_R(-, \mathcal{W}_I)$ exact. So $M \in \mathcal{A}_C(R)$ by Proposition 3.4, and hence every kernel and cokernel of $W_\bullet$ is in $\mathcal{A}_C(R)$ by Lemma 3.3. Thus $C \otimes_R W_\bullet$ is exact, moreover, $C \otimes_R M \in \mathcal{B}_C(S)$ by Lemma 3.9. On the other hand, we have

$$C \otimes_R W_\bullet \cong \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

with $C \otimes_R M = \ker(E^0 \to E^1)$. For each injective left S-module E, we have

$$\begin{aligned}\mathrm{Hom}_S(E, C \otimes_R W_\bullet) &\cong \mathrm{Hom}_R\big(\mathrm{Hom}_S(C, E), \mathrm{Hom}_S(C, C \otimes_R W_\bullet)\big) \quad \big(\text{by [16, Theorem 6.4]}\big)\\ &\cong \mathrm{Hom}_R\big(\mathrm{Hom}_S(C, E), W_\bullet\big).\end{aligned}$$

So $\mathrm{Hom}_S(E, C \otimes_R W_\bullet)$ is exact. Therefore $C \otimes_R M$ is Gorenstein injective.

The proof that $\mathrm{Hom}_S(C, -)$ maps *G-Inj* $\cap\, \mathcal{B}_C(S)$ to G_C-*Inj* is similar. Finally, we note that if $M \in G_C$-*Inj* and $N \in$ *G-Inj* $\cap\, \mathcal{B}_C(S)$, then there exist natural isomorphisms $M \xrightarrow{\simeq} \mathrm{Hom}_S(C, C \otimes_R M)$ and $C \otimes_R \mathrm{Hom}_S(C, N) \xrightarrow{\simeq} N$. Now the desired equivalences of categories follow. □

Let n be a nonnegative integer. In the following, we denote by G-$Proj_{\leqslant n}$ (G-$Inj_{\leqslant n}$, G_C-$Proj_{\leqslant n}$, G_C-$Inj_{\leqslant n}$) the class of modules with Gorenstein projective (Gorenstein injective, C-Gorenstein projective, C-Gorenstein injective) dimension at most n.

Corollary 3.12. *There are equivalences of categories*:

$$G_C - Inj_{\leqslant n} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} G - Inj_{\leqslant n} \cap \mathcal{B}_C(S),$$

$$G - Proj_{\leqslant n} \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} G_C - Proj_{\leqslant n}.$$

Proof. We only prove the first part and the second is dual. Let $M \in G_C$-$Inj_{\leqslant n}$. Then there exists an exact sequence $0 \to M \to G_0 \to \cdots \to G_{n-1} \to G_n \to 0$ with $G_i \in G_C$-Inj for $0 \leqslant i \leqslant n$, and so every kernel is in $\mathcal{A}_C(R)$ by Corollary 3.7. So we have the exact sequence $0 \to C \otimes_R M \to C \otimes_R G_0 \to \cdots \to C \otimes_R G_{n-1} \to C \otimes_R G_n \to 0$ with each $C \otimes_R G_i$ $(0 \leqslant i \leqslant n)$ Gorenstein injective and in $\mathcal{B}_C(S)$ by Theorem 3.11. Hence $C \otimes_R M \in G$-$Inj_{\leqslant n} \cap \mathcal{B}_C(S)$.

Conversely, let $M \in G$-$Inj_{\leqslant n} \cap \mathcal{B}_C(S)$. If $n = 0$ then $\mathrm{Hom}_S(C, M) \in G_C$-$Inj$ by Theorem 3.11. Next we assume $n \geqslant 1$, then by the dual of Proposition 2.12, there is an exact sequence $0 \to M \to G \to E^1 \to \cdots \to E^n \to 0$ such that G is Gorenstein injective and E^j is injective for $1 \leqslant j \leqslant n$. Let $L = \mathrm{coker}(M \to G)$, then $\mathrm{id}({}_S L) \leqslant n-1$, and so $L \in \mathcal{B}_C(S)$ by [16, Corollary 6.2]. Thus every kernel of the sequence above is in $\mathcal{B}_C(S)$. Therefore the sequence $0 \to \mathrm{Hom}_S(C, M) \to \mathrm{Hom}_S(C, G) \to \mathrm{Hom}_S(C, E^1) \to \cdots \to \mathrm{Hom}_S(C, E^n) \to 0$ is exact. Note that in the exact sequence $0 \to M \to G \to L \to 0$, we have $M, L \in \mathcal{B}_C(S)$. So $G \in \mathcal{B}_C(S)$ by Lemma 3.3, and hence $\mathrm{Hom}_S(C, G) \in G_C$-$Inj$ by Theorem 3.11. Thus $\mathrm{Hom}_S(C, M) \in G_C$-$Inj_{\leqslant n}$ by the dual of Proposition 2.12 again. The rest of the proof is similar to that of Theorem 3.11. □

Remark 3.13 *(Foxby equivalence).* Let ${}_R Proj$ (${}_S Inj$) be the class of projective left R-modules (injective left S-modules). By [16, Theorem 1], Theorem 3.11 and Corollaries 3.7 and 3.12, there are equivalences of categories

$$\begin{array}{ccc}
{}_R Proj & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{W}_P \\
\downarrow & & \downarrow \\
G - Proj \cap \mathcal{A}_C(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & G_C - Proj \\
\downarrow & & \downarrow \\
G - Proj_{\leqslant n} \cap \mathcal{A}_C(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & G_C - Proj_{\leqslant n} \\
\downarrow & & \downarrow \\
\mathcal{A}_C(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{B}_C(S) \\
\uparrow & & \uparrow \\
G_C - Inj_{\leqslant n} & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & G - Inj_{\leqslant n} \cap \mathcal{B}_C(S) \\
\uparrow & & \uparrow \\
G_C - Inj & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & G - Inj \cap \mathcal{B}_C(S) \\
\uparrow & & \uparrow \\
\mathcal{W}_I & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & {}_S Inj.
\end{array}$$

Theorem 3.14. *Let R be a commutative Noetherian ring. A finitely generated R-module M is C-Gorenstein projective if and only if there exists an exact sequence*

$$W_\bullet = \cdots \to C \otimes_R F_1 \to C \otimes_R F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots$$

with F_i, F^j finitely generated free for all $i, j \geqslant 0$ and $M = \ker(C \otimes_R F^0 \to C \otimes_R F^1)$ such that $W_\bullet$ is $\operatorname{Hom}_R(C, -)$ *and* $\operatorname{Hom}_R(-, C)$ *exact.*

Proof. "⇒" Let M be a finitely generated C-Gorenstein projective R-module. Then $M \in \mathcal{B}_C(R)$ by Proposition 3.5 and $\operatorname{Hom}_R(C, M) \in A_C(R)$ is (finitely generated) Gorenstein projective by Theorem 3.11. So there is a finitely generated free resolution

$$F_\bullet = \cdots \to F_1 \to F_0 \to \operatorname{Hom}_R(C, M) \to 0$$

of $\operatorname{Hom}_R(C, M)$ such that every kernel is in $\mathcal{A}_C(R)$ by Lemma 3.3. Thus

$$C \otimes_R F_\bullet \cong \cdots \to C \otimes_R F_1 \to C \otimes_R F_0 \to M \to 0 \qquad (\dagger)$$

is exact and every kernel is in $\mathcal{B}_C(R)$. Note that $B_C(R) \subseteq \mathcal{W}_P^{\perp}$ by Corollary 3.2(2), so every kernel of the sequence (†) is C-Gorenstein projective by Corollary 2.6(2).

On the other hand, since $\operatorname{Hom}_R(C, M)$ is finitely generated Gorenstein projective, by an argument similar to the proof of [5, Theorem 4.2.6], there exists a short exact sequence $0 \to \operatorname{Hom}_R(C, M) \to F^0 \to L \to 0$ with F^0 finitely generated free and L finitely generated Gorenstein projective, moreover, $L \in A_C(R)$ by Lemma 3.3 again. So we have that the sequence $0 \to C \otimes_R \operatorname{Hom}_R(C, M) \to C \otimes_R F^0 \to C \otimes_R L \to 0$, i.e., $0 \to M \to C \otimes_R F^0 \to C \otimes_R L \to 0$ is exact, and $C \otimes_R L$ is finitely generated C-Gorenstein projective by Theorem 3.11. Repeating the foregoing process, we obtain an exact sequence

$$0 \to M \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots \qquad (\ddagger)$$

with F^j finitely generated free for all $j \geqslant 0$ such that every cokernel is C-Gorenstein projective. By pasting (†) and (‡), we get the desired complex $W_\bullet$ which is $\operatorname{Hom}_R(\mathcal{W}_P, -)$ and $\operatorname{Hom}_R(-, \mathcal{W}_P)$ exact by Proposition 2.7. In particular, $W_\bullet$ is $\operatorname{Hom}_R(C, -)$ and $\operatorname{Hom}_R(-, C)$ exact.

"⇐" Because $\mathcal{W}_P = \operatorname{Add} C$ by Theorem 3.1(1), $W_\bullet$ is $\operatorname{Hom}_R(\mathcal{W}_P, -)$ exact if and only if it is $\operatorname{Hom}_R(C, -)$ exact, and $W_\bullet$ is $\operatorname{Hom}_R(-, \mathcal{W}_P)$ exact if and only if it is $\operatorname{Hom}_R(-, C)$ exact since $W_\bullet$ is a complex of finitely generated modules. This completes the proof. □

By Theorems 2.15 and 3.14 we immediately obtain the following corollary which generalizes [5, Theorem 4.2.6].

Corollary 3.15. *Let R be a commutative Noetherian ring and C a semidualizing R-module. A finitely generated R-module M is* add C*-Gorenstein if and only if it is* Add C*-Gorenstein.*

Let R be a commutative Noetherian ring and C a semidualizing R-module. The add C-Gorenstein resolution (add C resolution) dimension of an R-module M is denoted by $G_C\text{-}\dim M$ ($C\text{-}\dim M$). By [14, Fact 8] and [4, Corollary 3.2(a)] and Remark 2.13, we immediately obtain the following

Corollary 3.16. *Let R be a commutative Noetherian local ring and M a nonzero finitely generated R-module. Then*:

(1) *If $G_C\text{-}\dim M < \infty$, then $G_C\text{-}\dim M + \operatorname{depth} M = \operatorname{depth} C$.*
(2) *If $C\text{-}\dim M < \infty$, then $C\text{-}\dim M + \operatorname{depth} M = \operatorname{depth} C$.*

Remark 3.17. (1) Let R be a commutative Noetherian local ring and $M \neq 0$ a finitely generated R-module with finite G-dimension. Auslander and Bridger [2] proved that M satisfies an analogue of the Auslander–Buchsbaum formula: $G\text{-dim}\, M + \operatorname{depth} M = \operatorname{depth} R$. Note that $\operatorname{depth} C = \operatorname{depth} R$ by [4, Corollary 3.2(a)], so Corollary 3.16(1) bears strong analogy with the Auslander–Buchsbaum formula.

(2) Corollary 3.16(2) was obtained by J.R. Strooker in [20].

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