Partial regularity for minimizers of discontinuous quasi-convex integrals with degeneracy

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In this paper we are concerned with the regularity of minimizers \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) of quasi-convex integral functionals of the type

\[
\mathcal{F}[u] := \int_{\Omega} f(x, u, Du) \, dx.
\]

The crucial point here is that the integrand \( f \) admits very weak regularity properties. With respect to the gradient variable it satisfies degenerate/singular \( p \)-growth conditions without necessarily possessing a quasi-diagonal Uhlenbeck-type structure, and with respect to the \( x \)-variable the integrand might be even discontinuous. It is only assumed that a certain VMO-condition holds. Under those assumptions we prove partial Hölder continuity of minimizers, i.e. Hölder continuity of \( u \) for any Hölder exponent \( \alpha \in (0, 1) \) outside a set of measure zero. Under such weak assumptions regularity results for the gradient of minimizers is not expected to hold since even in the scalar case counterexamples to \( C^1 \)-regularity are known.

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1. Introduction

In this paper we consider minimizers \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) of quasi-convex integral functionals of the type

\[
\mathcal{F}[u] := \int_{\Omega} f(x, u, Du) \, dx,
\]

(1.1)
where $\Omega$ denotes a bounded, open set in $\mathbb{R}^n$ and the minimizers are possibly vector valued, i.e. $u : \Omega \to \mathbb{R}^N$, with $N \geq 1$. Thereby, the integrand $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ is supposed to satisfy degenerate ($p > 2$), respectively singular ($p < 2$) $p$-growth assumptions with respect to the gradient variable while with respect to the $x$-variable we only assume a weak VMO-condition; see Section 1.1 for the precise set of assumptions. The main novelty here is the fact that we are able to deal at the same time with a degenerate/singular problem not necessarily possessing a quasi-diagonal Uhlenbeck-type structure, i.e. integrands of the form $f(x, u, Du) = g(x, u, |Du|^2)$, while with respect to $x$ and $u$ only a very weak degree of regularity of the integrand $f$ is supposed, i.e. the partial map $x \mapsto f(x, u, \xi)$ is assumed to be merely VMO while $u \mapsto f(x, u, \xi)$ is continuous.

In order to understand the degree of regularity we can expect we first consider the scalar case $N = 1$. If the integrand $f$ is continuous with respect to $(x, u)$ then it is known that minimizers are Hölder continuous with any Hölder exponent $\alpha < 1$, cf. Cupini, Fusco and Petti [9]. This result is indeed optimal in the sense that $\alpha = 1$ cannot be achieved since there are examples of non-Lipschitz minimizers and solutions to PDEs [9, 24]. In the vectorial case the situation is more difficult since it is known that singularities may appear [11, 30] and solutions may be even discontinuous although the integrand is analytic [21, 27]. Therefore, everywhere Hölder continuity cannot be expected in the vectorial case and the best we can hope for is a result ensuring at least partial Hölder continuity, i.e. Hölder continuity outside a set of measure zero. In fact, we are able to establish that minimizers are locally Hölder continuous with any Hölder exponent $\alpha \in (0, 1)$ in an open subset $\Omega_0 \subseteq \Omega$ with full Lebesgue measure, i.e. $|\Omega \setminus \Omega_0| = 0$.

1.1. Assumptions and results

Throughout the paper we let $p > 1$. The integrand $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ is assumed to be Borel-measurable such that the partial map $\xi \mapsto f(\cdot, \cdot, \xi)$ is of class $C^2$ on $\mathbb{R}^{Nn}$ if $p \geq 2$ and on $\mathbb{R}^{Nn} \setminus \{0\} \cap 1 < p < 2$. Moreover, $f$ satisfies the following $p$-growth conditions:

\[
\begin{align*}
&|\nabla \xi|^p + f(x, u, 0) \leq f(x, u, \xi) \leq L(1 + |\xi|)^p, \\
&D_x^2 f(x, u, \xi) \leq L|\xi|^{p-2} \quad (|\xi| \neq 0 \text{ if } 1 < p < 2)
\end{align*}
\]

whenever $x \in \Omega$, $u \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{Nn}$ and constants $0 < \nu \leq 1 \leq L$. Furthermore, we assume that $f$ is strictly degenerate quasi-convex, i.e.

\[
\int_{(0,1)^n} f(x, u, \xi + D\varphi(y)) - f(x, u, \xi) \, dy \geq \nu \int_{(0,1)^n} (|\xi| + |D\varphi(y)|)^{p-2} |D\varphi(y)|^2 \, dy,
\]

for all $x \in \Omega$, $u \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{Nn}$ and any testing function $\varphi \in W_0^{1,p}((0,1)^n, \mathbb{R}^N)$. With respect to $x$ we merely assume that $x \mapsto f(x, u, \xi)/(1 + |\xi|)^p$ satisfies a VMO-condition, uniformly with respect to $(u, \xi)$, in the sense that

\[
|f(x, u, \xi) - (f(\cdot, u, \xi))_{x_0,r}| \leq \psi_{x_0}(x, r)|\xi|^p, \quad \text{for all } x \in B_r(x_0)
\]

holds whenever $x_0 \in \Omega$, $r \in (0, \varrho_0)$, $u \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{Nn}$, where $\varrho_0 > 0$ and $\psi_{x_0} : \mathbb{R}^n \times [0, \varrho_0] \to [0, 2L]$ are bounded functions satisfying

\[
\lim_{\varrho \searrow 0} \psi(\varrho) = 0, \quad \text{where } \psi(\varrho) := \sup_{x_0 \in \Omega} \sup_{0 < r \leq \varrho} \int_{B_r(x_0)} \psi_{x_0}(x, r) \, dx.
\]

Note that assuming $\psi_{x_0} \leq 2L$ is no restriction because of the growth assumption (1.2)1. Moreover, we impose the following continuity assumptions on $f$ and $D^2 f$. With respect to $u$ we assume that
Theorem 1.1. \(\frac{|f(x, u, \xi) - f(x, u_0, \xi)|}{(1.2)\text{--}(1.8)}\)
In this case, the assumptions \((1.4)\text{ and } (1.5)\) are satisfied with the choice \((1.7)\text{--}(1.8)\)

Moreover, for any \(\xi \in \Omega\) and \(u \in \mathbb{R}^N\), \(\xi, \xi_0 \in \mathbb{R}^N\). In the case \(p < 2\) \((1.7)\) is assumed only if \(|\xi| \neq 0 \neq |\xi_0|\).
Here, \(\tilde{\omega} : [0, \infty) \rightarrow [0, 1]\) is a nondecreasing, concave modulus of continuity with \(\lim_{s \downarrow 0} \tilde{\omega}(s) = 0 = \omega(0)\).
Finally, instead of assuming an Uhlenbeck-type diagonal structure for \(f\) we only require that \(\xi \mapsto Df(x, u, \xi)\) behaves asymptotically at zero as the \(p\)-Laplacian, i.e.

\[
\lim_{s \downarrow 0} \frac{Df(x, u, s \xi)}{s^{p-1}} = |\xi|^{p-2} \xi
\]

uniformly in \(\xi \in \mathbb{R}^N\): \(|\xi| = 1\) and uniformly for all \(x \in \Omega\) and \(u \in \mathbb{R}^N\).
Under this set of assumptions we have the following partial regularity result:

**Theorem 1.1.** Let \(p > 1\) and \(u \in W^{1,p}(\Omega, \mathbb{R}^N)\) be a minimizer of the functional \((1.1)\), where the assumptions \((1.2)\text{--}(1.8)\) are in force. Then, there exists an open subset \(\Omega_0 \subseteq \Omega\) such that

\[u \in C^{0,\alpha}_{\text{loc}}(\Omega_0, \mathbb{R}^N)\text{ for every } \alpha \in (0, 1) \text{ and } |\Omega \setminus \Omega_0| = 0.\]

Moreover, for any \(\gamma \in (0, 1)\) we have \(Du \in L^{p,\gamma}_{\text{loc}}(\Omega_0, \mathbb{R}^{Nn})\) (see \((2.1)\) below for the definition). Finally, the singular set satisfies \(\Omega \setminus \Omega_0 \subseteq \Sigma_1 \cup \Sigma_2\), where

\[
\Sigma_1 := \left\{ x_0 \in \Omega : \liminf_{\epsilon \searrow 0} \int_{B_\epsilon(x_0)} |Du - (Du)_{x_0, \epsilon}|^p \, dx > 0 \right\},
\]

\[
\Sigma_2 := \left\{ x_0 \in \Omega : \limsup_{\epsilon \searrow 0} |(Du)_{x_0, \epsilon}| = \infty \right\}.
\]

**Remark 1.2.** The case of continuous integrands \(f\), i.e. when \((1.4)\) is replaced by a continuity assumption is certainly implied by \((1.4)\). Moreover, our assumptions cover integrands of splitting type such as

\[
\mathcal{F}[u] = \int_{\Omega} c(x) f(u, Du) \, dx,
\]

where the map \((u, \xi) \mapsto f(u, \xi)\) satisfies the assumptions \((1.2), (1.3)\) and \((1.6)\text{--}(1.8)\) with \(\nu\) and \(L\) replaced by \(\sqrt{\nu}\) and \(\sqrt{L}\) and the coefficients \(c \in \text{VMO}(\Omega)\) satisfy

\[
0 < \sqrt{\nu} \leq c(x) \leq \sqrt{L} \text{ for all } x \in \Omega \quad \text{and} \quad \sup_{x_0 \in \Omega} \sup_{0 < \epsilon \leq \epsilon_0} \int_{B_\epsilon(x_0) \cap \Omega} |c(x) - c_{x_0, \epsilon}| \, dx \to 0 \quad \text{as } \epsilon \searrow 0.
\]

In this case, the assumptions \((1.4)\) and \((1.5)\) are satisfied with the choice \(\nu_{x_0}(x, r) := \sqrt{L}|c(x) - c_{x_0, r}|.\)
If the integrand is non-degenerate and depends continuously on \((x, u)\) partial Hölder continuity of minimizers has been established in the case \(p \geq 2\) by Foss and Mingione in [17]. To our knowledge the sub-quadratic case \(p \in (1, 2)\) had not yet been established even in the non-degenerate setting and it is thus also included in our result. Note that in the special case of low dimensions \(n \leq p + 2\) Campanato [7], for elliptic systems, and Kristensen and Mingione [25], for convex integral functionals, established (partial) Hölder continuity for a fixed Hölder exponent (which can be determined in dependence on the dimension \(n\) and the growth exponent \(p\)) together with an estimate for the Hausdorff dimension of the singular set. The case of a non-degenerate integrand satisfying a VMO-condition with respect to the \(x\) variable has been treated in [4]. Related results at the boundary were obtained by Beck [2] and for systems with non-standard \(p(x)\)-growth by Habermann [23]. Finally, a partial Hölder continuity result for solutions to parabolic systems with continuous coefficients has been established by Bögelein, Foss and Mingione [6].

In the case of a degenerate integrand with a quasi-diagonal Uhlenbeck structure and stronger assumptions with respect to \((x, u)\) there are the up to now classical and well-known (partial) regularity results by Uhlenbeck [31], Acerbi and Fusco [1] and Giaquinta and Modica [19]. The scalar case had been previously treated by Ural’tseva [32]. The case of a general autonomous degenerate integrands \(f(\xi)\), i.e. an integrand without Uhlenbeck structure and without dependence upon \((x, u)\), has been considered in [14] by Duzaar and Mingione; there the partial \(C^{1,\alpha}\)-regularity of minimizers for some \(\alpha < 1\) is established. Schmidt [28] obtained a similar result when \(f\) has \((p, q)\)-growth and Beck and Stroffolini [3] for differential forms depending on \((x, u)\) in a Hölder continuous way. Here it is worth to add that a continuous respectively VMO dependence upon \((x, u)\) is the obstacle which prevents us from proving partial \(C^1\)-regularity, i.e. a continuity result for the gradient. Recently, in [5] Bögelein, Duzaar and Mingione were able to obtain a partial \(C^1\)-regularity result for weak solutions to parabolic systems with degenerate diffusion. The proof in the parabolic setting is very delicate since it uses certain parabolic Lipschitz-truncation arguments.

1.2. Technical aspects

In this section we briefly comment on the techniques involved in the proof of our main result. The maybe surprising fact is that the re-scaled excess which is typically used for low-order regularity problems and the distinction whether the problem behaves degenerate, respectively non-degenerate perfectly match together. This shall be explained a bit more in detail in the following. When considering a ball \(B_\varrho(x_0) \subseteq \Omega\) we shall distinguish between the degenerate and non-degenerate regime. We say that we are in the degenerate regime if the excess functional

\[
\Phi(x_0, \varrho) := \int_{B_\varrho(x_0)} |D\ell_{x_0, \varrho}|(u - \ell_{x_0, \varrho})^2 \, dx
\]

of the minimizer \(u\) is large compared to \(|D\ell_{x_0, \varrho}|^p\). Here, \(\ell_{x_0, \varrho} : \mathbb{R}^n \to \mathbb{R}^N\) denotes the affine function minimizing \(\ell \mapsto \int_{B_\varrho(x_0)} |u - \ell|^2 \, dx\) amongst all affine functions; see Section 2.4 for the basic properties. The precise definition of the \(V_{\mu}\)-function is given in (2.3); \(|V_{\mu}|^2\) interpolates in a certain sense between \(|\xi|^2\) when \(|\xi|\) is small – and \(|\xi|^p\) when \(|\xi|\) is large – taking into account the parameter \(\mu\) which is a measure for the degeneration. To be more precise, the degenerate regime is characterized by

\[
\Phi(x_0, \varrho) \geq \mu |D\ell_{x_0, \varrho}|^p,
\]

for some fixed \(\mu \ll 1\). Then, via hypothesis (1.8) we can show that the solution has approximately \(p\)-Laplacian-type behavior on the ball \(B_\varrho(x_0)\) and we can prove that if the excess is small at radius \(\varrho\) it is also small at some smaller radius, i.e. there exist \(\varepsilon, \kappa \ll 1\) and \(\theta < 1\) such that for any \(\alpha \in (0, 1)\) there holds
\[ \Phi(x_0, \varrho) < \varepsilon \implies \Phi(x_0, \vartheta \varrho) < \varepsilon \quad \text{and} \quad \Psi_\alpha(x_0, \vartheta \varrho) < \kappa, \tag{1.10} \]

where

\[ \Psi_\alpha(x_0, \varrho) := \varrho^{-\alpha p} \int_{B_\varrho(x_0)} |u - (u)_{x_0, \varrho}|^p \, dx. \]

This can be iterated up to the scale \( \vartheta^k \varrho \) as long as (D) is satisfied with \( \vartheta^k \varrho \) instead of \( \varrho \). Now let \( k_0 \) denote the first integer where (D) fails. Then, \( \vartheta^{k_0} \varrho \) is called the switching radius, where the behavior of the minimizer switches from degenerate to non-degenerate. In the so-called **non-degenerate regime** we have that \( |D\ell_{x_0, \varrho}| \) is large in a certain sense and therefore we linearize the problem around \( D\ell_{x_0, \varrho} \). This is the point where the assumptions (1.4) and (1.6) come into the play. Unfortunately, they are too week to allow to further iterate the excess \( \Phi \) – because this would require Hölder continuity of \( f \) with respect to \( (x, u) \). Therefore, we have to re-scale the excess by the quantity \( |D\ell_{x_0, \varrho}|^p \) which cannot be controlled during the iteration (note that \( |D\ell_{x_0, \varrho}| \sim |(Du)_{x_0, \varrho}| \) might blow up). The re-scaled excess \( \Phi(x_0, \varrho)/|D\ell_{x_0, \varrho}|^p \) can in fact be iterated, i.e. it can be shown that there exists \( \vartheta < 1 \) such that

\[ \frac{\Phi(x_0, \vartheta^{k_0} \varrho)}{|D\ell_{x_0, \vartheta^{k_0} \varrho}|^p} < \mu \quad \text{and} \quad \Psi_\alpha(x_0, \vartheta^{k_0} \varrho) < \kappa \]

implies

\[ \frac{\Phi(x_0, \vartheta^{k_0} \varrho)}{|D\ell_{x_0, \vartheta^{k_0} \varrho}|^p} < \mu \quad \text{and} \quad \Psi_\alpha(x_0, \vartheta^{k_0} \varrho) < \kappa. \tag{1.11} \]

The crucial point here is that the first assumption in (1.11) is satisfied exactly when (D) fails and therefore we can proceed the iteration in the non-degenerate regime. Moreover, as pointed out before \( |D\ell_{x_0, \varrho}| \) might blow up in the iteration since we cannot expect \( C^1 \)-regularity; however the Campanato-type excess \( \Psi_\alpha(x_0, \varrho) \) stays bounded, exactly as it should be for a \( C^{0, \alpha} \)-regularity result. Hence, having arrived at the non-degenerate regime at level \( k_0 \), the behavior stays non-degenerate at any larger level \( k > k_0 \) and we can proceed the iteration. We have thus ensured the smallness of \( \Psi_\alpha \) at any level and by Campanato’s characterization of Hölder continuous functions we conclude the Hölder continuity of \( u \) in \( x_0 \) provided the excess functionals \( \Phi \) and \( \Psi_\alpha \) are small at some initial radius \( \varrho \) (note that the smallness of the excess is an open condition and therefore it is satisfied already on a neighborhood of \( x_0 \)). In a final step it is then ensured that such an initial smallness condition on the excess is indeed satisfied on the complement of \( \Sigma_1 \cup \Sigma_2 \) from Theorem 1.1.

Let us finally comment on the methods leading to the conclusions in (1.10) and (1.11). The main tools are the lemmata of \( p \)-harmonic approximation and \( A \)-harmonic approximation. Provided the minimizer is in a certain sense approximatively \( p \)-harmonic, respectively \( A \)-harmonic, these lemmas ensure the existence of a \( p \)-harmonic, respectively \( A \)-harmonic function which is \( L^p \)-close to the original minimizer. These kind of lemmas have their origin in De Giorgi’s harmonic approximation lemma [10] and were proved in their first versions in [16] and [13].

### 2. Preliminaries

#### 2.1. Notations

By \( B_\varrho(x_0) \) we denote the open ball in \( \mathbb{R}^n \) with radius \( \varrho > 0 \) and center \( x_0 \). For the mean value of a function \( v \in L^1(B_\varrho(x_0), \mathbb{R}^k) \) we write

\[ (v)_{x_0, \varrho} := \int_{B_\varrho(x_0)} v \, dx. \]
In the case \( x_0 = 0 \) we use the more compact notations \( B_\varrho \equiv B_\varrho (0) \) and \((v)_\varrho \equiv (v)_{0,\varrho} \). Moreover we identify \( \mathbb{R}^{Nn} \) with the space of linear functions \( \mathbb{R}^n \to \mathbb{R}^N \) and write \( \langle \cdot , \cdot \rangle \) for the Euclidean scalar product on \( \mathbb{R}^{Nn} \).

For \( 1 \leq p < \infty \) and \( \gamma \geq 0 \) we denote by \( \mathcal{L}^{p,\gamma} (\Omega, \mathbb{R}^k) \) the Campanato-space of functions \( v \in L^p (\Omega, \mathbb{R}^k) \) satisfying

\[
\sup_{\varrho > 0, x_0 \in \Omega} \varrho^{-\gamma} \int_{B_\varrho (x_0) \cap \Omega} |v - (v)_{x_0,\varrho}|^p \, dx < \infty, \tag{2.1}
\]

and \( \mathcal{L}^{p,\gamma}_{loc} (\Omega, \mathbb{R}^k) \) for the space of all functions \( v \in L^p_{loc} (\Omega, \mathbb{R}^k) \) with \( v|_U \in L^{p,\gamma} (U, \mathbb{R}^k) \) for every sub-domain \( U \subset \Omega \).

As mentioned before we consider both, the super-, and sub-quadratic case, i.e. \( p > 2 \) and \( 1 < p < 2 \). Thereby, some arguments are slightly different for the two cases, e.g. some terms only appear in one of the two cases. In order to provide a unified treatment for both cases we use the following notation:

\[
\chi_{p < 2} := \begin{cases} 
1 & \text{if } p < 2, \\
0 & \text{if } p \geq 2
\end{cases} \quad \text{and} \quad \chi_{p \geq 2} := \begin{cases} 
0 & \text{if } p < 2, \\
1 & \text{if } p \geq 2
\end{cases} \tag{2.2}
\]

### 2.2. The \( V \)-function

Since we are dealing with \( p \)-growth problems it will be convenient to use the function \( V_\mu : \mathbb{R}^k \to \mathbb{R}^k \), where \( \mu \geq 0 \) and \( k \in \mathbb{N} \), given by

\[
V_\mu (A) := (\mu^2 + |A|^2)^{\frac{p-2}{2}} A \quad \text{for } A \in \mathbb{R}^k. \tag{2.3}
\]

In the following we shall provide some useful properties of the \( V \)-function. The first lemma collects some algebraic properties, cf. [14, Lemma 1].

**Lemma 2.1.** Let \( p > 1, k \in \mathbb{N} \) and \( \mu \geq 0 \). Then, for any \( A, B \in \mathbb{R}^k \) there holds

\[
(\mu^2 + |A|^2)^{\frac{p-2}{2}} |A| |B| \leq c(p) \left( |V_\mu (A)| + |V_\mu (B)| \right) \leq (\mu^2 + |A|^2)^{\frac{p-2}{2}} |B| |A| \tag{2.4}
\]

and

\[
|V_\mu (A + B)| \leq c(p) \left( |V_\mu (A)| + |V_\mu (B)| \right) \leq 2^{\frac{p-2}{2}} \max \{ \mu^2, |A|^2 \} |A| |B| \tag{2.5}
\]

and

\[
|V_\mu (\sigma A)| \leq \max \{ \sigma, \sigma^{p/2} \} |V_\mu (A)|, \quad \text{for any } \sigma > 0, \tag{2.6}
\]

and

\[
\begin{cases}
2^{\frac{p-2}{2}} \leq \frac{|V_\mu (A)|}{\min \{ \mu^{\frac{p-2}{2}} |A|, |A|^{\frac{p}{2}} \}} \leq 1 & \text{if } 1 < p < 2, \\
1 \leq \frac{|V_\mu (A)|}{\max \{ \mu^{\frac{p-2}{2}} |A|, |A|^{\frac{p}{2}} \}} \leq 2^{\frac{p-2}{2}} & \text{if } p \geq 2.
\end{cases} \tag{2.7}
\]
Next, we state a slightly generalized variant of [22, Lemma 3.7], see also [8].

**Lemma 2.2.** Let $p \geq 1$, $\mu \geq 0$, $\vartheta \in (0, 1)$, $A, B, C \geq 0$, $\beta > 0$, $v \in L^p(B_\varrho(x_0), \mathbb{R}^N)$ and $\phi : [r, \varrho] \to [0, \infty)$ be a bounded function satisfying

$$\phi(s) \leq \vartheta \phi(t) + A \int_{B_\varrho(x_0)} \left| V_\mu \left( \frac{v}{t-s} \right) \right|^2 dz + B(t-s)^{-\beta} + C,$$

for all $r \leq s < t \leq \varrho$. Then, there exists a constant $c = c(\vartheta, p, \beta)$ such that

$$\phi(r) \leq c \left[ A \int_{B_\varrho(x_0)} \left| V_\mu \left( \frac{v}{\varrho-r} \right) \right|^2 dx + B(\varrho-r)^{-\beta} + C \right].$$

The following Poincaré inequality for the $V$-function can be found in [14, Lemma 8] for $p \in (1, 2)$ (see also [12, Theorem 2]). Note that in the case $p \geq 2$ it follows from the usual Poincaré inequality.

**Lemma 2.3.** Let $p > 1$, $k \in \mathbb{N}$, $\mu \geq 0$ and $u \in W^{1,p}(B_\varrho(x_0), \mathbb{R}^k)$. Then, there exists a constant $c = c(n, N, p)$ such that

$$\int_{B_\varrho(x_0)} \left| V_\mu \left( \frac{u-(u)_{x_0,\varrho}}{\varrho} \right) \right|^2 dx \leq c \int_{B_\varrho(x_0)} \left| V_\mu(Du) \right|^2 dx,$$

where $V_\mu(\cdot)$ is defined according to (2.3).

The following algebraic fact can be retrieved from [1].

**Lemma 2.4.** For every $\sigma \in (-1/2, 0)$, $k \in \mathbb{N}$ and $\mu \geq 0$ we have

$$1 \leq \int_0^1 \left( \frac{\mu^2 + |A + s(B - A)|^2}{\mu^2 + |A|^2 + |B|^2} \right)^{\sigma} ds \leq \frac{8}{2\sigma + 1},$$

for any $A, B \in \mathbb{R}^k$, not both zero if $\mu = 0$.

**Remark 2.5.** From Lemma 2.4 we easily deduce: For every $\sigma \in (-1/2, 0)$ and $\mu \geq 0$ we have

$$\int_0^1 \left( \frac{\mu^2 + |A + sB|^2}{\mu^2 + |A|^2 + |B|^2} \right)^{\sigma} ds \leq \frac{24}{2\sigma + 1} \left( \frac{\mu^2 + |A|^2 + |B|^2}{\mu^2 + |A|^2 + |B|^2} \right)^{\sigma},$$

for any $A, B \in \mathbb{R}^{Nn}$, not both zero if $\mu = 0$. 
2.3. Basic deductions from the structure conditions

From the growth assumption (1.2)2 and (2.8) we conclude that

\[
|D_\xi f(x, u, \xi) - D_\xi f(x, u, \xi_0)| \leq \int_0^1 |D_\xi^2 f(x, u, \xi_0 + s(\xi - \xi_0))| \, ds \, |\xi - \xi_0|
\]

\[
\leq L \int_0^1 |\xi_0 + s(\xi - \xi_0)|^{p-2} \, ds \, |\xi - \xi_0|
\]

\[
\leq c(p) L (|\xi_0|^2 + |\xi - \xi_0|^2)^{\frac{p-2}{2}} |\xi - \xi_0| \tag{2.9}
\]

for all \( x \in \Omega, u \in \mathbb{R}^N \) and \( \xi, \xi_0 \in \mathbb{R}^{N_n} \). Note the first and second identities in (2.9) need to be justified in the case \( p < 2 \) since then the mapping \( \xi \mapsto f(\cdot, \cdot, \xi) \) is of class \( C^2 \) only on \( \mathbb{R}^{N_n} \setminus \{0\} \). Therefore it is enough to consider \( \xi, \xi_0 \in \mathbb{R}^{N_n} \) with \( \xi \neq \xi_0 \), since otherwise the term on the left-hand side is zero. In this case we shall first justify the identity

\[
D_\xi f(x, u, \xi) - D_\xi f(x, u, \xi_0) = \int_0^1 D_\xi^2 f(x, u, \xi_0 + s(\xi - \xi_0)) \, ds \, (\xi - \xi_0). \tag{2.10}
\]

To this aim we consider the function \( [0, 1] \ni s \mapsto g(s) := D_\xi^2 f(x, u, \xi_0 + s(\xi - \xi_0)) \in \mathbb{R}^{N_n} \). We first note that (2.10) is easily seen to hold if the segment \( [\xi, \xi_0] \) does not contain the origin of \( \mathbb{R}^{N_n} \) because then \( g(\cdot) \) is differentiable on \( [0, 1] \). Therefore, we can assume that there exists one parameter value \( \tilde{s} \in [0, 1] \) such that \( \xi_0 + \tilde{s}(\xi - \xi_0) = 0 \). In the case \( \tilde{s} \in (0, 1) \) we know that \( g(\cdot) \) is differentiable on \( [0, \tilde{s}] \) and on \( (\tilde{s}, 1] \) and hence, for any \( 0 < \varepsilon < \min\{\tilde{s}, 1 - \tilde{s}\} \) the following identities are valid:

\[
g(1) - g(\tilde{s} + \varepsilon) = \int_{\tilde{s} + \varepsilon}^1 D_\xi^2 f(x, u, \xi_0 + s(\xi - \xi_0)) \, ds \, (\xi - \xi_0)
\]

and

\[
g(\tilde{s} - \varepsilon) - g(0) = \int_0^{\tilde{s} - \varepsilon} D_\xi^2 f(x, u, \xi_0 + s(\xi - \xi_0)) \, ds \, (\xi - \xi_0).
\]

Since the function \( g \) is continuous we now can recover (2.10) by passing to the limit \( \varepsilon \downarrow 0 \) in the previous identities, noting that the integrals converge due to the growth condition (1.2)2, i.e. \( |D_\xi^2 f(x, u, \xi_0 + s(\xi - \xi_0))| \leq L |\xi_0 + s(\xi - \xi_0)|^{p-2} \) and the fact that \( p - 2 > -1 \). The cases \( \tilde{s} = 0 \) and \( \tilde{s} = 1 \) are similar. This proves (2.10) and hence the first inequality in (2.9). The second one is then justified in a similar way, noting that (1.2)1 provides a pointwise bound for the integrand which ensures the finiteness of the integral.

Taking into account that \( D_\xi f(\cdot, \cdot, 0) = 0 \) by (1.8) the inequality (2.9) implies

\[
|Df(x, u, \xi)| \leq L |\xi|^{p-1}, \tag{2.11}
\]

for any \( x \in \Omega, u \in \mathbb{R}^N \) and \( \xi \in \mathbb{R}^{N_n} \). Finally, from (2.11) we immediately conclude that
\[ |f(x, u, \xi) - f(x, u, \xi_0)| \leq \int_0^1 |D_{\xi} f(x, u, \xi_0 + s(\xi - \xi_0))| \, ds \, |\xi - \xi_0| \]
\[ \leq L(|\xi| + |\xi - \xi_0|)^{p-1} |\xi - \xi_0| \]
\[ \leq c(p)L(|\xi|^p + |\xi_0|^p). \quad (2.12) \]

Hypothesis (1.8) implies that there exists a function \( \eta : (0, 1) \to (0, 1) \) such that for any \( x \in \Omega, u \in \mathbb{R}^N \) and \( \delta \in (0, 1) \) we have
\[ |Df(x, u, \xi) - |\xi|^{p-2}\xi| \leq \delta |\xi|^{p-1}, \quad \text{for any } \xi \in \mathbb{R}^N \text{ with } |\xi| \leq \eta(\delta). \quad (2.13) \]

In order to “re-absorb” certain terms, we will use the following iteration lemma, which is a standard tool and can for instance be found in [20, Lemma 6.1].

**Lemma 2.6.** Let \( 0 < \vartheta < 1, A, B, C \geq 0 \) and \( 0 < \beta < \alpha \). Then there exists a constant \( c = c(\alpha, \vartheta) \) such that the following holds: For any \( 0 < r < \varrho \) and any non-negative, bounded function \( \phi : [r, \varrho] \to [0, \infty) \) satisfying
\[ \phi(s) \leq \vartheta \phi(t) + \frac{A}{(t-s)^\alpha} + \frac{B}{(t-s)^\beta} + C \quad \text{for all } r \leq s < t \leq \varrho, \]
we have
\[ \phi(r) \leq c \left( \frac{A}{(\varrho-r)^\alpha} + \frac{B}{(\varrho-r)^\beta} + C \right). \]

### 2.4. Minimizing affine functions

In order to measure the oscillation of a function certain related affine functions play a crucial role. Especially, affine functions constructed from mean values of the function and those minimizing the \( L^2 \)-distance – or \( L^p \)-distance – to the function are typically of interest. In this section we summarize some basic properties of those minimizing affine functions. Let us consider a ball \( B_\varrho(x_0) \subset \mathbb{R}^n \), with \( x_0 \in \mathbb{R}^n \) and \( \varrho > 0 \). For \( v \in L^1(B_\varrho(x_0), \mathbb{R}^N) \) we denote by \( \ell_{x_0,\varrho} \equiv \ell_{v;x_0,\varrho} : \mathbb{R}^n \to \mathbb{R}^N \) the affine function defined by
\[ \ell_{x_0,\varrho}(x) := \xi_{x_0,\varrho} + A_{x_0,\varrho}(x - x_0). \quad (2.14) \]

where
\[ \xi_{x_0,\varrho} := (v)_{x_0,\varrho} \quad \text{and} \quad A_{x_0,\varrho} := \frac{n+2}{\varrho^2} \int_{B_\varrho(x_0)} v \otimes (x - x_0) \, dz. \quad (2.15) \]

For notational convenience we omit the center point \( x_0 \) in our notation when \( x_0 = 0 \), writing for instance \( \ell_\varrho \equiv \ell_{0,\varrho} \). It is a well-known fact that if \( v \in L^2(B_\varrho(x_0), \mathbb{R}^N) \), then \( \ell_{x_0,\varrho} \) is the unique affine map minimizing
\[ \ell \mapsto \int_{B_\varrho(x_0)} |v - \ell|^2 \, dx \quad (2.16) \]
amongst all affine maps $\ell(x) = \xi + A(x - x_0)$, with $\xi \in \mathbb{R}^N$ and $A \in \mathbb{R}^{Nn}$. For more details we refer to [26].

In the following we collect some useful properties of the map $\ell_{v, x_0, q}$. First of all, we recall that for any $A \in \mathbb{R}^{Nn}$ and $\xi \in \mathbb{R}^N$ there holds

$$|A_{x_0, q} - A| \leq \frac{n + 2}{q} \int_{B_q(x_0)} |v - \xi - A(x - x_0)| \, dx, \tag{2.17}$$

which can be deduced by a slight modification of the proof of [26, Lemma 2].

The following lemma ensures that $\ell_{x_0, q}$ is an almost minimizer of the functional $\ell \mapsto \int_{B_q(x_0)} |v - \ell|^p \, dx$ amongst the affine functions $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^k$, cf. [4, formula (2.7)].

**Lemma 2.7.** Let $p \geq 1$, $k \in \mathbb{N}$, $B_q(x_0)$ be a ball in $\mathbb{R}^n$ with $q > 0$ and $v \in L^p(B_q(x_0), \mathbb{R}^k)$. Then, we have

$$\int_{B_q(x_0)} |v - \ell_{x_0, q}|^p \, dx \leq c(n, p) \int_{B_q(x_0)} |v - \ell|^p \, dx$$

for any affine function $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^k$.

The next lemma ensures that $\ell_{x_0, q}$ is also an almost minimizer of the functional $\ell \mapsto \int_{B_q(x_0)} |V_\mu (\frac{v - \ell}{Q})|^2 \, dx$ amongst the affine functions $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^k$. The argument is quite similar to [29, Lemma 6.2].

**Lemma 2.8.** Let $p \geq 1$, $k \in \mathbb{N}$, $\mu \geq 0$, $B_q(x_0)$ be a ball in $\mathbb{R}^n$ with $q > 0$ and $v \in L^p(B_q(x_0), \mathbb{R}^k)$. Then, we have

$$\int_{B_q(x_0)} \left| V_\mu \left( \frac{v - \ell_{x_0, q}}{Q} \right) \right|^2 \, dx \leq c(p) \int_{B_q(x_0)} \left| V_\mu \left( \frac{v - \ell}{Q} \right) \right|^2 \, dx$$

for any affine function $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^k$.

**Proof.** Without loss of generality we assume $x_0 = 0$. From (2.5) we have

$$\int_{B_q} \left| V_\mu \left( \frac{v - \ell}{Q} \right) \right|^2 \, dx \leq c \left[ \int_{B_q} \left| V_\mu \left( \frac{v - \ell}{Q} \right) \right|^2 \, dx + \int_{B_q} \left| V_\mu \left( \frac{\ell - \ell}{Q} \right) \right|^2 \, dx \right]. \tag{2.18}$$

with $c = c(p)$. At this point it remains to estimate the second term on the right-hand side. From (2.17) and the fact that $\int_{B_q} D\ell \, dx = 0$ we conclude that for any $x \in B_q$ there holds

$$|\ell_q(x) - \ell(x)| \leq |(v)_q - \ell(0)| + |D\ell_q|_{Q} |x| \leq (n + 3) \int_{B_q} |v - \ell| \, dx.$$

Using the preceding estimate, the monotonicity of $V_\mu$ and (2.6) we infer

$$\int_{B_q} \left| V_\mu \left( \frac{\ell_q - \ell}{Q} \right) \right|^2 \, dx \leq c \int_{B_q} \left| V_\mu \left( \frac{v - \ell}{Q} \right) \right|^2 \, dx. \tag{2.19}$$
The claim would follow, if we could apply Jensen’s inequality to the right-hand side of the preceding inequality. But, unfortunately \( V_\mu \) is not convex if \( p \geq 1 \) is too small. To overcome this lack of convexity, we define

\[
W_\mu(A) := (\mu + |A|)^{\frac{p-2}{2}} A \quad \text{for} \quad A \in \mathbb{R}^k.
\]

Then, on the one hand there holds

\[
c(p)^{-1} W_\mu(A) \leq V_\mu(A) \leq c(p) W_\mu(A) \quad \text{for all} \quad A \in \mathbb{R}^k
\]

and on the other hand the mapping \( A \mapsto |W_\mu(A)|^2 \) is convex which can be verified by the use of one-dimensional calculus. Therefore, we can apply Jensen’s inequality to \( W_\mu(\cdot) \) and proceed to estimate the right-hand side of (2.19) as follows:

\[
\int_{B_\varrho} \left| \nabla \left( \frac{\varrho - \ell}{\varrho} \right) \right|^2 dx \leq c \int_{B_\varrho} \left| W_\mu \left( \frac{\varrho - \ell}{\varrho} \right) \right|^2 dx \leq c \int_{B_\varrho} \left| V_\mu \left( \frac{\varrho - \ell}{\varrho} \right) \right|^2 dx.
\]

Joining this with (2.18) yields the claim. \( \square \)

Finally, in the case \( p \geq 2 \) we can show that \( \ell_{x_0,0} \) is also an almost minimizer of \( \ell \mapsto \int_{B_\varrho(x_0)} |D\ell| \left( \frac{v - \ell_\varrho}{\varrho} \right) |^2 \right) dx. \]

**Lemma 2.9.** Let \( p \geq 2, k \in \mathbb{N}, B_\varrho(x_0) \) be a ball in \( \mathbb{R}^n \) with \( \varrho > 0 \) and \( v \in L^p(B_\varrho(x_0), \mathbb{R}^k) \). Then, we have

\[
\int_{B_\varrho(x_0)} \left| V_{D\ell}(\frac{v - \ell_\varrho}{\varrho}) \right|^2 dx \leq c(n, p) \int_{B_\varrho(x_0)} \left| V_{D\ell}(\frac{v - \ell_\varrho}{\varrho}) \right|^2 dx.
\]

for any affine function \( \ell : \mathbb{R}^n \to \mathbb{R}^k \).

**Proof.** For simplicity we assume \( x_0 = 0 \). From Lemma 2.8 and (2.7) we obtain

\[
\int_{B_\varrho} \left| V_{D\ell}(\frac{v - \ell_\varrho}{\varrho}) \right|^2 dz \leq c(p) \int_{B_\varrho} \left| V_{D\ell}(\frac{v - \ell_\varrho}{\varrho}) \right|^2 dz \leq c(p) \int_{B_\varrho} \left| D\ell(\frac{v - \ell_\varrho}{\varrho}) \right| |^2 dz + \int_{B_\varrho} \left| D\ell(\frac{v - \ell_\varrho}{\varrho}) \right|^p dz.
\]

Moreover, from (2.17) we infer

\[
|D\ell_\varrho| \leq |D\ell_\varrho - D\ell| + |D\ell| \leq (n+2) \int_{B_\varrho} \left| \frac{v - \ell_\varrho}{\varrho} \right| dx + |D\ell|.
\]

Inserting this above and applying Hölder’s inequality we deduce the claim. \( \square \)
2.5. $\mathcal{A}$-harmonic and $p$-harmonic functions

Given a bilinear form $\mathcal{A}$ on $\mathbb{R}^{Nn}$ that is strictly elliptic in the sense of Legendre–Hadamard, with ellipticity constant $\nu$ and upper bound $L$, i.e.

$$\mathcal{A}(\zeta \otimes \eta, \zeta \otimes \eta) \geq \nu |\zeta|^2 |\eta|^2 \quad \text{and} \quad \mathcal{A}(\xi, \tilde{\xi}) \leq L |\xi| |\tilde{\xi}|, \quad (2.20)$$

for all $\zeta \in \mathbb{R}^N$, $\eta \in \mathbb{R}^n$ and $\xi, \tilde{\xi} \in \mathbb{R}^{Nn}$. Here and in the following, we say that a map $h \in W^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ is $\mathcal{A}$-harmonic on $B_\rho(x_0) \subset \mathbb{R}^n$ if and only if

$$\int_{B_\rho(x_0)} \mathcal{A}(Dh, D\varphi) \, dx = 0, \quad \text{for all } \varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N).$$

By the classical theory it is well known that $h$ is smooth in the interior of $B_\rho(x_0)$, and also up to the boundary provided that the boundary data is smooth enough. Moreover, it satisfies the following estimate that we shall use later on:

$$\sup_{B_\rho(2)(x_0)} |Dh|^2 + \varepsilon^2 \sup_{B_\rho(2)(x_0)} |D^2h|^2 \leq c(n, N, \nu, L) \int_{B_\rho(x_0)} |Dh|^2 \, dx. \quad (2.21)$$

Moreover, given $p > 1$ we say that a map $h \in W^{1,p}(B_\rho(x_0), \mathbb{R}^N)$ is $p$-harmonic on $B_\rho(x_0) \subset \mathbb{R}^n$ if and only if

$$\int_{B_\rho(x_0)} |Dh|^{p-2} Dh \cdot D\varphi \, dx = 0, \quad \text{for all } \varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N).$$

By the famous result of Uhlenbeck [31] it is known that any $p$-harmonic function is locally of class $C^{1,\alpha}$ with some $\alpha \in (0,1)$. The following version of this result can be found in [18, Theorem 4.2 and Corollary 4.3] for the case $p > 2$, respectively in [1, Proposition 2.13] for the case $p < 2$ (note that by [1, Lemma 2.2] the excess functional used in [1] is equivalent to the one in (2.23)).

**Theorem 2.10.** Let $p > 1$. There exist constants $a_0 = a_0(n, N, p) \in (0,1)$ and $c = c(n, N, p) \geq 1$ such that the following holds: Whenever $h \in W^{1,p}(U, \mathbb{R}^N)$ is $p$-harmonic on $U$ and $B_R(x_0) \subset U$ then for any $r \in (0, R]$ there holds

$$\sup_{R/2(x_0)} |Dh|^p \leq c \int_{B_R(x_0)} |Dh|^p \, dx \quad (2.22)$$

and

$$\int_{B_r(x_0)} |V_{\mathcal{A}_k}(Dh) - \mathcal{A}_k(x_0, r)|^2 \, dx \leq c \left( \frac{r}{R} \right)^{2a_0} \int_{B_R(x_0)} |V_{\mathcal{A}_k}(Dh) - \mathcal{A}_k(x_0, r)|^2 \, dx, \quad (2.23)$$

where $\mathcal{A}_k \in \mathbb{R}^{Nn}$, $\varepsilon > 0$, is defined by

$$\begin{cases} 
\mathcal{A}_k(x_0, r) = (Dh)_{x_0, \varepsilon} & \text{if } p \geq 2, \\
V_0(\mathcal{A}_k, \varepsilon) = (V_0(Dh))_{x_0, \varepsilon} & \text{if } p < 2.
\end{cases} \quad (2.24)$$
2.6. Harmonic type approximation lemmas

Later on, we are going to compare in the non-degenerate case the minimizer of our integral functional with solutions to constant coefficient elliptic systems. In the case \( p \geq 2 \) this will be achieved by the following version of the \( \mathcal{A} \)-harmonic approximation lemma which can be retrieved from the corresponding parabolic version in [15, Lemma 3.2], cf. [4, Lemma 2.1] after a scaling argument (i.e. one applies the lemma to \( \tilde{w} \equiv w/\gamma \) instead of \( w \); note that the lemma holds trivially if \( \gamma = 0 \)). In the case \( p = 2 \) the lemma was initially proved in [16, Lemma 3.3].

**Lemma 2.11.** Let \( 0 < \nu \leq L \) and \( p \geq 2 \) be given. For every \( \varepsilon > 0 \), there are constants \( \delta_0 = \delta_0(n, N, p, \nu, L, \varepsilon) \in (0, 1) \) and \( c = c(n, p) \geq 1 \) such that the following holds: Assume that \( \gamma \in [0, 1] \) and that \( \mathcal{A} \) is a bilinear form on \( \mathbb{R}^N \) satisfying (2.20). Furthermore, let \( w \in W^{1, p}(B_{\delta}(x_0), \mathbb{R}^N) \) be an approximately \( \mathcal{A} \)-harmonic map in the sense that there holds

\[
\left| \int_{B_{\delta}(x_0)} \mathcal{A}(Dw, D\varphi) \, dx \right| \leq \delta_0 \gamma \sup_{B_{\delta}(x_0)} |D\varphi| \tag{2.25}
\]

for all \( \varphi \in C_0^\infty(B_{\delta}(x_0), \mathbb{R}^N) \) that satisfies

\[
\int_{B_{\delta}(x_0)} \left| V_1(Dw) \right|^2 \, dx \leq \gamma^2. \tag{2.26}
\]

Then there exists an \( \mathcal{A} \)-harmonic map \( h \in C^\infty(B_{\delta/2}(x_0), \mathbb{R}^N) \) satisfying

\[
\int_{B_{\delta/2}(x_0)} \left| V_1(Dh) \right|^2 \, dx \leq c \gamma^2
\]

and

\[
\int_{B_{\delta/2}(x_0)} \left| V_1\left( \frac{w-h}{\gamma} \right) \right|^2 \, dx \leq \varepsilon \gamma^2. \tag{2.27}
\]

The following variant of the \( \mathcal{A} \)-harmonic approximation lemma shall be used in the sub-quadratic case \( 1 < p < 2 \) and can be retrieved from [12, Lemma 6] (after a scaling argument, i.e. applying the lemma with \( \tilde{w} \equiv w/\gamma \) instead of \( w \) and taking into account that \( c(p)^{-1} |V(\cdot)| \leq |W(\cdot)| \leq c(p) |V(\cdot)| \)).

**Lemma 2.12.** Let \( 0 < \nu \leq L \) and \( 1 < p < 2 \) be given. For every \( \varepsilon > 0 \), there are constants \( \delta_0 = \delta_0(n, N, p, \nu, L, \varepsilon) \in (0, 1) \) and \( c = c(n, p) \geq 1 \) such that the following holds: Assume that \( \gamma \in [0, 1] \) and \( \mathcal{A} \) is a bilinear form on \( \mathbb{R}^N \) satisfying (2.20). Furthermore, let \( w \in W^{1, p}(B_{\delta}(x_0), \mathbb{R}^N) \) be an approximately \( \mathcal{A} \)-harmonic map in the sense of (2.25) satisfying (2.26). Then there exists an \( \mathcal{A} \)-harmonic map \( h \in C^\infty(B_{\delta/2}(x_0), \mathbb{R}^N) \) satisfying

\[
\int_{B_{\delta/2}(x_0)} \left| V_1(\gamma^{-1} Dh) \right|^2 \, dx \leq c
\]

(respectively \( h \equiv 0 \) in the case \( \gamma = 0 \)) and (2.27).
The following \( p \)-harmonic approximation lemma was proved in [13, Lemma 1]. It allows to approximate “almost \( p \)-harmonic functions” by \( p \)-harmonic functions.

**Lemma 2.13.** For any \( \varepsilon > 0 \) there exists a positive constant \( \delta_0 \in (0, 1] \), depending only on \( n, N, p, \varepsilon \), such that the following is true: whenever \( w \in W^{1,p}(B_c(x_0), \mathbb{R}^N) \) with

\[
\int_{B_c(x_0)} |Dw|^p \, dx \leq 1
\]

is approximatively \( p \)-harmonic in the sense that

\[
\left| \int_{B_c(x_0)} |Dw|^{p-2} Dw \cdot D\varphi \, dx \right| \leq \delta_0 \sup_{B_c(x_0)} |D\varphi|
\]

holds for all \( \varphi \in C^1_0(B_c(x_0), \mathbb{R}^N) \), then there exists a \( p \)-harmonic function \( h \in W^{1,p}(B_c(x_0), \mathbb{R}^N) \) such that

\[
\int_{B_c(x_0)} |Dh|^p \, dx \leq 1 \quad \text{and} \quad \int_{B_c(x_0)} \left| \frac{w - h}{\varepsilon} \right|^p \, dx \leq \varepsilon.
\]

### 2.7. Ekeland’s variational principle

The following version of Ekeland’s variational principle will be used for the construction of suitable comparison maps within the linearization procedure. A proof can be found e.g. in [20, Theorem 5.6].

**Lemma 2.14.** Let \((X, d)\) be a complete metric space, and assume that \( G: X \rightarrow [0, \infty) \) is not identically \( \infty \) and lower semicontinuous with respect to the metric topology on \( X \). If for some \( u \in X \) and some \( \kappa > 0 \), there holds

\[
G(u) \leq \inf_X G + \kappa,
\]

then there exists \( v \in X \) with the properties

\[
d(u, v) \leq 1 \quad \text{and} \quad G(v) \leq G(w) + \kappa d(v, w) \quad \forall w \in X.
\]

### 3. Partial continuity

This section is devoted to the proof of the main result.

#### 3.1. Caccioppoli inequality and higher integrability

We first state a zero order Caccioppoli inequality. Thereby it is crucial that no additive constant appears on the right-hand side of the Caccioppoli inequality. For the sake of completeness we give the proof, although it can essentially be deduced from the one of [20, Theorem 6.5].
**Lemma 3.1.** Let \( p > 1 \). There is a constant \( c = c(p, L/\nu) \), such that the following holds. Assume that \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) is a minimizer of the functional (1.1), under the assumption (1.2). Then, for any \( u_0 \in \mathbb{R}^N \) and \( x_0 \in \Omega \) and all radii \( 0 < \varrho < \text{dist}(x_0, \partial \Omega) \) and \( r \in [\frac{\varrho}{2}, \varrho) \) there holds:

\[
\int_{B_r(x_0)} |Du|^p \, dx \leq c \int_{B_\varrho(x_0)} \left| \frac{u - u_0}{\varrho - r} \right|^p \, dx.
\]

**Proof.** For the sake of brevity we write \( B_s \equiv B_s(x_0) \) for any \( s > 0 \). For radii \( r \leq s < t \leq \varrho \) we choose a standard cut-off function \( \eta \in C_c^\infty(B_t, [0, 1]) \) with \( \eta \equiv 1 \) on \( B_s \) and \( |D\eta| \leq \frac{2}{t-s} \). Then, we define the function \( \varphi := \eta(u - u_0) \in W^{1,p}_0(B_s, \mathbb{R}^N) \). Using (1.2) and the fact that \( D\varphi \equiv Du \) on \( B_s \) we obtain

\[
v \int_{B_t} |D\varphi|^p \, dx \leq \int_{B_t} f(x, u, D\varphi) - f(x, u, 0) \, dx
\]

\[
= \int_{B_t} f(x, u, Du) - f(x, u, 0) \, dx + \int_{B_t \setminus B_s} f(x, u, D\varphi) - f(x, u, Du) \, dx
\]

\[
=: I + II,
\]

with the obvious meaning of \( I \) and \( II \). For the estimate of \( I \) we use the minimality of \( u \), (2.12) and the fact that \( Du - D\varphi \equiv 0 \) on \( B_s \). This leads us to

\[
I \leq \int_{B_t} f(x, u, Du - D\varphi) - f(x, u, 0) \, dx
\]

\[
\leq L \int_{B_t \setminus B_s} |Du - D\varphi|^p \, dx \leq c(p) L \int_{B_t \setminus B_s} |Du|^p + |D\varphi|^p \, dx.
\]

Using (2.12) again we estimate \( II \) as follows:

\[
II \leq c(p) L \int_{B_t \setminus B_s} |Du|^p + |D\varphi|^p \, dx.
\]

Joining the preceding estimates for \( I \) and \( II \) with (3.1) we arrive at

\[
v \int_{B_t} |D\varphi|^p \, dx \leq c(p) L \int_{B_t \setminus B_s} |Du|^p + |D\varphi|^p \, dx.
\]

Recalling that \( D\varphi \equiv Du \) on \( B_s \) and \( D\varphi \leq c[Du + \left| \frac{u - u_0}{t-s} \right|] \) this leads us to

\[
\int_{B_s} |Du|^p \, dx \leq \hat{c}(p, L/\nu) \int_{B_t \setminus B_s} |Du|^p + \left| \frac{u - u_0}{t-s} \right|^p \, dx.
\]
Now, we add on both sides of the preceding inequality the quantity \( \hat{c} \int_{B_r} |Du|^p \, dx \) and divide by \( \hat{c} + 1 \). This yields

\[
\int_{B_s} |Du|^p \, dx \leq \vartheta \int_{B_s} |Du|^p + \left| \frac{u - u_0}{\ell - s} \right|^p \, dx,
\]

with \( \vartheta = \frac{\hat{c}}{\hat{c} + 1} < 1 \) depending only on \( p, L/\nu \). At this point the asserted Caccioppoli inequality immediately follows from the iteration Lemma 2.6. \( \square \)

From Lemma 3.1 we deduce in a standard way with the help of Gehring’s Lemma the following higher integrability result.

**Lemma 3.2.** Let \( p > 1 \). There exist an exponent \( q_0 = q_0(n, N, p, L/\nu > p \) and a constant \( c = c(n, N, p, L/\nu) \) such that the following holds. Assume that \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) is a minimizer of the variational functional (1.1) where the assumptions (1.2) are in force. Then, \( u \in W^{1,q_0}_{\text{loc}}(\Omega, \mathbb{R}^N) \). Moreover, for any \( x_0 \in \Omega \) and all radii \( 0 < \varrho < \text{dist}(x_0, \partial \Omega) \) and \( r \in [\frac{\varrho}{2}, \varrho) \) the quantitative estimate

\[
\int_{B_r(x_0)} |Du|^q \, dx \leq c \left( \frac{\varrho}{\varrho - r} \right)^{\frac{n(q - p)}{p}} \left( \int_{B_{\varrho}(x_0)} |Du|^p \, dx \right)^{\frac{q}{p}}
\]

holds, for all \( q \in (0, q_0] \).

Furthermore, we will need the following global higher integrability result.

**Lemma 3.3.** Let \( p > 1 \) and \( u \in W^{1,q_0}(B_r(x_0), \mathbb{R}^N) \) for some \( q_0 > p \). Then there exist an exponent \( q = q(n, N, p, L/\nu, q_0) \in (p, q_0] \) and a constant \( c = c(n, N, p, L/\nu) \) such that there holds: Whenever \( v \in u + W^{1,p}_0(B_r(x_0), \mathbb{R}^N) \) is a minimizer of the integral functional \( G[v] := \int_{B_r(x_0)} g(Dv) \, dx \) with a \( C^1 \)-integrand \( g : \mathbb{R}^N \to \mathbb{R} \) satisfying the growth assumptions

\[
v|\xi|^p \leq g(\xi) \leq L(1 + |\xi|)^p \quad \text{and} \quad |Dg(\xi)| \leq L|\xi|^{p-1}
\]

for all \( \xi \in \mathbb{R}^N \), then we have \( v \in W^{1,q}(B_r(x_0), \mathbb{R}^N) \) and moreover,

\[
\int_{B_r(x_0)} |Dv|^q \, dx \leq c \left( \int_{B_r(x_0)} |Dv|^p \, dx \right)^{\frac{q}{p}} + c \left( \int_{B_r(x_0)} |Du|^{q_0} \, dx \right)^{\frac{q}{q_0}}.
\]

**Remark 3.4.** We now fix the exponent \( q_0 = q_0(n, N, p, L/\nu) > p \) from Lemma 3.2. From now on we shall denote by \( q = q(n, N, p, L/\nu) \in (p, q_0] \) the exponent determined by Lemma 3.3 applied with the exponent \( q_0 \).

The crucial point in the following Caccioppoli inequality is based on the fact that the constant appearing on the right-hand side depends only upon the structural constants of the elliptic system, but is independent from \( |D\ell| \), where \( \ell : \mathbb{R}^N \to \mathbb{R}^N \) is any affine function. Later on, we will apply this inequality with the minimizing affine function \( \ell = \ell_{x_0,r} : \mathbb{R}^n \to \mathbb{R}^N \) introduced in the definition (2.14).
Lemma 3.5. Let $p > 1$. There is a constant $c = c(n, N, p, v, L) \geq 1$, such that the following holds: Whenever $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a minimizer of the functional (1.1), under the assumptions (1.2)-(1.6) and $\ell : \mathbb{R}^n \to \mathbb{R}^N$ is an affine function, then for any ball $B_r(x_0) \subseteq \Omega$ with $r \leq q_0$ there holds:

$$
\int_{B_{r/2}(x_0)} |V|_{D\ell}(Du - D\ell)|^2 \, dx \\
\leq c \int_{B_{r/2}(x_0)} V|D\ell| \left( \frac{u - \ell}{q} \right)^2 \, dx + c|D\ell|^p \left[ \omega \left( \int_{B_{r/2}(x_0)} |u - \ell(x_0)| + |u - \ell| \, dx \right)^{\frac{q-p}{q}} + \mathbf{V}(q) \frac{q}{q-p} \right],
$$

where $q$ is the exponent determined by Remark 3.4.

Proof. Without loss of generality we assume $x_0 = 0$. For radii $\frac{r}{2} \leq r \leq s \leq \frac{3r}{4}$ with $s := \frac{r+t}{2}$ we choose a standard cut-off function $\eta \in C_0^\infty(B_s,[0,1])$ with $\eta \equiv 1$ on $B_r$ and $|D\eta| \leq \frac{2}{s-r} = \frac{4}{t-r}$ on $B_s$. With these choices we define the functions $\varphi := \eta(u - \ell) \in W^{1,p}_0(B_s, \mathbb{R}^N)$ and $\psi := (1-\eta)(u - \ell) \in W^{1,p}(B_s, \mathbb{R}^N)$. From the quasi-convexity (1.3) we obtain for any $y \in B_s$ that

$$
v \int_{B_{s}} \left( |D\ell| + |D\varphi| \right)^{p-2} |D\varphi|^2 \, dx \\
\leq \int_{B_{s}} \left[ f(y, \ell(0), D\ell + D\varphi(x)) - f(y, \ell(0), D\ell) \right] \, dx.
$$

Integrating the preceding inequality with respect to $y$ over $B_s$, taking mean values and finally applying Fubini’s theorem yields

$$
v \int_{B_{s}} \left( |D\ell| + |D\varphi| \right)^{p-2} |D\varphi|^2 \, dx \\
\leq \int_{B_{s}} \int_{B_{s}} \left[ f(y, \ell(0), D\ell + D\varphi(x)) - f(y, \ell(0), D\ell) \right] \, dx \, dy \\
= \int_{B_{s}} \left[ (f(\cdot, \ell(0), D\ell + D\varphi(x)) \right] \, dx \\
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7,
$$

where we have abbreviated (note that $\ell + \varphi = u - \psi$):

$$
I_1 := \int_{B_{s}} \left[ (f(\cdot, \ell(0), Du(x) - D\psi(x)) \right] \, dx, \\
I_2 := \int_{B_{s}} \left[ (f(\cdot, \ell(0), Du(x)) \right] \, dx, \\
I_3 := \int_{B_{s}} \left[ (f(\cdot, u(x), Du(x)) \right] \, dx.
$$
\[ I_4 := \int_{B_s} \left[ f(x, u(x), Du(x)) - f(x, u(x) - \varphi(x), Du(x) - D\varphi(x)) \right] dx, \]

\[ I_5 := \int_{B_s} \left[ f(x, u(x) - \varphi(x), D\ell + D\varphi(x)) - f(x, \ell(0), D\ell + D\varphi(x)) \right] dx, \]

\[ I_6 := \int_{B_s} \left[ f(x, \ell(0), D\ell + D\varphi(x)) - (f(\cdot, \ell(0), D\ell + D\varphi(x))_x) \right] dx, \]

\[ I_7 := \int_{B_s} \left[ (f(\cdot, \ell(0), D\ell + D\varphi(x))_x - (f(\cdot, \ell(0), D\ell))_x \right] dx. \]

In the following we will infer bounds for the terms \( I_1 \)–\( I_7 \). First, we observe that the minimizing property of \( u \) yields \( I_4 \leq 0 \). From the definition of the partial mean of \( f \) with respect to the first variable and the continuity assumption (1.6) with respect to the second one, we obtain

\[ I_2 = \int_{B_s} \int_{B_s} \left[ f(y, \ell(0), Du(x)) - f(y, u(x), Du(x)) \right] dy \, dx \]

\[ \leq L \int_{B_s} \omega(|u - \ell(0)|)|Du|^p \, dx, \]

while in a similar way we infer from (1.4) that

\[ I_3 \leq \int_{B_s} \omega(\cdot, s)|Du|^p \, dx. \]

For the estimate of \( I_5 \), we use the continuity property (1.6) as well as \( |\varphi| \leq |u - \ell| \) and the choice of \( \eta \) to find that

\[ I_5 \leq L \int_{B_s} \omega(|u - \varphi - \ell(0)|)|D\ell + D\varphi|^p \, dx \]

\[ \leq c(p)L \int_{B_s} \omega(|u - \ell(0)| + |u - \ell|)(|D\ell| + |Du| + \frac{|u - \ell|}{|u - \ell|})^p \, dx. \]

In order to bound \( \mathcal{U} := \frac{u - \ell}{|u - \ell|} \) in terms of \( V_{|D\ell|}(\mathcal{U}) \) and \( |D\ell| \) we distinguish the cases when \( |\mathcal{U}| \geq |D\ell| \) and \( |\mathcal{U}| < |D\ell| \). Using that we have \( |\mathcal{U}|^p \leq 2|V_{|D\ell|}(\mathcal{U})|^2 \) in the first case while \( |\mathcal{U}|^p \leq |D\ell|^p \) in the second one, we observe that

\[ \left| \frac{u - \ell}{|u - \ell|} \right|^p \leq 2 \left| V_{|D\ell|} \left( \frac{u - \ell}{|u - \ell|} \right) \right|^2 + |D\ell|^p. \tag{3.3} \]

Keeping in mind that \( \omega \leq 1 \) this leads us to

\[ I_5 \leq c \int_{B_s} \left| V_{|D\ell|} \left( \frac{u - \ell}{|u - \ell|} \right) \right|^p \, dx + c \int_{B_s} \omega(|u - \ell(0)| + |u - \ell|)(|D\ell| + |Du|)^p \, dx, \]
for a constant \( c = c(p, L) \). Similarly, we use the VMO-condition (1.4) on \( f \) and (3.3) to estimate \( I_6 \) as follows:

\[
I_6 \leq \int_{B_s} v_0(\cdot, s)|D\ell + D\psi|^p \, dx
\]

\[
\leq c \int_{B_s} \left| V_{|D\ell|} \left( \frac{u - \ell}{t - s} \right) \right|^p \, dx + c \int_{B_s} v_0(\cdot, s)(|D\ell| + |Du|)^p \, dx,
\]

for a constant \( c = c(p) \). It remains to consider the terms \( I_1 \) and \( I_7 \) which can be combined to

\[
I_7 + I_1 = \int \int \int_{B_s \times B_s \times 0} \left[ D_\xi f(y, \ell(0), D\ell + \tau D\psi(x)) - D_\xi f(y, \ell(0), D\ell), D\psi(x) \right] d\tau dy \, dx
\]

\[
+ \int \int \int_{B_s \times B_s \times 0} \left[ D_\xi f(y, \ell(0), D\ell) - D_\xi f(y, \ell(0), Du(x) - (1 - \tau) D\psi(x)), D\psi(x) \right] d\tau dy \, dx
\]

\[
=: I'_7 + I'_1,
\]

with the obvious meaning of \( I'_7 \) and \( I'_1 \). For the estimate of \( I'_7 \) we use (2.9), (2.8) and the fact that \( D\psi \equiv 0 \) on \( B_r \) to infer

\[
I'_7 \leq c(p)L \int_{B_s} \int_0^1 \left( |D\ell|^2 + \tau^2 |D\psi|^2 \right)^{\frac{p-2}{2}} |\tau| |D\psi|^2 \, d\tau \, dx
\]

\[
\leq c(p)L \int_{B_s \setminus B_r} \left| V_{|D\ell|}(D\psi) \right|^2 \, dx.
\]

To estimate \( I'_1 \) we proceed in a similar way. We first recall that \( Du - (1 - \tau) D\psi = D\ell + D\varphi + \tau D\psi \). Then, we use (2.9) and the fact that \( D\psi \equiv 0 \) on \( B_r \). Finally, we apply Young’s inequality (2.4) for \( V_{|D\ell|} \), the triangle inequality from (2.5) and the fact that \( \tau \mapsto |V_{|D\ell|}(\tau B)| \) is increasing. This leads us to

\[
I'_1 \leq c(p)L \int \int_0^1 \left( |D\ell|^2 + |D\varphi + \tau D\psi|^2 \right)^{\frac{p-2}{2}} |D\varphi + \tau D\psi| \, d\tau \, |D\psi| \, dx
\]

\[
\leq c(p)L \int \int_0^1 \left| V_{|D\ell|}(D\varphi) \right|^2 + \left| V_{|D\ell|}(D\psi + \tau D\psi) \right|^2 \, d\tau \, dx
\]

\[
\leq c(p)L \int \left| V_{|D\ell|}(D\varphi) \right|^2 + \left| V_{|D\ell|}(D\psi) \right|^2 \, dx.
\]
At this point we observe from (2.4) and (2.6) the following estimate:

\[
\left| V_{D\ell}(D\varphi) \right| \leq c(p) \left[ \left| V_{D\ell}(\eta(Du - D\ell)) \right| + \left| V_{D\ell}(\nabla\eta \otimes (u - \ell)) \right| \right]
\]

\[
\leq c(p) \left[ \left| V_{D\ell}(Du - D\ell) \right| + \left| V_{D\ell}\left( \frac{u - \ell}{t - r} \right) \right| \right],
\]

and similarly:

\[
\left| V_{D\ell}(D\psi) \right| \leq c(p) \left[ \left| V_{D\ell}(Du - D\ell) \right| + \left| V_{D\ell}\left( \frac{u - \ell}{t - r} \right) \right| \right],
\]

so that

\[
I_1 + I_7 = I'_1 + I'_7 \leq c \int_{B_s \setminus B_r} \left| V_{D\ell}(Du - D\ell) \right|^2 \, dx + c \int_{B_s} \left| V_{D\ell}\left( \frac{u - \ell}{t - r} \right) \right|^2 \, dx,
\]

where \( c = c(p, L) \). Finally, we observe that since \( \varphi = u - \ell \) on \( B_r \), the left-hand side of (3.2) can be bounded from below by

\[
v \int_{B_s} (|D\ell| + |D\varphi|)^{p-2} |D\varphi|^2 \, dx \geq v \int_{B_r} (|D\ell| + |Du - D\ell|)^{p-2} |Du - D\ell|^2 \, dx
\]

\[
\geq \frac{v}{2} \int_{B_r} \left| V_{D\ell}(Du - D\ell) \right|^2 \, dx.
\]

Joining the preceding estimate and the bounds for \( I_1-I_7 \) with (3.2) and keeping in mind that \( I_4 \leq 0 \) and \( s \in \left[ \frac{\rho}{2}, \rho \right] \), we infer

\[
\int_{B_r} \left| V_{D\ell}(Du - D\ell) \right|^2 \, dx
\]

\[
\leq \hat{c} \int_{B_s \setminus B_r} \left| V_{D\ell}(Du - D\ell) \right|^2 \, dx + \hat{c} \int_{B_s} \left| V_{D\ell}\left( \frac{u - \ell}{t - r} \right) \right|^2 \, dx
\]

\[
+ \hat{c} \int_{B_s} \left( \omega(|u - \ell(0)| + |u - \ell|) + \nu_0(., s) (|D\ell| + |Du|)^p \right) \, dx,
\]

with a constant \( \hat{c} > 0 \) depending only on \( p, v, L \). Now we add \( \hat{c} \) times the left-hand side to both sides of the estimate and divide by \( 1 + \hat{c} \). This yields

\[
\int_{B_r} \left| V_{D\ell}(Du - D\ell) \right|^2 \, dx
\]

\[
\leq \vartheta \int_{B_s} \left| V_{D\ell}(Du - D\ell) \right|^2 \, dx + \int_{B_s} \left| V_{D\ell}\left( \frac{u - \ell}{t - r} \right) \right|^2 \, dx
\]
\[ + \int_{B_s} (\omega(|u - \ell(0)| + |u - \ell|) + v_0(\cdot, s))(|D\ell| + |Du|^p) \, dx \]
\[ =: II_1 + II_2 + II_3, \quad (3.4) \]

where \( \vartheta := \frac{c}{1 + c} \in (0, 1) \). Our next aim is to bound the term \( II_3 \) further. This is achieved with the help of Hölder's inequality, the bounds \( \omega \leq 1 \) and \( v_0 \leq 2L \), the concavity of \( \omega \), Jensen’s inequality and the higher integrability estimate from Lemma 3.2 as follows:

\[ II_3 \leq c |B_s| \left( \int_{B_0} \omega\left(|u - \ell(0)| + |u - \ell|\right)^{\frac{q}{q-p}} \, dx + \int_{B_s} v_0(\cdot, s)^{\frac{q}{q-p}} \, dx \right)^{\frac{q-p}{q}} \]
\[ \cdot \left( \int_{B_s} |D\ell|^q + |Du|^q \, dx \right)^{\frac{p}{q}} \]
\[ \leq c \left( \frac{t}{t-r} \right)^{\beta n^p} \left[ \omega \left( \int_{B_0} |u - \ell(0)| + |u - \ell| \, dx \right)^{\frac{q-p}{q}} + V(s)^{\frac{q-p}{q}} \right] \]
\[ \cdot \int_{B_t} |D\ell|^p + |Du|^p \, dx \]
\[ \leq \frac{cQ^\beta}{(t-r)^\beta} \cdot B, \]

where \( \beta := n \frac{q-p}{p} \) and \( c = c(n, N, p, v, L) \) and

\[ B := \left[ \omega \left( \int_{B_0} |u - \ell(0)| + |u - \ell| \, dx \right)^{\frac{q-p}{q}} + V(Q)^{\frac{q-p}{q}} \right] \cdot \int_{B_{3Q/4}} |D\ell|^p + |Du|^p \, dx. \]

Here we enlarged in the first and in the last step the domain of integration from \( B_s \) to \( B_0 \), respectively from \( B_t \) to \( B_{3Q/4} \). Combining the preceding estimate with (3.4), we arrive at

\[ \int_{B_r} \left| V_{|D\ell|}(Du - D\ell) \right|^2 \, dx \]
\[ \leq \vartheta \int_{B_t} \left| V_{|D\ell|}(Du - D\ell) \right|^2 \, dx + c \int_{B_0} \left| V_{|D\ell|} \left( \frac{u - \ell}{t-r} \right) \right|^2 \, dx + \frac{cQ^\beta}{(t-r)^\beta} \cdot B, \]

and this estimate holds for arbitrary radii \( r, t \) with \( \frac{Q}{2} \leq r < t \leq 3Q/4 \). We note that the constant \( c \) depends only on \( n, N, p, v, L \) and \( \vartheta < 1 \). Therefore, we may apply Lemma 2.2 to infer

\[ \int_{B_{Q/2}} \left| V_{|D\ell|}(Du - D\ell) \right|^2 \, dx \leq c \int_{B_0} \left| V_{|D\ell|} \left( \frac{u - \ell}{Q} \right) \right|^2 \, dx + cB. \]
In order to further estimate the term involving $|Du|^p$ contained in $B$ we apply the zero order Caccioppoli inequality from Lemma 3.1 and (3.3) with $\varrho$ instead of $t-s$ yielding that
\[
\int_{B_{3/4}} |Du|^p \, dx \leq c \int_{B_0} \left| \frac{u - \ell(0)}{\varrho} \right|^p \, dx
\]
\[
\leq c \left[ \int_{B_0} \left| \frac{u - \ell}{Q} \right|^p \, dx + |D\ell|^p \right]
\]
\[
\leq c(p, L/\nu) \int_{B_0} \left| V_{|D\ell|} \left( \frac{u - \ell}{Q} \right) \right|^2 \, dx + |D\ell|^p.
\]
Joining this with the second last inequality and using $\omega \leq 1$ as well as $V(R) \leq 2L$ we finally arrive at
\[
\int_{B_{0/2}} \left| V_{|D\ell|}(Du - D\ell) \right|^2 \, dx
\]
\[
\leq c \int_{B_0} \left| V_{|D\ell|} \left( \frac{u - \ell}{Q} \right) \right|^2 \, dx
\]
\[
+ cQ^n |D\ell|^p \left[ \omega \left( \int_{B_0} \left| u - \ell(0) \right| + \left| u - \ell \right| \, dx \right)^{\frac{q-p}{q}} + V(\varrho)^{\frac{q-p}{q}} \right].
\]
where $c = c(n, N, p, v, L)$. Taking means on both sides of the preceding inequality we deduce the desired Caccioppoli inequality.

Later on, the Caccioppoli inequality will be applied with the choice $\ell \equiv \ell_{x_0, \varrho}$, where $\ell_{x_0, \varrho} : \mathbb{R}^n \to \mathbb{R}^N$ is the affine function defined in (2.14). This motivates the definition of the following excess functionals. For $x_0 \in \Omega$, $\varrho \in (0, \text{dist}(x_0, \partial\Omega))$ we set
\[
\Phi(x_0, \varrho) \equiv \Phi(x_0, \varrho, \ell_{x_0, \varrho}) := \int_{B_\varrho(x_0)} \left| V_{|D\ell_{x_0, \varrho}|} \left( \frac{u - \ell_{x_0, \varrho}}{\varrho} \right) \right|^2 \, dx
\]
(3.5)
and
\[
\Psi_\alpha(x_0, \varrho) := \varrho^{-\alpha p} \int_{B_\varrho(x_0)} \left| u - (u)_{x_0, \varrho} \right|^p \, dx, \quad \text{for } \alpha \in [0, 1].
\]
(3.6)
Moreover, we define the following hybrid excess functional:
\[
\Phi_*(x_0, \varrho) := \Phi(x_0, \varrho) + |D\ell_{x_0, \varrho}|^p H(x_0, \varrho)^{\min \left\{ 1 - \frac{1}{p}, \frac{1}{q} \right\}},
\]
(3.7)
where
\[
H(x_0, \varrho) := \omega \left( \Psi_0(x_0, \varrho)^{\frac{1}{q}} \right)^{\frac{q-p}{q}} + V(\varrho)^{\frac{q-p}{q}}.
\]
(3.8)
In the case $x_0 = 0$ we omit the reference point in the notation of the various excess functionals, writing for instance $\Phi(\varrho) \equiv \Phi(0, \varrho)$. The reason for using the exponent $\min \{ 1 - \frac{1}{p}, \frac{1}{p} \}$ of $H$ in the definition of $\Phi_\varrho$, which does not appear in the Caccioppoli inequality, will become clear during the linearization procedure in Lemma 3.9 (see estimate (3.21)).

With the preceding definitions we can re-formulate the Caccioppoli 3.1 as follows.

**Corollary 3.6.** Under the assumptions of Lemma 3.5, there exists a constant $c = c(n, N, p, \nu, L) \geq 1$ such that there holds

$$
\int_{B_{\varrho/2}(x_0)} \left| V |D \ell_{x_0, \varrho}|(Du - D \ell_{x_0, \varrho}) \right|^2 \, dx \leq c \Phi_\varrho(x_0, \varrho). 
$$

(3.9)

**Proof.** Recalling from (2.15) that $\ell_{x_0, \varrho}(x_0) = (u)_{x_0, \varrho}$ and using (2.17) with the choices $A = 0$ and $\xi = (u)_{x_0, \varrho}$ and Hölder’s inequality we obtain

$$
\int_{B_{\varrho}(x_0)} |u - \ell_{x_0, \varrho}| \, dx \leq \int_{B_{\varrho}(x_0)} |u - (u)_{x_0, \varrho}| \, dx + \varrho |D \ell_{x_0, \varrho}| \leq (n + 3) \Psi_\varrho(x_0, \varrho) \frac{1}{\varrho^p}.
$$

Joining this with the Caccioppoli inequality from Lemma 3.5 and using the sub-linearity of the concave function $\omega$ we get

$$
\int_{B_{\varrho/2}(x_0)} \left| V |D \ell_{x_0, \varrho}|(Du - D \ell_{x_0, \varrho}) \right|^2 \, dx \leq c \Phi(x_0, \varrho) + c |D \ell_{x_0, \varrho}|^p H(x_0, \varrho).
$$

Since $\omega \leq 1$ and $V \leq 2L$ we have $H(x_0, \varrho) \leq 1 + (2L)^{\frac{q-p}{p}}$ and hence

$$
H(x_0, \varrho) \leq \left( 1 + (2L)^{\frac{q-p}{p}} \right)^{\max \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\}} H(x_0, \varrho)^{\min \left\{ 1 - \frac{1}{p}, \frac{1}{p} \right\}}.
$$

Inserting this above and recalling the definition of $\Phi_\varrho$, this proves the claim. □

### 3.2. Approximate $A$-harmonicity and $p$-harmonicity

The aim of this section is to provide two different linearization strategies for the minimization problem. We will show that on the one hand the minimizer is an almost $A$-harmonic function (see Lemma 3.9), and on the other hand it is an almost $p$-harmonic function (see Lemma 3.11). Later on, these results will be the starting point for the application of the $A$-harmonic approximation lemma, respectively the $p$-harmonic approximation lemma.

In the course of the proofs of these two lemmas we will need suitable comparison functions. These will be constructed with the help of Ekeland’s variational principle in the following lemma.

**Lemma 3.7.** Let $p > 1$ and assume that $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a minimizer of the functional (1.1), under the assumptions (1.2)–(1.6). Furthermore, let $B_{\varrho}(x_0) \subseteq \Omega$ with $\varrho \leq \varrho_0$ and abbreviate

$$
g(\xi) \equiv g_{x_0, \varrho}(\xi) := (f(\cdot, (u)_{x_0, \varrho}, \xi))_{x_0, \varrho} \quad \text{for all } \xi \in \mathbb{R}^{Nn}.
$$

(3.10)

Then there exists a minimizer $v \in u + W^{1,p}_0(B_{\varrho/2}(x_0), \mathbb{R}^N)$ of the functional
that satisfies
\[
\int_{B_{\rho/2}(x_0)} |Dv - Du|^p \, dx \leq c_p^* K(x_0, \varrho), \tag{3.11}
\]
with a constant \(c^* = c^*(n, N, p, \nu, L)\). Here, we abbreviated
\[
K(x_0, \varrho) := H(x_0, \varrho) \Psi_1(x_0, \varrho), \tag{3.12}
\]
where \(\Psi_\alpha(x_0, \varrho)\), for \(\alpha \in [0, 1]\), is defined in (3.6) and \(H(x_0, \varrho)\) in (3.8).

Remark 3.8. Later on we shall make use of the minimality property of \(v\), i.e.
\[
\tilde{G}[v] \leq \tilde{G}[v + \varphi] \text{ for all } \varphi \in C_0^\infty(B_{\rho/2}, \mathbb{R}^N) \tag{3.13}
\]
in the following way. Taking the derivative with respect to \(t\) at 0 from above and below in:
\[
\tilde{G}[v + t\varphi] = \int_{B_{\rho/2}} g(Dv + tD\varphi) \, dx + t|K(\varrho)|^{1-\frac{1}{p}} \left( \int_{B_{\rho/2}} |D\varphi|^p \, dx \right)^{\frac{1}{p}},
\]
the minimality property (3.13) ensures that \(v\) also satisfies the following associated Euler–Lagrange variational inequality:
\[
\left| \int_{B_{\rho/2}} (Dg(Dv), D\varphi) \, dx \right| \leq K(\varrho)^{1-\frac{1}{p}} \left( \int_{B_{\rho/2}} |D\varphi|^p \, dx \right)^{\frac{1}{p}}, \tag{3.14}
\]
for all \(\varphi \in C_0^\infty(B_{\rho/2}, \mathbb{R}^N)\).

Proof of Lemma 3.7. For simplicity we assume that \(x_0 = 0\). First, we recall that from the Caccioppoli inequality of Lemma 3.1 we have
\[
\int_{B_{3\rho/4}} |Du|^p \, dx \leq c \int_{B_\rho} \left| \frac{u - (u)_\varrho}{\varrho} \right|^p \, dx = c(p, L/\nu) \Psi_1(\varrho). \tag{3.15}
\]
By \(\tilde{v} \in u + W_0^{1,p}(B_{\rho/2}, \mathbb{R}^N)\) we denote a minimizer of the functional
\[
G[v] := \int_{B_{\rho/2}} g(Dv) \, dx
\]
in the Dirichlet class \(u + W_0^{1,p}(B_{\rho/2}, \mathbb{R}^N)\), which can be constructed by the direct method, using the assumptions (1.2) and (1.3). Using the minimality of \(\tilde{v}\), the growth condition (1.2)_1 and (2.12) we obtain
\[
\int_{B_{\rho/2}} |D\tilde{v}|^p \, dx \leq \frac{1}{\nu} \int_{B_{\rho/2}} g(D\tilde{v}) - g(0) \, dx \\
\leq \frac{1}{\nu} \int_{B_{\rho/2}} g(Du) - g(0) \, dx \leq \frac{c(p)L}{\nu} \int_{B_{\rho/2}} |Du|^p \, dx.
\] (3.16)

Together with Poincaré's inequality and (3.15) this yields
\[
\int_{B_{\rho/2}} \frac{1}{\rho/2} \left| \tilde{v} - (u)_\rho \right|^p \, dx \leq 2^{p-1} \int_{B_{\rho/2}} \left| \tilde{v} - u \right|^p + \left| \tilde{v} - (u)_\rho \right|^p \, dx \\
\leq c_0^p \int_{B_{\rho/2}} |D\tilde{v}|^p + |Du|^p \, dx \leq c_0^p \Psi_1(\rho) = c_0(\rho).
\] (3.17)

for a constant \( c = c(n, p, \nu, L) \geq 1 \). Moreover, the application of Lemmas 3.2 and 3.3 and the estimates (3.16) and (3.15) yield the following higher integrability result
\[
\left( \int_{B_{\rho/2}} |D\tilde{v}|^q \, dx \right)^{\frac{p}{q}} \leq c \int_{B_{3\rho/4}} |Du|^p \, dx \leq c \Psi_1(\rho),
\] (3.18)

where \( c = c(n, N, p, \nu, L) \) and \( q = q(n, N, p, \nu, L) > p \) denotes the exponent determined in Remark 3.4.

Now, we want to show that \( u \) is in some sense an almost minimizer of the functional \( G \). To this aim we use the minimality of \( u \) and the assumptions (1.6) and (1.4) to infer
\[
\int_{B_{\rho/2}} f(x, u, Du) \, dx - G[\tilde{v}] \leq \int_{B_{\rho/2}} f(x, \tilde{v}, D\tilde{v}) \, dx - G[\tilde{v}] \\
= \int_{B_{\rho/2}} f(x, \tilde{v}, D\tilde{v}) - (f(\cdot, \tilde{v}, D\tilde{v}))_\rho \, dx \\
+ \int_{B_{\rho/2}} \left[ (f(\cdot, \tilde{v}, D\tilde{v}))_\rho - (f(\cdot, (u)_\rho, D\tilde{v}))_\rho \right] \, dx \\
\leq c(L) \int_{B_{\rho/2}} \left[ \nu_0(\cdot, \rho) + \omega(\left| \tilde{v} - (u)_\rho \right|) \right] |D\tilde{v}|^p \, dx.
\]

Using Hölder’s inequality, the bounds \( \omega \leq 1 \) and \( \nu_0 \leq 2L \), the concavity and sub-linearity of \( \omega \), and finally (3.17), (3.18) and Hölder’s inequality we obtain from the preceding estimate
\[
\int_{B_{\rho/2}} f(x, u, Du) \, dx - G[\tilde{v}] \\
\leq c \left[ \omega \left( \int_{B_{\rho/2}} \left| \tilde{v} - (u)_\rho \right| \, dx \right)^{\frac{q-p}{q}} + V(\rho)^{\frac{q-p}{q}} \right] \left( \int_{B_{\rho/2}} |D\tilde{v}|^q \, dx \right)^{\frac{p}{q}} \\
\leq c \left[ \omega(\Psi_0(\rho))^{\frac{q-p}{q}} + V(\rho)^{\frac{q-p}{q}} \right] \Psi_1(\rho) = cK(\rho).
\]
for a constant $c = c(n, N, p, \nu, L, q)$. Similarly, we can estimate

$$G[u] - \int_{B_{\rho/2}} f(x, u, Du) \, dx$$

$$\leq c \left[ \omega \left( \int_{B_{\rho/2}} |u - (u)_{\rho}| \, dx \right)^{\frac{q-p}{q}} + \mathbf{V}(\rho)^{\frac{q-p}{q}} \right] \left( \int_{B_{\rho/2}} |Du|^q \, dx \right)^{\frac{p}{q}}$$

$$\leq c \left[ \omega \left( \Psi_0(\rho)^{\frac{1}{p}} \right)^{\frac{q-p}{q}} + \mathbf{V}(\rho)^{\frac{q-p}{q}} \right] \int_{B_{\rho/2}} |Du|^p \, dx$$

$$\leq c \left[ \omega \left( \Psi_0(\rho)^{\frac{1}{p}} \right)^{\frac{q-p}{q}} + \mathbf{V}(\rho)^{\frac{q-p}{q}} \right] \Psi_1(\rho) = cK(\rho),$$

where $c$ depends only upon $n, N, p, \nu, L$ and $q$. Adding the last two estimates and recalling the minimality of $\tilde{v}$, we conclude

$$G[u] \leq G[\tilde{v}] + c_s K(\rho) = \min_{u + W^{1,p}_0(B_{\rho/2}, \mathbb{R}^N)} G + c_s K(\rho),$$

with some constant $c_s = c_s(n, N, p, \nu, L, q)$. Due to the dependency of $q$ upon the structural parameters $n, N, p, \nu$ and $L$, this amounts to the dependencies of $c_s$ on $n, N, p, \nu$ and $L$. Now we define the metric

$$d(v_1, v_2) := \frac{1}{c_s} \left( \frac{1}{K(\rho)} \int_{B_{\rho/2}} |Dv_1 - Dv_2|^p \, dx \right)^{\frac{1}{p}}$$

for $v_1, v_2 \in X := u + W^{1,p}_0(B_{\rho/2}, \mathbb{R}^N)$. The application of Ekeland’s variational principle stated in Lemma 2.14, with the choice $\kappa = c_s K(\rho)$ yields the existence of a function $v \in u + W^{1,p}_0(B_{\rho/2}, \mathbb{R}^N)$ with the desired properties. \qed

We are now in the position to prove the approximate $A$-harmonicity of a minimizer to (1.1). Later on, this will be the starting point for the application of the $A$-harmonic approximation lemma.

**Lemma 3.9.** For $p > 1$ there exists a constant $c = c(n, N, p, \nu, L) \geq 1$ such that the following holds: Assume that $u \in W^{1,p}_0(\Omega, \mathbb{R}^N)$ is a minimizer of the functional (1.1), under the assumptions (1.2)–(1.7) and that for a ball $B_\rho(x_0) \subseteq \Omega$ the smallness assumptions

$$\Phi(x_0, \rho) \leq |D(\ell_{x_0, \rho})|^p \quad \text{and} \quad \rho \leq \rho_0$$

are satisfied. Then, $u$ is approximately $A$-harmonic on the ball $B_{\rho/2}(x_0)$ in the sense that
\[
\left| \int_{B_{\rho/2}(x_0)} A(Du - D\ell_{x_0, \varphi}, D\varphi) \, dx \right| \\
\leq c |D\ell_{x_0, \varphi}| \left[ \Phi_* (x_0, \varphi) \right]^\frac{1}{p} \left[ \sup_{B_{\rho/2}(x_0)} |D\varphi| \right]^\frac{1}{p-2} |D\ell_{x_0, \varphi}|^{p-2}
\]

holds for all \( \varphi \in C^\infty_0 (B_{\rho/2}(x_0), \mathbb{R}^N) \). Here we used the short-hand notation

\[
A := \frac{D^2 g(D\ell_{x_0, \varphi})}{|D\ell_{x_0, \varphi}|^{p-2}} = \frac{(D^2 f(\cdot, (u)_{x_0, \varphi}, D\ell_{x_0, \varphi}))_{x_0, \varphi}}{|D\ell_{x_0, \varphi}|^{p-2}}.
\]

**Remark 3.10.** We point out that by (1.2)_2 and (1.3), the bilinear form \( A \) on \( \mathbb{R}^{Nn} \) satisfies the ellipticity and boundedness conditions (2.20). Note that the above ellipticity condition (2.20)_1 in the sense of Legendre–Hadamard holds because the quasi-convexity condition (1.3) implies rank-one-convexity, cf. [20, Proposition 5.2]. Thus, \( A \) satisfies the assumptions of the \( \tilde{A} \)-harmonic approximation Lemma 3.9.

**Proof of Lemma 3.9.** For convenience in notation, we assume once again \( x_0 = 0 \). Moreover, we assume without loss of generality that \( |D\varphi| \leq 1 \) on \( B_{\rho/2} \). We define \( g, K \) and \( c_* = c_*(n, N, p, v, L) \) according to Lemma 3.7. The application of the lemma together with Remark 3.8 yields a function \( v \in u + W^{1,p}_0(B_{\rho/2}, \mathbb{R}^N) \) satisfying (3.11) and (3.14).

Next, we want to infer a bound for the term \( \Psi_1(\ell) \) appearing in the definition of \( K(\ell) \) in (3.12) in terms of \( |D\ell|^p \). For this aim we first recall that \( \ell_0(0) = (u)_{\ell} \) to infer

\[
\Psi_1(\ell) = \int_{B_{\rho}} \left| \frac{u - \ell_0(0)}{\varphi} \right|^p \, dx \leq 2^{p-1} \left[ \int_{B_{\rho}} \left| \frac{u - \ell_0}{\varphi} \right|^p \, dx + |D\ell_0|^p \right].
\]

Using (2.7) and the assumption (3.19) we obtain for the first term on the right-hand side of the preceding inequality (recall the definition of \( \chi_{p<2} \) in (2.2))

\[
\int_{B_{\rho}} \left| \frac{u - \ell_0}{\varphi} \right|^p \, dx \leq c \int_{B_{\rho}} V_{|D\ell_0|^{1/2}} \left| \frac{u - \ell_0}{\varphi} \right|^p \, dx + \chi_{p<2}|D\ell_0|^\frac{p(p-1)}{2} \left[ \frac{p(p-1)}{2} \Phi(\ell, \varphi) \right] \leq c(p)|D\ell_0|^p.
\]

Joining this with the second last inequality we obtain

\[
\Psi_1(\ell) \leq c(p)|D\ell_0|^p. \quad (3.20)
\]

Using (3.20) and recalling the definitions of \( K(\ell) \) and \( \Phi_* (\ell) \) in (3.12) and (3.7) we infer for \( \theta \geq \min \{1 - \frac{1}{p}, 1 \} \) that

\[
K(\ell)^\theta \leq c(p)|D\ell_0|^p \left[ \omega(\Psi)(^\frac{1}{p} \frac{q-p}{q} + V(\varphi) \frac{q-p}{q}) \right]^\theta \\
= c(p)|D\ell_0|^p |D\ell_0|^{p(\theta-1)} \left[ \omega(\Psi)(^\frac{1}{p} \frac{q-p}{q} + V(\varphi) \frac{q-p}{q}) \right]^\theta \\
\leq c(p)|D\ell_0|^p |D\ell_0|^{p(\theta-1)} \Phi_* (\ell). \quad (3.21)
\]
Now, we fix an arbitrary function \( \varphi \in C^\infty_0(\frac{B_\rho}{2}, \mathbb{R}^N) \) with \( |D\varphi| \leq 1 \). Keeping in mind the definition of \( A \) and
\[
\int_0^1 D^2 g(D\ell_\varrho + s(Dv - D\ell_\varrho))(Dv - D\ell_\varrho)\, ds = Dg(Dv) - Dg(D\ell_\varrho)
\]
as well as
\[
\int_{B_{\rho/2}} [Dg(D\ell_\varrho), D\varphi] \, dx = 0
\]
we now re-write
\[
\int_{B_{\rho/2}} A(Du - D\ell_\varrho, D\varphi) \, dx = \int_{B_{\rho/2}} A(Du - Dv, D\varphi) \, dx
\]
\[
+ \frac{1}{|D\ell_\varrho|^{p-2}} \int_{B_{\rho/2}} \int_0^1 \left[ D^2 g(D\ell_\varrho) - D^2 g(D\ell_\varrho + s(Dv - D\ell_\varrho)) \right](Dv - D\ell_\varrho, D\varphi) \, ds \, dx
\]
\[
+ \frac{1}{|D\ell_\varrho|^{p-2}} \int_{B_{\rho/2}} [Dg(Dv), D\varphi] \, dx =: I + II + III,
\]
with the obvious labeling of \( I \text{--} III \). For the estimate of the first term we use \( |D\varphi| \leq 1, (2.20), \) Hölder's inequality, (3.11) and (3.21) with the choice \( \theta = \frac{1}{p} \) to infer that
\[
|I| \leq L \int_{B_{\rho/2}} |Du - Dv| \, dx \leq c K(\varrho)^{\frac{1}{p}} \leq c(n, N, p, \nu, L)|D\ell_\varrho| \frac{\Phi_*(\varrho)}{|D\ell_\varrho|^p}.
\]
To estimate the second term we use \( |D\varphi| \leq 1 \), the definition of \( g \) in (3.10) and the continuity assumption (1.7) for \( D^2 f \) to infer that
\[
|II| \leq \frac{L}{|D\ell_\varrho|^{p-2}} \int_{B_{\rho/2}} \tilde{\omega}\left( \frac{|Dv - D\ell_\varrho|}{|D\ell_\varrho|} \right) \chi(\cdot)|Dv - D\ell_\varrho| \, dx,
\]
where
\[
\chi(x) := \begin{cases}
\int_0^1 (|D\ell_\varrho| + |D\ell_\varrho + s(Dv(x) - D\ell_\varrho)|)^{p-2} \, ds & \text{if } p \geq 2, \\
\int_0^1 \left( \frac{|D\ell_\varrho| + |D\ell_\varrho + s(Dv(x) - D\ell_\varrho)|}{|D\ell_\varrho + s(Dv(x) - D\ell_\varrho)|} \right)^{2-p} \, ds & \text{if } p < 2.
\end{cases}
\]
In the case $p \geq 2$ we immediately have $X \leq (2|D\ell_\ell| + |DV - D\ell_\ell|)^{p-2}$, while in the case $p < 2$ we obtain from Lemma 2.4 that $X \leq c(p)|D\ell_\ell|^{p-2}$. Inserting this above yields

$$
|II| \leq \frac{c(p)L}{|D\ell_\ell|^{p-2}} \int_{B_{\ell/2}} \tilde{\omega} \left( \frac{|DV - D\ell_\ell|}{|D\ell_\ell|} \right) \left( |D\ell_\ell| + \chi_{p>2}|DV - D\ell_\ell| \right)^{p-2} |DV - D\ell_\ell| \, dx.
$$

We now infer pointwise estimates for the integrand distinguishing those points in $x \in B_{\ell/2}$ where $|DV(x) - D\ell_\ell|$ is smaller, respectively larger than $|D\ell_\ell|$. In the case $|DV - D\ell_\ell| \leq |D\ell_\ell|$ we have $|DV - D\ell_\ell| \leq c(p)|D\ell_\ell|^{\frac{2-p}{2}} |V|_{D\ell_\ell}(DV - D\ell_\ell)|$ by (2.7), and hence

$$
\tilde{\omega} \left( \frac{|DV - D\ell_\ell|}{|D\ell_\ell|} \right) \left( |D\ell_\ell| + \chi_{p>2}|DV - D\ell_\ell| \right)^{p-2} |DV - D\ell_\ell|
\leq c(p)\tilde{\omega} \left( \frac{|DV - D\ell_\ell|}{|D\ell_\ell|} \right) |D\ell_\ell|^{p-2} |DV - D\ell_\ell|
\leq c(p)|D\ell_\ell|^{p-1} \tilde{\omega} \left( \frac{|V|_{D\ell_\ell}(DV - D\ell_\ell)|}{|D\ell_\ell|^p} \right) \frac{|V|_{D\ell_\ell}(DV - D\ell_\ell)|}{|D\ell_\ell|^p},
$$

where we also used the sub-linearity of $\tilde{\omega}$. On the other hand, in the case $|DV - D\ell_\ell| > |D\ell_\ell|$ we have $|DV - D\ell_\ell| \leq |V|_{D\ell_\ell}(DV - D\ell_\ell)^2$. Using also that $\tilde{\omega} \leq 1$ we find

$$
\tilde{\omega} \left( \frac{|DV - D\ell_\ell|}{|D\ell_\ell|} \right) \left( |D\ell_\ell| + \chi_{p>2}|DV - D\ell_\ell| \right)^{p-2} |DV - D\ell_\ell|
\leq c(p)\left[ |D\ell_\ell|^{p-2} |DV - D\ell_\ell| + \chi_{p>2}|DV - D\ell_\ell|^{p-1} \right]
\leq c(p)|D\ell_\ell|^{p-1} \frac{|DV - D\ell_\ell|^p}{|D\ell_\ell|^p} \leq c(p)|D\ell_\ell|^{p-1} \frac{|V|_{D\ell_\ell}(DV - D\ell_\ell)|^2}{|D\ell_\ell|^p}.
$$

Combining both cases we arrive at

$$
|II| \leq c(p)L|D\ell_\ell| \left[ \int_{B_{\ell/2}} \tilde{\omega} \left( \frac{|V|_{D\ell_\ell}(DV - D\ell_\ell)|}{|D\ell_\ell|^p} \right) \frac{|V|_{D\ell_\ell}(DV - D\ell_\ell)|}{|D\ell_\ell|^p} \, dx \right]
+ \int_{B_{\ell/2}} \frac{|V|_{D\ell_\ell}(DV - D\ell_\ell)|^2}{|D\ell_\ell|^p} \, dx.
$$

With the help of Hölder's and Jensen's inequalities (recall that $\tilde{\omega} \leq 1$ is concave) we obtain

$$
|II| \leq c|D\ell_\ell| \left( \int_{B_{\ell/2}} \tilde{\omega}^2 \left( \frac{|V|_{D\ell_\ell}(DV - D\ell_\ell)|}{|D\ell_\ell|^\frac{p}{2}} \right) \, dx \right)^\frac{1}{2} \left( \int_{B_{\ell/2}} \frac{|V|_{D\ell_\ell}(DV - D\ell_\ell)|^2}{|D\ell_\ell|^p} \, dx \right)^\frac{1}{2}
+ c|D\ell_\ell| \int_{B_{\ell/2}} \frac{|V|_{D\ell_\ell}(DV - D\ell_\ell)|^2}{|D\ell_\ell|^p} \, dx.
$$
\[
\begin{align*}
&\leq c|D\ell_\varrho|\tilde{\omega}\left(\int_{B_{\varrho/2}} \frac{|V|\cdot|D\ell_\varrho|}{|D\ell_\varrho|^p} \, dx\right)^{1/2} \left(\int_{B_{\varrho/2}} \frac{|V|\cdot|D\ell_\varrho|^2}{|D\ell_\varrho|^p} \, dx\right)^{1/2} \\
&+ c|D\ell_\varrho|\int_{B_{\varrho/2}} \frac{|V|\cdot|D\ell_\varrho|^2}{|D\ell_\varrho|^p} \, dx,
\end{align*}
\]

where \( c = c(p, L) \). Now, we use (2.5), (2.7), the Caccioppoli inequality (3.9), (3.11) and (3.21) with the choice \( \theta = 1 \) and in the case \( p > 2 \) also with \( \theta = \frac{2}{p} \) to deduce

\[
\begin{align*}
&\int_{B_{\varrho/2}} |V|\cdot|D\ell_\varrho|\cdot|D\varphi| \, dx \\
&\leq c(p)\int_{B_{\varrho/2}} |V|\cdot|D\ell_\varrho|\cdot|D\varphi| \, dx + c(p)\int_{B_{\varrho/2}} |V|\cdot|Du - D\ell_\varrho| \, dx \\
&\leq c\int_{B_{\varrho/2}} |D\varphi| + \chi_{p>2}|D\ell_\varrho|^p - 2|D\varphi| \, dx + c\Phi_*(\varrho) \\
&\leq c\left[\chi_{p>2}|D\ell_\varrho|^p - 2(c_*\cdot K(\varrho))^\frac{2}{p}\right] + c\Phi_*(\varrho) \leq c\Phi_*(\varrho),
\end{align*}
\]

(3.23)

where \( c = c(n, p, v, L) \). Inserting this into the second last inequality and using also H"{o}lder's inequality we arrive at

\[
|I| \leq c(n, p, k, v, L)|D\ell_\varrho|\tilde{\omega}\left[\frac{\Phi_*(\varrho)}{|D\ell_\varrho|^p} \right]^{1/2} \left[\frac{\Phi_*(\varrho)}{|D\ell_\varrho|^p} + \frac{\Phi_*(\varrho)}{|D\ell_\varrho|^p} \right].
\]

For the third term in (3.22), we have from (3.14), \(|D\varphi| \leq 1\) and (3.21) applied with the choice \( \theta = 1 - \frac{1}{p} \) that

\[
|III| \leq \frac{K(\varrho)}{|D\ell_\varrho|^p - 2} \left(\int_{B_{\varrho/2}} |D\varphi|^p \, dx\right)^{\frac{1}{p}} \leq \frac{K(\varrho)}{|D\ell_\varrho|^p - 2} \leq c(p)|D\ell_\varrho|\Phi_*(\varrho).\]

Joining the estimates for I–III with (3.22) we conclude that

\[
\left|\int_{B_{\varrho/2}} A(D\varphi - D\ell_\varrho, D\varphi) \, dx\right| \leq c|D\ell_\varrho|\tilde{\omega}\left[\frac{\Phi_*(\varrho)}{|D\ell_\varrho|^p}\right]^{1/2} \left[\frac{\Phi_*(\varrho)}{|D\ell_\varrho|^p} + \frac{\Phi_*(\varrho)}{|D\ell_\varrho|^p}\right]^{1/2},
\]

for a constant \( c = c(n, p, v, L) \). This proves the assertion of the lemma. \( \Box \)

The following lemma ensures that any minimizer to (1.1) is in a certain sense approximately \( p \)-harmonic. Later on, this will be the starting point for the application of the \( p \)-harmonic approximation lemma.
Lemma 3.11. For $p > 1$ there exists a constant $c = c(n, N, p, \nu, \mu) \geq 1$ such that the following holds: Assume that $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a minimizer of the functional (1.1), under the assumptions (1.2)–(1.6) and (1.8). Then, $u$ is approximately $p$-harmonic on any ball $B_{\delta/2}(x_0)$ satisfying $B_\delta(x_0) \subseteq \Omega$ in the sense that for any $\delta > 0$ there holds

$$\left| \int_{B_{\delta/2}(x_0)} |Du|^{p-2} Du \cdot D\varphi \, dx \right| \leq c\Psi_1(x_0, \varphi)^{1-\frac{1}{p}} \left[ \delta + H(x_0, \varphi) \min\left\{ 1 - \frac{1}{p}, \frac{\psi_1(x_0, \varphi)^{\frac{1}{p}}}{\eta(\delta)} \right\} \right] \sup_{B_{\delta/2}(x_0)} |D\varphi|,$$

for all $\varphi \in C_0^\infty(B_{\delta/2}(x_0), \mathbb{R}^N)$.

Proof. Again we assume without loss of generality that $x_0 = 0$ and $|D\varphi| \leq 1$ in $B_{\delta/2}$. We define $g$, $K$ and $c_\ast = c_\ast(n, N, p, \nu, \mu)$ according to Lemma 3.7. The application of the lemma together with Remark 3.8 yields a function $v \in u + W^{1,p}_0(B_{\delta/2}, \mathbb{R}^N)$ satisfying (3.11) and (3.14). We now re-write the considered integral as follows:

$$\int_{B_{\delta/2}} |Du|^{p-2} Du \cdot D\varphi \, dx = \int_{B_{\delta/2}} \left[ |Du|^{p-2} Du - Dg(Du) \right] \cdot D\varphi \, dx$$

$$+ \int_{B_{\delta/2}} Dg(Du) - Dg(Dv) \cdot D\varphi \, dx$$

$$+ \int_{B_{\delta/2}} Dg(Dv) \cdot D\varphi \, dx =: I + II + III,$$  

with the obvious meaning of $I$–$III$. For the estimate of $I$ we first recall the definition of $g$ in (3.10) and $|D\varphi| \leq 1$ to see that

$$|I| \leq \int_{B_{\delta/2}} \left( |Du|^{p-2} Du - \left( Df(\cdot, (u)_\varphi, Du) \right)_\varphi \right) \, dx.$$  

We now decompose the domain of integration into the set where $|Du| \leq \eta(\delta)$ and its complement. Therefore, we define

$$S_1 := \left\{ x \in B_{\delta/2} : |Du(x)| \leq \eta(\delta) \right\} \quad \text{and} \quad S_2 := \left\{ x \in B_{\delta/2} : |Du(x)| > \eta(\delta) \right\}.$$

On $S_1$ we first use the assumption that $Df$ behaves like the $p$-Laplacian at the origin, in the form (2.13). Subsequently we use Hölder’s inequality and Caccioppoli’s inequality from Lemma 3.1 to deduce

$$\frac{1}{|B_{\delta/2}|} \int_{S_1} \left| |Du|^{p-2} Du - \left( Df(\cdot, (u)_\varphi, Du) \right)_\varphi \right| \, dx$$

$$\leq \delta \int_{B_{\delta/2}} |Du|^{p-1} \, dx \leq \delta \left( \int_{B_{\delta/2}} |Du|^p \, dx \right)^{1-\frac{1}{p}} \leq c\psi_1(\varphi)^{1-\frac{1}{p}},$$  

(3.26)
where \( c = c(p, L/\nu) \). From the definition of the set \( S_2 \) we have

\[
|S_2| \leq \frac{1}{\eta(\delta)^p} \int_{B_{\rho/2}} |Du|^p \, dx,
\]

which together with Hölder’s inequality and the Caccioppoli inequality from Lemma 3.1 leads us to

\[
\frac{1}{|B_{\rho/2}|} \int_{S_2} |Du|^{p-2} Du - (Df(\cdot, (u)_{\rho}, Du))_{\rho},\]

\[
\leq L + \frac{1}{|B_{\rho/2}|} \int_{S_2} |Du|^{p-1} \, dx \leq \frac{L + 1}{|B_{\rho/2}|} |S_2|^{\frac{1}{p}} \left( \int_{B_{\rho/2}} |Du|^p \, dx \right)^{1 - \frac{1}{p}}
\]

\[
\leq \frac{L + 1}{\eta(\delta)} \int_{B_{\rho/2}} |Du|^p \, dx \leq \frac{c}{\eta(\delta)} \Psi_1(q),
\]

(3.27)

where \( c = c(p, \nu, L) \). Inserting (3.26) and (3.27) into (3.25) we find

\[
|I| \leq c(p, \nu, L) \left[ \delta \Psi_1(q) \right]^{1 - \frac{1}{p}} + \frac{\Psi_1(q)}{\eta(\delta)}.
\]

We now start considering the second term in (3.24). Recalling the definition of \( g \) and \( |D\varphi| \leq 1 \) and using (2.9) we obtain

\[
|II| \leq c \int_{B_{\rho/2}} (|Du|^2 + |Dv - Du|^2)^{\frac{p-2}{2}} |Dv - Du| \, dx
\]

\[
\leq c(p, L) \int_{B_{\rho/2}} |Dv - Du|^{p-1} + \chi_{p>2}|Du|^{p-2}|Dv - Du| \, dx.
\]

We proceed to estimate the right-hand side by the use of Hölder’s inequality, (3.11) and the Caccioppoli inequality from Lemma 3.1 to infer

\[
|II| \leq c \left( \int_{B_{\rho/2}} |Dv - Du|^p \, dx \right)^{1 - \frac{1}{p}}
\]

\[
+ \chi_{p>2} c \left( \int_{B_{\rho/2}} |Du|^p \, dx \right)^{\frac{p-2}{p}} \left( \int_{B_{\rho/2}} |Dv - Du|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq cK(q)^{1 - \frac{1}{p}} + \chi_{p>2} c \left( \int_{B_{\rho/2}} |Du|^p \, dx \right)^{\frac{p-2}{p}} K(q)^{\frac{1}{p}}
\]

\[
\leq c(n, N, p, \nu, L) \Psi_1(q)^{1 - \frac{1}{p}} \left[ H(q)^{1 - \frac{1}{p}} + H(q)^{\frac{1}{p}} \right].
\]

Finally, for the estimate of the last term appearing on the right-hand side of (3.24) we use (3.14) to infer that
\[ |\text{III}| \leq K(\varrho)^{1 - \frac{1}{p}} \left( \int_{B_{\varrho/2}} |D\varphi|^p \, dx \right)^{\frac{1}{p}} \leq K(\varrho)^{1 - \frac{1}{p}} = [H(\varrho)\Psi_1(\varrho)]^{1 - \frac{1}{p}}. \]

Inserting the estimates for I–III into (3.24) we finally arrive at

\[ \left| \int_{B_{\varrho/2}} |Du|^p - 2 Du \cdot D\varphi \, dx \right| \leq c \Psi_1(\varrho)^{1 - \frac{1}{p}} \left[ \delta + H(\varrho)^{\min\{1 - \frac{1}{p}, \frac{1}{p}\}} \right] + \Psi_1(\varrho)^{\frac{1}{p}} \eta(\delta), \]

where \( c = c(n, N, p, \nu, L) \). This proves the claim. \( \Box \)

3.3. The non-degenerate regime

Now we are in the position to establish excess improvement estimates under certain smallness conditions. We start with the non-degenerate regime which is characterized by (3.29) below, i.e. the fact that \( \Phi(x_0, \varrho) \) is small compared to \( |D\ell_{x_0, \varrho}|^p \). The strategy of the proof is to approximate the given minimizer by \( A \)-harmonic functions, for which suitable decay estimates are available from the classical theory.

Lemma 3.12. Let the assumptions of Theorem 1.1 be in force. For \( \vartheta \in (0, \frac{1}{2}] \) we denote by \( \delta_0 = \delta_0(n, N, p, \nu, L, \vartheta^{n+p+4}) \in (0, 1] \) the constant from Lemma 2.11 if \( p \geq 2 \), resp. Lemma 2.12 if \( p < 2 \). Assumed that for some ball \( B_\varrho(x_0) \subseteq \Omega \) with \( \varrho \leq \varrho_0 \) and \( |D\ell_{x_0, \varrho}| \neq 0 \) the smallness assumptions

\[ \left[ \tilde{\omega} \left( \frac{\Phi_a(x_0, \varrho)}{|D\ell_{x_0, \varrho}|^p} \right) + \left( \frac{\Phi_a(x_0, \varrho)}{|D\ell_{x_0, \varrho}|^p} \right)^2 \right] \leq \delta_0 \]  

and

\[ \frac{\Phi(x_0, \vartheta \varrho)}{|D\ell_{x_0, \varrho}|^p} \leq \left( \frac{\vartheta^{n+1}}{8(n+2)} \right)^{2p} \]  

are satisfied, then the following excess improvement estimate holds:

\[ \Phi(x_0, \vartheta \varrho) \leq c \vartheta^2 \Phi_a(x_0, \varrho) \]

with a constant \( c \) depending on \( n, N, p, \nu \) and \( L \). Here, \( \ell_{x_0, \varrho} \) and \( \ell_{x_0, \varrho} \) denote the affine functions introduced in (2.14).

Proof. For simplicity we assume \( x_0 = 0 \). We define the re-scaled function

\[ w := \frac{u - \ell_\varrho}{c_2|D\ell_\varrho|} \quad \text{and} \quad \gamma := \sqrt{\frac{\Phi_a(\varrho)}{|D\ell_\varrho|^p}} \leq 1, \]

where \( c_2 := \max\{c_{\text{cap}}^{1/p}, c_{\text{cap}}^{1/2}, c_1 \} \geq 1 \), with \( c_{\text{cap}} \) and \( c_1 \) denoting the constants from (3.9) and Lemma 3.9 depending on \( n, N, p, \nu, L \). This amounts in a dependence of the constant \( c_2 \) upon the same parameters \( n, N, p, \nu, L \). Note that \( \gamma \leq 1 \) is a consequence of (3.28). Next we observe that the assumption (3.19) of Lemma 3.9 is implied by (3.29). The application of the lemma together with (3.28) ensures that \( w \) is approximately \( A \)-harmonic in the sense that...
\[
\int_{B_{r/2}} A(Dw, D\varphi) \, dx \leq \frac{c_1 \gamma}{c_2} \left[ \tilde{c}_0 \left( \frac{\Phi_+ (Q)}{|D\ell_Q|^p} \right) + \left( \frac{\Phi_+(Q)}{|D\ell_Q|^p} \right) \right] \|D\varphi\|_{L^\infty(B_{r/2})} \\
\leq \delta_0 \gamma \|D\varphi\|_{L^\infty(B_{r/2})}. \tag{3.31}
\]

Moreover, from (2.6), the definition of \( \gamma \) and \( c_2 \) and Caccioppoli’s inequality (3.9) we deduce the following energy bound for \( w \):

\[
\int_{B_{\rho/2}} |V_1(Dw)|^2 \, dx = |D\ell_Q|^{-p} \int_{B_{\rho/2}} |V_1|_{D\ell_Q} \left( \frac{Du - D\ell_Q}{c_2} \right) |D\ell_Q| \, dx \\
\leq \max \{ c_2^{-p}, c_2^{-2} \} |D\ell_Q|^{-p} \int_{B_{\rho/2}} |V_1|_{D\ell_Q} (Du - D\ell_Q) |D\ell_Q| \, dx \\
\leq \frac{\Phi_+(Q)}{|D\ell_Q|^p} \leq \gamma^2. \tag{3.32}
\]

At this point we distinguish the cases \( p \geq 2 \) and \( p < 2 \). When \( p \geq 2 \) we apply the \( A \)-harmonic approximation Lemma 2.11. In view of (2.20), (3.31) and (3.32) the assumptions of the lemma are satisfied for the choice \( \varepsilon = \vartheta \rho^{n+p+4} \) and with \( \rho/2 \) instead of \( \rho \). The application of the lemma yields the existence of an \( A \)-harmonic function \( h \in C^\infty(B_{\rho/4}, \mathbb{R}^N) \) with the properties

\[
\int_{B_{\rho/4}} |V_1(Dh)|^2 \, dx \leq c(n, p) \gamma^2 \tag{3.33}
\]

and

\[
\int_{B_{\rho/4}} \left| V_1 \left( \frac{w - h}{Q/4} \right) \right|^2 \, dx \leq \vartheta^{n+p+4} \gamma^2. \tag{3.34}
\]

Since \( h \) is an \( A \)-harmonic function on \( B_{\rho/4} \) it satisfies (2.21) with \( (B_\rho, B_{\rho/2}) \) replaced by \( (B_{\rho/4}, B_{\rho/8}) \). Therefore, using (2.21), (2.7) and (3.33) we obtain

\[
\vartheta^2 \sup_{B_{\rho/8}} |D^2 h|^2 \leq c \int_{B_{\rho/4}} |Dh|^2 \, dx \leq c \int_{B_{\rho/4}} |V_1(Dh)|^2 \, dx \leq c(n, N, p, \nu, L) \gamma^2.
\]

For \( \vartheta \in (0, \frac{1}{8}) \) we hence conclude that

\[
\sup_{x \in B_{\rho/8}} |h(x) - \ell^{(h)}(x)| \leq c \vartheta^2 Q \gamma, \quad \text{where } \ell^{(h)}(x) := h(0) + Dh(0) x \tag{3.35}
\]

and \( c = c(n, N, p, \nu, L) \). Together with (2.6) this implies

\[
\int_{B_{\vartheta \rho}} \left| V_1 \left( \frac{h - \ell^{(h)}}{\vartheta Q} \right) \right|^2 \, dx \leq \vartheta^2 \gamma^2 \int_{B_{\vartheta \rho}} \left| V_1 \left( \frac{h - \ell^{(h)}}{\vartheta^2 Q \gamma} \right) \right|^2 \, dx \leq c \vartheta^2 \gamma^2, \tag{3.36}
\]
with a constant \( c = c(n, N, p, v, L) \). Using (2.5), (2.6), (3.34) and (3.36), we thus conclude
\[
\int_{B_{\varrho \varepsilon}} \left| V_1 \left( \frac{w - \ell(h)}{\vartheta Q} \right) \right|^2 \, dx \leq c \int_{B_{\varrho \varepsilon}} \left| V_1 \left( \frac{w - h}{\vartheta Q} \right) \right|^2 \, dx + c \int_{B_{\varrho \varepsilon}} \left| V_1 \left( \frac{h - \ell(h)}{\vartheta Q} \right) \right|^2 \, dx
\]
\[
\leq c(4\vartheta)^{-n-p} \vartheta^{n+p+4} \gamma^2 + c\vartheta^2 \gamma^2
\]
\[
\leq c(n, N, p, v, L) \vartheta^2 \gamma^2. \tag{3.37}
\]

In the case \( p < 2 \) we apply the sub-quadratic \( A \)-harmonic approximation Lemma 2.12. As in the super-quadratic case the assumptions of the lemma are satisfied in view of (2.20), (3.31) and (3.32) for the choice \( \varepsilon = \vartheta^{n+p+4} \) and with \( \varrho / 2 \) instead of \( \varrho \). Therefore, the application of the lemma yields an \( A \)-harmonic function \( h \in C^\infty(B_{\varrho / 4}. \mathbb{R}^N) \) satisfying
\[
\int_{B_{\varrho / 4}} \left| V_1 \left( \frac{\gamma^{-1} Dh}{\vartheta} \right) \right|^2 \, dx \leq c(n, p) \tag{3.38}
\]
and
\[
\int_{B_{\varrho / 4}} \left| V_1 \left( \frac{w - h}{\varrho / 4} \right) \right|^2 \, dx \leq \vartheta^{n+p+4} \gamma^2. \tag{3.39}
\]
Since \( h \) is an \( A \)-harmonic function we can use (2.21) as in the case \( p > 2 \). Therefore, combining (2.21), (2.7) and (3.38) we find
\[
\vartheta^2 \gamma^{-2} \sup_{B_{\varrho \varepsilon / 8}} |D^2 h|^2 \leq c \gamma^{-2} \int_{B_{\varrho / 4}} |Dh|^2 \, dx
\]
\[
\leq c \int_{B_{\varrho / 4}} \left| V_1 \left( \gamma^{-1} Dh \right) \right|^2 + \left| V_1 \left( \gamma^{-1} Dh \right) \right| \frac{\gamma}{2} \, dx \leq c(n, N, p, v, L).
\]

For \( \vartheta \in (0, \frac{1}{8}] \) we hence obtain (3.35), as in the case \( p \geq 2 \). Together with (2.7) we find
\[
\int_{B_{\varrho \varepsilon}} \left| V_1 \left( \frac{h - \ell(h)}{\vartheta Q} \right) \right|^2 \, dx \leq \int_{B_{\varrho \varepsilon}} \left| \frac{h - \ell(h)}{\vartheta Q} \right|^2 \, dx \leq c \vartheta^2 \gamma^2, \tag{3.40}
\]
with a constant \( c = c(n, N, p, v, L) \). Using (2.5), (2.6), (3.39) and (3.40), we thus conclude
\[
\int_{B_{\varrho \varepsilon}} \left| V_1 \left( \frac{w - \ell(h)}{\vartheta Q} \right) \right|^2 \, dx \leq c \int_{B_{\varrho \varepsilon}} \left| V_1 \left( \frac{w - h}{\vartheta Q} \right) \right|^2 \, dx + c \int_{B_{\varrho \varepsilon}} \left| V_1 \left( \frac{h - \ell(h)}{\vartheta Q} \right) \right|^2 \, dx
\]
\[
\leq c(4\vartheta)^{-n-2} \vartheta^{n+p+4} \gamma^2 + c\vartheta^2 \gamma^2
\]
\[
\leq c(n, N, p, v, L) \vartheta^2 \gamma^2, \tag{3.41}
\]
which is the counterpart to the estimate (3.37) for the case \( p \geq 2 \).
Scaling back to $u$ with the help of the definition of $w$ and $\gamma$ in (3.30) and using (2.6) we infer the following lower bound for the left-hand side of (3.37) for the case $p \geq 2$, respectively (3.41) for the case $p < 2$:

$$
\int_{B_{\rho Q}} \left| V_1 \left( \frac{w - \ell^{(h)}}{\partial Q} \right) \right|^2 \, dx = |D\ell_Q|^{-p} \int_{B_{\rho Q}} \left| V_{|D\ell_Q|} \left( \frac{u - \ell_Q - |D\ell_Q|c_2 \ell^{(h)}}{\partial Q} \right) \right|^2 \, dx
$$

$$
\geq \frac{|D\ell_Q|^{-p}}{\max\{c_2^2, c_p^2\}} \int_{B_{\rho Q}} \left| V_{|D\ell_Q|} \left( \frac{u - \ell_Q - |D\ell_Q|c_2 \ell^{(h)}}{\partial Q} \right) \right|^2 \, dx,
$$

which in any case leads us to

$$
\int_{B_{\rho Q}} \left| V_{|D\ell_Q|} \left( \frac{u - \ell_Q - |D\ell_Q|c_2 \ell^{(h)}}{\partial Q} \right) \right|^2 \, dx \leq c \partial^2 \gamma^2 |D\ell_Q|^p = c \partial^2 \Phi_*(Q),
$$

for a constant $c = c(n, N, p, \nu, L)$. In view of Lemma 2.8 we can replace in the preceding inequality the affine function $\ell_Q + |D\ell_Q|c_2 \ell^{(h)}$ by $\ell_{\partial Q}$. We therefore get

$$
\int_{B_{\rho Q}} \left| V_{|D\ell_Q|} \left( \frac{u - \ell_{\partial Q}}{\partial Q} \right) \right|^2 \, dx \leq c(n, N, p, \nu, L) \partial^2 \Phi_*(Q). \tag{3.42}
$$

Here, we want to replace the term $V_{|D\ell_Q|}(\ldots)$ on the left-hand side by $V_{|D\ell_{\partial Q}|}(\ldots)$. For this we use (2.17), (2.7) and (3.29) in order to estimate

$$
|D\ell_Q - D\ell_{\partial Q}| \leq (n + 2) \int_{B_{\rho Q}} \left| \frac{u - \ell_{\partial Q}}{\partial Q} \right| \, dx \leq \frac{n + 2}{\partial^{n+1}} \int_{B_{\rho}} \left| \frac{u - \ell_Q}{Q} \right| \, dx
$$

$$
\leq \frac{2(n + 2)}{\partial^{n+1}} \int_{B_{\rho}} \left| V_{|D\ell_Q|} \left( \frac{u - \ell_Q}{Q} \right) \right|^2 \, dx + \chi_{p < 2} |D\ell_Q|^{\frac{2-p}{2}} \int_{B_{\rho}} \left| V_{|D\ell_Q|} \left( \frac{u - \ell_Q}{Q} \right) \right| \, dx
$$

$$
\leq \frac{2(n + 2)}{\partial^{n+1}} \left[ \Phi(Q)^\frac{1}{p} + \chi_{p < 2} |D\ell_Q|^{\frac{2-p}{2}} \Phi(Q)^\frac{1}{2} \right] \leq \frac{1}{2} |D\ell_Q|,
$$

which in turn implies

$$
|D\ell_{\partial Q}| \leq |D\ell_Q| + |D\ell_Q - D\ell_{\partial Q}| \leq |D\ell_Q| + \frac{1}{2} |D\ell_Q| \leq 2|D\ell_Q| \tag{3.43}
$$

and

$$
|D\ell_{\partial Q}| \leq |D\ell_Q| + |D\ell_Q - D\ell_{\partial Q}| \leq |D\ell_{\partial Q}| + \frac{1}{2} |D\ell_Q|,
$$

which after re-absorbing the last term of the right-hand side into the left, implies

$$
|D\ell_{\partial Q}| \leq 2|D\ell_{\partial Q}|. \tag{3.44}
$$
Now, using (3.43) in the case $p < 2$ and (3.44) in the case $p \geq 2$ we can estimate $V|_\mathcal{D}_\varrho|(...) \leq c(p)V|_\mathcal{D}_\varrho|(...)$. Plugging this into (3.42), we deduce

$$
\Phi(\partial Q) = \int_{\mathcal{D}_\varrho} \left| V|_\mathcal{D}_\varrho| \left( \frac{u - \ell \partial Q}{\partial Q} \right) \right|^2 \, dx \leq c(n, N, p, \nu, L) \partial^2 \Phi(Q).
$$

This proves the claim and therefore finishes the proof of the lemma. □

In the following lemma we will iterate the excess-decay estimate from Lemma 3.12. This is possible since, within the iteration scheme, we can ensure that the smallness condition (3.28) and the assumption (3.29) – which characterizes the non-degenerate regime – are also satisfied on any smaller ball $B_{\varrho^\ell}(x_0)$, $\ell \in \mathbb{N}$.

**Lemma 3.13.** Suppose that the assumptions of Theorem 1.1 are satisfied and let $\alpha \in (0, 1)$. Then, there are constants $\mu_* = \mu_*(n, N, p, \nu, L, \alpha, \omega(\cdot)) \in (0, 1)$, $\kappa_* = \kappa_*(n, N, p, \nu, L, \alpha, \omega(\cdot), \omega(\cdot)) \in (0, 1)$ and $\varrho_* = \varrho_*(n, N, p, \nu, L, \alpha, \omega(\cdot), \tilde{\omega}(\cdot), \mathbf{V}(\cdot)) \in (0, \varrho_0]$ and $\vartheta = \vartheta(n, N, p, \nu, L, \alpha) \in (0, \frac{1}{2})$, such that the conditions

$$
\frac{\Phi(x_0, \varrho)}{|\mathcal{D}_\varrho|} < \mu_* \quad \text{and} \quad \psi_\alpha(x_0, \varrho) < \kappa_* ,
$$

(3.47)

for $B_{\varrho}(x_0) \subseteq \Omega$ with $\varrho \in (0, \varrho_*)$ imply

$$
\frac{\Phi(x_0, \varrho^\ell \varrho)}{|\mathcal{D}_\varrho|} < \mu_* \quad \text{and} \quad \psi_\alpha(x_0, \varrho^\ell \varrho) < \kappa_* ,
$$

(3.47)

for every $\ell \in \mathbb{N}_0$.

**Proof.** For notational convenience we once more assume $x_0 = 0$. We start by choosing the constants $\vartheta, \mu_*, \kappa_*$ and $\varrho_*$ as follows. First, we define

$$
\vartheta := \min\left\{ \frac{1}{8}, (3c_{\text{dec}})^{-\frac{1}{2}} \left[ 2(n + 2) \right]^{-\frac{1}{2}} \right\},
$$

(3.45)

where $c_{\text{dec}} = c_{\text{dec}}(n, N, p, \nu, L)$ denotes the constant from Lemma 3.12. This yields the dependencies $\vartheta = \vartheta(n, N, p, \nu, L, \alpha).$ With $\delta_0 = \delta_0(n, N, p, \nu, L, \varrho_{n+p} + 4) = \delta_0(n, N, p, \nu, L, \alpha)$ being the constant from Lemma 2.11, applied with the choice $\varepsilon = \vartheta^{n+p+4}$, we choose $\mu_* > 0$ so small that

$$
\mu_* \leq \frac{1}{3} \left( \frac{\vartheta^{n+1}}{8(n+2)} \right)^{2p} \quad \text{and} \quad \left[ \tilde{\omega}(3\mu_*) + 3\mu_* \right]^{\frac{1}{2}} \leq \delta_0.
$$

(3.46)

This yields a constant $\mu_*$ depending only on $n, N, p, \nu, L, \alpha$ and $\tilde{\omega}(\cdot)$. Next, we fix the constant $\kappa_* > 0$ so small that

$$
\omega(\kappa_*^{\frac{1}{q}}) \frac{\vartheta^p - \min\left\{ 1 - \frac{1}{p}, \frac{1}{p} \right\}}{1 - \frac{1}{p}} < \mu_* ,
$$

(3.47)

where $q = q(n, N, p, \nu, L)$ is determined in Remark 3.4. Finally, we choose the radius $\varrho_* > 0$ sufficiently small to guarantee

$$
\varrho_* \leq \varrho_0 \quad \text{and} \quad \mathbf{V}(\varrho_*) \frac{\varrho_q^p - \min\left\{ 1 - \frac{1}{p}, \frac{1}{p} \right\}}{1 - \frac{1}{p}} < \mu_* .
$$

(3.48)
Consequently, \( \kappa_* \) depends on \( n, N, p, v, I, \alpha, \tilde{\omega}(\cdot) \) and \( \omega(\cdot) \), and \( \varrho_\star \) additionally on \( \mathbf{V}(\cdot) \). Now we will prove the assertion \((N_\ell)\) by induction. Note that \((N_0)\) is implied by the hypothesis of the lemma. Now, we assume that we have already established \((N_\ell)\) up to some \( \ell \in \mathbb{N}_0 \) and want to derive \((N_{\ell+1})\). We begin with proving the first part of the assertion \((N_\ell)\), i.e. the one concerning \( \Phi(\partial^{\ell+1} \varrho) \). For this we want to ensure that the assumptions for the excess improvement in Lemma 3.12 are satisfied on the level \( \partial^\ell \varrho \) instead of \( \varrho \). First, we observe that from \((N_\ell)\) and the choices of \( \kappa_* \) and \( \varrho_\star \) in (3.47) and (3.48) we have

\[
\frac{\Phi_*(\partial^\ell \varrho)}{|D \partial^\ell \varrho|} = \frac{\Phi(\partial^\ell \varrho)}{|D \partial^\ell \varrho|} + \left[ \omega(\varphi, (\partial^\ell \varrho)^{\frac{1}{p}}) \frac{q_\star - p}{q_\star} + \mathbf{V}(\partial^\ell \varrho)^{\frac{q_\star - p}{q_\star}} \right] \min[1 - \frac{1}{p}, \frac{1}{q_\star}] < \mu_* + \omega(\kappa_*^{\frac{1}{p}}) \frac{q_\star - p}{q_\star} \min[1 - \frac{1}{p}, \frac{1}{q_\star}] + \mathbf{V}(\varrho_\star \partial^\ell \varrho)^{\frac{q_\star - p}{q_\star}} \min[1 - \frac{1}{p}, \frac{1}{q_\star}] < 3 \mu_* \quad (3.49)
\]

By our choice of \( \mu_* \) from (3.46) the preceding inequality ensures that the smallness assumptions of Lemma 3.12 are satisfied on the level \( \partial^\ell \varrho \), i.e. that there holds

\[
\left[ \tilde{\omega} \left( \sqrt{\frac{\Phi_*(\partial^\ell \varrho)}{|D \partial^\ell \varrho|}} \right) + \left( \frac{\Phi_*(\partial^\ell \varrho)}{|D \partial^\ell \varrho|} \right) \right]^{\frac{1}{2}} \leq \left[ \tilde{\omega}(3 \mu_\star) + 3 \mu_\star \right]^{\frac{1}{2}} \leq \delta_0,
\]

and

\[
\frac{\Phi_*(\partial^\ell \varrho)}{|D \partial^\ell \varrho|} < 3 \mu_* \leq \left( \frac{\phi^{n+1}}{8(n+2)} \right)^{2p}.
\]

We can thus apply Lemma 3.12 with the radius \( \partial^\ell \varrho \) instead of \( \varrho \), which yields

\[
\Phi(\partial^{\ell+1} \varrho) \leq c_{\text{dec}} \phi^2 \Phi_*(\partial^\ell \varrho) < 3c_{\text{dec}} \mu_* \phi^2 |D \partial^\ell \varrho| \leq \mu_* |D \partial^\ell \varrho|^p,
\]

by (3.49) and the choice of \( \phi \) in (3.45). We have thus established the first part of the assertion \((N_{\ell+1})\) and it remains to prove the second one, i.e. the one concerning \( \Psi_{\alpha}(\partial^{\ell+1} \varrho) \). For this aim, we first use (2.7) and \((N_\ell)\) to infer

\[
\int_{B_\varrho^\ell} \left| \frac{u - \kappa_{\partial^\ell \varrho}^p}{\partial^\ell \varrho} \right|^p \, dx \leq 2 \Phi(\partial^\ell \varrho) + \chi_{p < 2} |D \kappa_{\partial^\ell \varrho}^p| \varrho_\star^2 \Phi(\partial^\ell \varrho)^\frac{p}{2} \leq 2 \mu_* |D \partial^\ell \varrho|^p + \chi_{p < 2} 2 \mu_*^\phi |D \partial^\ell \varrho|^p \leq 4 \sqrt{\mu_* |D \partial^\ell \varrho|^p}.
\]

Recalling from (2.14) that \( \kappa_{\partial^\ell \varrho}(x) = (u)_{\partial^\ell \varrho} + D \kappa_{\partial^\ell \varrho}^p x \), we can estimate

\[
\Psi_{\alpha}(\partial^{\ell+1} \varrho) \leq c(p) (\partial^{\ell+1} \varrho)^{-\alpha p} \int_{B_{\partial^{\ell+1} \varrho}} |u - (u)_{\partial^\ell \varrho}|^p \, dx
\]

\[
\leq (\partial^{\ell+1} \varrho)^{-\alpha p} \left[ \int_{B_{\partial^{\ell+1} \varrho}} |u - \kappa_{\partial^\ell \varrho}^p|^p \, dx + (\partial^{\ell+1} \varrho)^p |D \kappa_{\partial^\ell \varrho}^p|^p \right]
\]

\[
\leq (\partial^{\ell+1} \varrho)^{-\alpha p} \left[ \varrho^{-n} \int_{B_{\partial^\ell \varrho}} |u - \kappa_{\partial^\ell \varrho}^p|^p \, dx + (\partial^{\ell+1} \varrho)^p |D \kappa_{\partial^\ell \varrho}^p|^p \right]
\]
we deduce exist constants via the Lemma 3.14. due to Uhlenbeck to the original minimizer.

We now define the re-scaled function which yields the bound

\[ \Psi \quad \text{Proof.} \]

for \( B \) by (3.50)2 below, i.e. by the fact that

\[ \ell \]

This establishes the remaining part of the assertion \( N_{\ell+1} \) and thus completes the proof of the lemma. \( \Box \)

3.4. The degenerate regime

We now establish an excess improvement estimate for the degenerate case which is characterized by (3.50)2 below, i.e. by the fact that \( \Phi(x_0, \varrho) \) is large compared to \( |D\ell x_0, \varrho| \). This will be achieved via the \( p \)-harmonic approximation lemma which allows to approximate the original minimizer by a \( p \)-harmonic function. In turn, this allows to transfer the a priori estimates for \( p \)-harmonic functions due to Uhlenbeck to the original minimizer.

Lemma 3.14. Suppose that the assumptions of Theorem 1.1 are satisfied and let \( \alpha, \kappa, \mu \in (0, 1) \). Then there exist constants \( \varepsilon_2 = \varepsilon_2(n, N, p, \nu, L, \alpha, \kappa, \mu, \eta(\cdot)) \in (0, 1) \) and \( \varrho_2 = \varrho_2(n, N, p, \nu, L, \alpha, \kappa, \mu, \omega(\cdot), \vartheta(\cdot)) \in (0, \varrho_0] \) and \( \vartheta = \vartheta(n, N, p, \nu, L, \alpha, \kappa, \mu) \in (0, \frac{1}{2}] \) such that the conditions

\[ \Phi(x_0, \varrho) < \varepsilon_2 \quad \text{and} \quad \mu |D\ell x_0, \varrho| < \Phi(x_0, \varrho) \]  

(3.50)

for \( B_0(x_0) \subseteq \Omega \) with \( \varrho \in (0, \varrho_2] \) imply

\[ \Phi(x_0, \vartheta \varrho) < \varepsilon_2 \quad \text{and} \quad \Psi_{\alpha}(x_0, \vartheta \varrho) < \kappa. \]  

(3.51)

Proof. As several times before we assume \( x_0 = 0 \) without loss of generality. Initially we will estimate \( \Psi_1(\varrho) \) in terms of \( \Phi(\varrho) \). Indeed, from (2.7) and (3.50)2 we deduce

\[ \Psi_1(\varrho) \leq 2^{p-1} \left[ \int_{B_0} \left| \frac{u - \ell \varrho}{\varrho} \right|^p \right] \]

\[ \leq 2 \left[ \int_{B_0} \left| V_{|D\ell \varrho|} \left( \frac{u - \ell \varrho}{\varrho} \right) \right|^2 + \chi_{p<2} |D\ell \varrho|^{\frac{p(2-p)}{2}} \left| V_{|D\ell \varrho|} \left( \frac{u - \ell \varrho}{\varrho} \right) \right|^p \right] \]

\[ \leq 2^p (1 + \chi_{p<2} \mu \frac{2-p}{2} + \mu^{-1}) \Phi(\varrho), \]

which yields the bound

\[ \Psi_1(\varrho) \leq c_1 \Phi(\varrho), \quad \text{where} \quad c_1 = 2^p (1 + \mu^{-1}). \]  

(3.52)

We now define the re-scaled function
where $c_2 \geq 1$ is a constant that will be fixed later in the proof. Using the Caccioppoli inequality from Lemma 2.13 and the inequality (3.52) we find

\[
\int_{B_{\rho/2}} |Dw|^p \, dx = \frac{1}{c_2^p \Phi(Q)} \int_{B_{\rho/2}} |Du|^p \, dx \leq \frac{c_{\text{cap}} \Psi_1(Q)}{c_2^p \Phi(Q)} \leq \frac{c_{\text{cap}} c_1}{c_2^p} \leq 1, \tag{3.53}
\]

provided we have chosen $c_2 \geq (c_{\text{cap}} c_1)^{\frac{1}{p}}$. Recalling that $c_{\text{cap}} = c_{\text{cap}}(p, L/v)$ and $c_1 = c_1(p, \mu)$ we see that $c_2$ depends on $p$, $L/v$, $\mu$. Moreover, from Lemma 3.11 we know that $u$ is approximatively $p$-harmonic in the sense that for any $\delta > 0$ and $\varphi \in C_0^1(B_{\rho/2}, \mathbb{R}^N)$ there holds

\[
\left| \int_{B_{\rho/2}} |Du|^{p-2} Du \cdot D\varphi \, dx \right| \leq c_{\text{ap}} \Psi_1(Q)^{\frac{1}{p} - \frac{1}{p^*}} \left[ \delta H(Q)^{\min\{1 - \frac{1}{p}, \frac{1}{p^*}\}} + \frac{\Psi_1(Q)^{\frac{1}{p}}}{\eta(\delta)} \right] \sup_{B_{\rho/2}} |D\varphi| \leq c_1 c_{\text{ap}} \Phi(Q)^{\frac{1}{p} - \frac{1}{p^*}} \left[ \delta H(Q)^{\min\{1 - \frac{1}{p}, \frac{1}{p^*}\}} + \frac{\Phi(Q)^{\frac{1}{p}}}{\eta(\delta)} \right] \sup_{B_{\rho/2}} |D\varphi|,
\]

where $c_{\text{ap}} = c_{\text{ap}}(n, N, p, v, L)$ is the constant from Lemma 3.11. By the definition of $w$ we infer from the preceding inequality

\[
\left| \int_{B_{\rho/2}} |Dw|^{p-2} Dw \cdot D\varphi \, dx \right| \leq \left[ \delta H(Q)^{\min\{1 - \frac{1}{p}, \frac{1}{p^*}\}} + \frac{\Phi(Q)^{\frac{1}{p}}}{\eta(\delta)} \right] \sup_{B_{\rho/2}} |D\varphi|, \tag{3.54}
\]

provided we have chosen $c_2$ large enough to ensure $c_2^{p-1} \geq c_1 c_{\text{ap}}$. Note that this can be achieved by enlarging $c_2$. Then, $c_2$ depends upon the parameters $n$, $N$, $p$, $v$, $L$, $\mu$. Now let $\epsilon \in (0, \frac{1}{2})$ to be fixed later. We set $\epsilon := \theta^{n+2p}$ and let $\delta_0 = \delta_0(n, N, p, \epsilon) \in (0, 1]$ be the corresponding constant from Lemma 2.13. Note that by the choice of $\epsilon$ the constant $\delta_0$ ultimately depends upon $n$, $N$, $p$, $\theta$. Next, we fix $\delta > 0$ such that

\[
\delta \leq \frac{\delta_0}{2}.
\]

Note that $\delta$ has the same dependencies as $\delta_0$, i.e. $\delta = \delta(n, N, p, \theta)$. This also fixes $\eta(\delta)$. Assuming that

\[
\frac{\Phi(Q)^{\frac{1}{p}}}{\eta(\delta)} \leq \frac{\delta_0}{4} \quad \text{and} \quad H(Q)^{\min\{1 - \frac{1}{p}, \frac{1}{p^*}\}} \leq \frac{\delta_0}{4}, \tag{3.55}
\]

we infer from (3.53) and (3.54) that the assumptions of the $p$-harmonic approximation Lemma 2.13 are satisfied for $w$ and the choice $\epsilon = \theta^{n+2p}$. The application of the lemma yields a $p$-harmonic function $h \in W^{1,p}(B_{\rho/2}, \mathbb{R}^N)$ such that
\[ \int_{B_{r/2}} |Dh|^p \, dx \leq 1 \quad \text{and} \quad \int_{B_{r/2}} \left| \frac{W - h}{Q} \right|^p \, dx \leq \varepsilon = \theta^{n+2p}. \]  
(3.56)

We now define \( \mathcal{H}_{\theta Q} \) according to (2.24). From (3.56) and Theorem 2.10 we infer

\[ |\mathcal{H}_{\theta Q}|^p \leq \sup_{B_{r/2}} |Dh|^p \leq c \int_{B_{r/2}} |Dh|^p \leq c(n, N, p). \]  
(3.57)

From now on we distinguish the cases \( p \geq 2 \) and \( p < 2 \). In the first case, i.e. when \( p \geq 2 \) we use (2.5), (2.7), Poincaré's inequality in the form (2.3), (3.57), Hölder's inequality, Theorem 2.10 and (3.56) to infer the following excess estimate for \( w \):

\[ -\int_{B_{r/2}} |D\mathcal{H}_{\theta Q}| \left| V \left| \frac{W - (h)_{\theta Q} - \mathcal{H}_{\theta Q} x}{\theta Q} \right| \right|^2 \, dx \leq c \left[ \int_{B_{r/2}} |D\mathcal{H}_{\theta Q}| \left| \frac{W - h}{\theta Q} \right|^2 + \left| \frac{W - h}{\theta Q} \right|^p \, dx + \int_{B_{r/2}} |D\mathcal{H}_{\theta Q}| \left( \frac{h - (h)_{\theta Q} - \mathcal{H}_{\theta Q} x}{\theta Q} \right) \right|^2 \, dx \right] \]

\[ \leq c \left[ \theta^{-2n-p-2} \varepsilon^{\frac{1}{p}} + \theta^{-n-p} + \left| V \left| \mathcal{H}_{\theta Q} \left( \frac{Dh - \mathcal{H}_{\theta Q}}{\theta Q} \right) \right|^2 \right| \right] \]

\[ \leq c \left[ \theta^2 + \theta^p + \theta^{2\alpha_0} \int_{B_{r/2}} |\mathcal{H}_{\theta Q}|^p + |Dh|^p \, dx \right] \leq c(n, N, p) \theta^{2\alpha_0}, \]

where \( \alpha_0 = \alpha_0(n, N, p) \in (0, 1) \) denotes the constant from Theorem 2.10. Re-scaling back to \( u \) we can re-write the preceding inequality as follows:

\[ \int_{B_{\theta r/2}} \left| V \left| \frac{u - \ell}{\theta Q} \right| \right|^2 \, dx \leq cc_2^2 \theta^{2\alpha_0} \Phi(Q), \]  
(3.58)

where we have abbreviated

\[ \mathbb{R}^n \ni x \mapsto \ell(x) := c_2 \Phi(Q)^{\frac{1}{p}} \left( (h)_{\theta Q} + \mathcal{H}_{\theta Q} x \right) \in \mathbb{R}^N. \]  
(3.59)

By Lemma 2.9 and (3.50) this implies

\[ \Phi(\theta Q) \leq c \int_{B_{\theta r/2}} \left| V \left| \frac{u - \ell}{\theta Q} \right| \right|^2 \, dx \leq c \theta^{2\alpha_0} \Phi(Q) < c_3 \theta^{2\alpha_0} \varepsilon^2, \]  
(3.60)

for a constant \( c_3 = c_3(n, N, p, \nu, L, \mu) \). Moreover, from (2.7), (3.58) and the fact that \( |D\ell| \leq c \Phi(Q)^{\frac{1}{p}} \) – which is a consequence of (3.57) and the definition of \( \ell \) – we get
\[ \Psi_\alpha(\theta \varrho) \leq 2^p (\theta \varrho)^{-\alpha p} \int_{B_{\theta \varrho}} |u - \ell(0)|^p \, dx \]
\[ \leq 4^p (\theta \varrho)^{(1-\alpha)p} \left[ \int_{B_{\theta \varrho}} \frac{|u - \ell|}{\theta \varrho} \, dx + |D\ell|^p \right] \]
\[ \leq 4^p (\theta \varrho)^{(1-\alpha)p} \left[ \int_{B_{\theta \varrho}} \left| V_{|D\ell|} \left( \frac{u - \ell}{\theta \varrho} \right) \right|^2 \, dx + c \Phi(\varrho) \right] \]
\[ \leq c(\theta \varrho)^{(1-\alpha)p} \Phi(\varrho) \leq c_4 \theta^{(1-\alpha)p}, \quad (3.61) \]

where \( c_4 = c_4(n, N, p, v, L, \mu). \)

Now we turn our attention to the case \( p < 2. \) Using Poincaré’s inequality, (3.56), (2.7), (3.57), Hölder’s inequality and Theorem 2.10 we infer the following estimate for \( w: \)

\[ - \int_{B_{\theta \varrho}} \left| \frac{w - (h)_{\theta \varrho} - H_{\theta \varrho} x}{\theta \varrho} \right|^p \, dx \]
\[ \leq c \left[ \int_{B_{\theta \varrho}} \left| \frac{w - h}{\varrho} \right|^p \, dx + \int_{B_{\theta \varrho}} \left| \frac{h - (h)_{\theta \varrho} - H_{\theta \varrho} x}{\theta \varrho} \right|^p \, dx \right] \]
\[ \leq c \theta^{-n-p} \int_{B_{\theta \varrho}} \left| \frac{w - h}{\varrho} \right|^p \, dx + \int_{B_{\theta \varrho}} \left| Dh - H_{\theta \varrho} \right|^p \, dx \]
\[ \leq c \left[ \theta^{-n-p} \varepsilon + \int_{B_{\theta \varrho}} \left| V_{|H_{\theta \varrho}|} (Dh - H_{\theta \varrho}) \right|^2 + \left| H_{\theta \varrho} \right|^{\frac{p(2-p)}{2}} \left| V_{|H_{\theta \varrho}|} (Dh - H_{\theta \varrho}) \right|^p \, dx \right] \]
\[ \leq c \left[ \theta^p + \int_{B_{\theta \varrho}} \left| V_{|H_{\theta \varrho}|} (Dh - H_{\theta \varrho}) \right|^2 \, dx + \left( \int_{B_{\theta \varrho}} \left| V_{|H_{\theta \varrho}|} (Dh - H_{\theta \varrho}) \right|^2 \, dx \right)^{\frac{p}{2}} \right] \]
\[ \leq c \left[ \theta^p + \theta^{2\alpha_0} \int_{B_{\theta \varrho}} \left| V_{|H_{\varrho}|} (Dh - H_{\varrho}) \right|^2 \, dx + \theta^{p\alpha_0} \left( \int_{B_{\theta \varrho}} \left| V_{|H_{\varrho}|} (Dh - H_{\varrho}) \right|^2 \, dx \right)^{\frac{p}{2}} \right] \]
\[ \leq c \left[ \theta^p + \theta^{2\alpha_0} + \theta^{p\alpha_0} \right] \leq c(n, N, p) \theta^{\alpha_0}. \]

Re-scaling back to \( u \) we can re-write the preceding inequality as follows:

\[ \int_{B_{\theta \varrho}} \left| \frac{u - \ell}{\theta \varrho} \right|^p \, dx \leq cc_2^p \theta^{\alpha_0} \Phi(\varrho), \quad (3.62) \]

where we used the abbreviation for \( \ell \) introduced in (3.59). By Lemma 2.7 we can replace \( \ell \) by \( \ell_{\theta \varrho} \) in the preceding inequality. Using also (2.7) and (3.50) we obtain the following chain of inequalities:
\[ \Phi(\theta Q) \leq c \int_{B_{\delta Q}} \frac{|u - \ell \theta Q|}{\theta Q}^p \, dx \leq c \int_{B_{\delta Q}} \frac{|u - \ell|}{\theta Q}^p \, dx \leq c \theta^\alpha \Phi(Q) < c_3 \theta^\alpha \varepsilon_z, \]  

(3.63)

for a constant \( c_3 = c_3(n, N, p, v, L, \mu) \). Moreover, from (3.62) and the fact that \(|D \ell| \leq c \Phi(Q)^{\frac{1}{p}} - \) which is a consequence of (3.57) and the definition of \( \ell - \) we get

\[ \Psi_{\alpha}(\theta Q) \leq 2^p(\theta Q)^{-\alpha p} \int_{B_{\delta Q}} |u - \ell(0)|^p \, dx \leq 4^p(\theta Q)^{(1-\alpha)p} \left( \int_{B_{\delta Q}} \frac{|u - \ell|}{\theta Q}^p \, dx + |D \ell|^p \right) \leq c(\theta Q)^{(1-\alpha)p} \Phi(Q) \leq c_4(\theta Q)^{(1-\alpha)p}, \]  

(3.64)

where \( c_4 = c_4(n, N, p, v, L, \mu) \).

In any case we now choose \( \theta \) small enough to ensure that

\[ c_3 \theta^\alpha \leq 1 \quad \text{and} \quad c_4(\theta Q)^{(1-\alpha)p} < \kappa. \]

Then, \( \theta \) depends on \( n, N, p, v, L, \mu, \alpha, \kappa \). By the choice of \( \theta \) we infer from (3.60) and (3.61) in the case \( p > 2 \), respectively (3.63) and (3.64) in the case \( p < 2 \) that

\[ \Psi_{\alpha}(\theta Q) < \kappa \quad \text{and} \quad \Phi(\theta Q) < \varepsilon_z, \]

and this proves the assertion of the lemma, provided the smallness assumption (3.55) is satisfied. This will be achieved by the following choices of \( \varepsilon_z \) and \( \varrho_z \). We first choose \( \varepsilon_z \in (0, 1] \) such that

\[ \varepsilon_z \leq \left( \frac{\delta_0 \eta(\delta)}{4} \right)^p, \]

which ensures by (3.50) the validity of (3.55)\(_1\). For (3.55)\(_2\) we choose \( \varrho_z \in (0, \varrho_0] \) according to

\[ \left[ \omega(c_1 \varrho_z)^{\frac{q-p}{q}} + V(\varrho_z)^{\frac{q-p}{q}} \right]^{\min\left[ 1 - \frac{1}{p}, \frac{1}{p} \right]} \leq \frac{\delta_0}{4}. \]

From (3.52), (3.50) and the fact that \( \varepsilon_z \leq 1 \) we get

\[ \Psi_0(Q) = \phi \Psi_1(Q) \leq c_1 \phi \Phi(Q) \leq c_1 \varrho_z, \]

which together with the definition of \( H(Q) \) in (3.8) yields that

\[ H(Q)^{\min\left[ 1 - \frac{1}{p}, \frac{1}{p} \right]} \leq \left[ \omega(c_1 \varrho_z)^{\frac{q-p}{q}} + V(\varrho_z)^{\frac{q-p}{q}} \right]^{\min\left[ 1 - \frac{1}{p}, \frac{1}{p} \right]} \leq \frac{\delta_0}{4}, \]

ensuring the validity of (3.55)\(_2\). Finally, we comment on the dependency of \( \varepsilon_z \) and \( \varrho_z \) on the various parameters. By the choices from above \( \varepsilon_z \) initially depends on \( p, \delta, \delta_0, \eta(\cdot) \) and \( \varrho_z \) depends on \( p, q, \delta_0, c_1, \omega(\cdot), V(\cdot) \). Since \( c_1 = c_1(p, \mu) \), \( q = q(n, N, p, v, L) \) and \( \delta, \delta_0 \) depend on \( n, N, p, \theta \) and \( \theta \) depends on \( n, N, p, v, L, \mu, \alpha, \kappa \) we see that \( \varepsilon_z \) finally depends on \( n, N, p, v, L, \mu, \alpha, \kappa, \eta(\cdot) \) and \( \varrho_z \) depends on \( n, N, p, v, L, \mu, \alpha, \kappa, \omega(\cdot), V(\cdot) \). This completes the proof of the lemma. \( \square \)
3.5. Concluding the proof

In the following lemma we combine the degenerate and the non-degenerate regime provided in Lemmas 3.13 and 3.14. The difficulty results in the fact that within the iteration scheme the behavior of the solution in each point might change from degenerate to non-degenerate and we do not know when this occurs. On the other hand, the switching cannot occur the other way round, i.e. if the solution behaves non-degenerate for some radius, it stays non-degenerate for any smaller radius (see the proof of Lemma 3.13). Therefore the strategy will be as follows. We consider a sequence of nested balls and on each scale we check if we are in the degenerate or in the non-degenerate regime. If we are in the degenerate regime we apply Lemma 3.14 and continue the procedure. If we are in the non-degenerate regime we may apply Lemma 3.13 and stop the procedure.

**Lemma 3.15.** Suppose that the assumptions of Theorem 1.1 are satisfied and let $\alpha \in (0, 1)$. Then there exist constants $\varepsilon_\ast = \varepsilon_\ast (n, N, p, v, L, \alpha, \omega(\cdot), \tilde{\omega}(\cdot), \eta(\cdot)) \in (0, 1)$, $\kappa_\ast = \kappa_\ast (n, N, p, v, L, \alpha, \omega(\cdot), \tilde{\omega}(\cdot)) \in (0, 1)$, $\varrho_1 = \varrho_1 (n, N, p, v, L, \alpha, \omega(\cdot), \tilde{\omega}(\cdot), \mathbf{V}(\cdot)) \in (0, \varrho_0]$ and $c = c(n, N, p, v, L, \alpha, \omega(\cdot), \tilde{\omega}(\cdot))$ such that the conditions

$$
\Phi(x_0, \varrho) < \varepsilon_\ast \quad \text{and} \quad \Psi_\alpha(x_0, \varrho) < \kappa_\ast,
$$

for $B_\varrho(x_0) \subseteq \Omega$ with $\varrho \in (0, \varrho_1]$ imply

$$
\sup_{r \in (0, \varrho]} r^{p(1-\alpha)} \int_{B_r} |Du - (Du)_r|^p \, dx \leq c \quad \text{and} \quad \sup_{r \in (0, \varrho]} \Psi_\alpha(x_0, r) \leq c. \tag{3.66}
$$

**Proof.** Without loss of generality we assume $x_0 = 0$. We now fix the various constants in Lemmas 3.13 and 3.14. By $\mu_\ast = \mu_\ast (n, N, p, v, L, \alpha, \omega(\cdot)) \in (0, 1)$, $\kappa_\ast = \kappa_\ast (n, N, p, v, L, \alpha, \omega(\cdot), \tilde{\omega}(\cdot)) \in (0, 1)$, $\varrho_1 = \varrho_1 (n, N, p, v, L, \alpha, \omega(\cdot), \tilde{\omega}(\cdot), \mathbf{V}(\cdot)) \in (0, \varrho_0]$ and $\vartheta = \vartheta (n, N, p, v, L, \alpha) \in (0, \frac{1}{2}]$ we denote the corresponding constants from Lemma 3.13. Next, we choose $\mu = \mu_\ast$ and $\kappa = \kappa_\ast$ in Lemma 3.14. This fixes the constants $\varepsilon_\ast = \varepsilon_\ast (n, N, p, v, L, \alpha, \kappa_\ast, \mu_\ast, \eta(\cdot)) \in (0, 1)$, $\varrho_1 = \varrho_1 (n, N, p, v, L, \alpha, \kappa_\ast, \mu_\ast, \omega(\cdot), \tilde{\omega}(\cdot), \mathbf{V}(\cdot)) \in (0, \varrho_0]$ and $\vartheta = \vartheta (n, N, p, v, L, \alpha, \kappa_\ast, \mu_\ast) \in (0, \frac{1}{2}]$. Finally, we set $\varrho_1 : = \min \{ \varrho_1, \varrho_2 \}$. Note that tracing back the dependencies of the constants $\varepsilon_\ast, \kappa_\ast$ and $\varrho_1$ from above, they are exactly the ones stated in the lemma.

Now, we introduce the set of integers

$$
\mathbb{S} := \left\{ k \in \mathbb{N}_0: \mu_\ast |Df_{\theta^k \varrho}|^p \leq \Phi(\theta^k \varrho) \right\}.
$$

We distinguish the cases $\mathbb{S} = \mathbb{N}_0$ and $\mathbb{S} \neq \mathbb{N}_0$.

In the case $\mathbb{S} = \mathbb{N}_0$, we first prove by induction that

$$
\Phi(\theta^k \varrho) < \varepsilon_\ast \quad \text{and} \quad \Psi_\alpha(\theta^k \varrho) < \kappa_\ast \tag{D_k}
$$

holds for every $k \in \mathbb{N}_0$. Note that $(D_0)$ trivially holds by assumption $(3.65)$. Suppose now that $(D_k)$ holds for some $k \in \mathbb{N}_0$. Since $k \in \mathbb{S} = \mathbb{N}_0$ the assumptions $(3.50)$ of Lemma 3.14 are satisfied for $\varrho$ replaced by $\theta^k \varrho$ (recall that $\mu = \mu_\ast$). Therefore, the application of the lemma with $\kappa = \kappa_\ast$ ensures the validity of $(D_{k+1})$. Therefore, by induction $(D_k)$ is valid for any $k \in \mathbb{N}_0$.

Now, we come to the proof of $(3.66)$. For any $r \in (0, \varrho]$ we find some $k \in \mathbb{N}_0$ such that $\theta^{k+1} \varrho < r \leq \theta^k \varrho$. Then, by the second inequality in $(D_k)$ we have

$$
\Psi_\alpha(r) \leq 2^p \theta^{-n-\alpha p} \Psi_\alpha(\theta^k \varrho) < 2^p \theta^{-n-\alpha p} \kappa_\ast,
$$
proving the second assertion in (3.66). Moreover, with the help of (2.17), Hölder's inequality and the second inequality in (Dk) we infer

\[
|D \ell_{\dot{\theta}^k \dot{Q}}| \leq \frac{n + 2}{\theta^k \dot{Q}} |D \ell_{\dot{\theta}^k \dot{Q}}|^{\frac{1}{q^*}} \leq \frac{n + 2}{\theta^k \dot{Q}} \kappa^*.
\]  

(3.67)

Hence, using the Caccioppoli inequality from (3.9) and (Dk) and the preceding estimate we find

\[
\int_{B_{\dot{\theta}^k \dot{Q}/2}} |V| |D \ell_{\dot{\theta}^k \dot{Q}}|(Du - D \ell_{\dot{\theta}^k \dot{Q}})|^2 dx 
\leq c \left[ \Phi(\theta^k \dot{Q}) + |D \ell_{\dot{\theta}^k \dot{Q}}|^p \left[ \omega(\Psi_0(\theta^k \dot{Q})^{\frac{1}{q'}} + V(\theta^k \dot{Q})^{\frac{q-p}{q'}} \right] \min \{1 - \frac{1}{p'}, \frac{1}{p'} \} \right] 
\leq c \left[ \varepsilon + c(\theta^k \dot{Q})^{\frac{p}{q'}} + V(\theta^k \dot{Q})^{\frac{q-p}{q'}} \right] \min \{1 - \frac{1}{p'}, \frac{1}{p'} \} \right] 
\leq c \left[ \varepsilon + \frac{(n + 2)^p \kappa^*}{\theta^k \dot{Q}} \left[ \omega(\Psi_0(\theta^k \dot{Q})^{\frac{1}{q'}} + V(\theta^k \dot{Q})^{\frac{q-p}{q'}} \right] \min \{1 - \frac{1}{p'}, \frac{1}{p'} \} \right] 
\leq c(\theta^k \dot{Q})^{(\alpha - 1)p}.
\]

Now, for \( r \in (0, \dot{Q}/2) \) we choose \( k \in \mathbb{N}_0 \) such that \( \theta^{k+1} \dot{Q}/2 < r \leq \theta^k \dot{Q}/2 \). Then, (2.7) and the preceding estimate imply

\[
\int_{B_r} |Du - (Du)_r|^p dx \leq 2^p \theta^{-n} \int_{B_{\dot{\theta}^k \dot{Q}/2}} |V| |D \ell_{\dot{\theta}^k \dot{Q}}|(Du - D \ell_{\dot{\theta}^k \dot{Q}})|^2 dx 
\leq c \theta^{-n} \int_{B_{\dot{\theta}^k \dot{Q}/2}} |V| |D \ell_{\dot{\theta}^k \dot{Q}}|(Du - D \ell_{\dot{\theta}^k \dot{Q}})|^2 dx 
\leq c \theta^k \dot{Q}^{(\alpha - 1)p} \leq cr^{(\alpha - 1)},
\]

where \( c = c(n, N, p, \nu, \omega, \alpha, \omega_0, \tilde{\omega}(\cdot)) \). This proves the second claim in (3.66) (note that for \( r \in (\dot{Q}/2, \dot{Q}) \) we trivially obtain a similar estimate by enlarging the domain of integration from \( B_r \) to \( B_{\dot{Q}} \) and finishes the proof of the lemma in the case \( S = \mathbb{N}_0 \).

In the case \( S \neq \mathbb{N}_0 \) there exists \( k_0 := \min \mathbb{N}_0 \setminus S \). Since \( k \in S \) for any integer \( k < k_0 \) we can iterate as in the case \( S = \mathbb{N}_0 \) for \( k = 0, 1, \ldots, k_0 - 1 \) to infer that (Dk) holds for any \( k \leq k_0 \). By the definition of \( S \) we have

\[
\Phi(\theta^{k_0} \dot{Q}) < \mu_* |D \ell_{\dot{\theta}^{k_0} \dot{Q}}|^p,
\]

which together with the second inequality in (Dk) ensures that the assumptions (N0) of Lemma 3.13 are satisfied for \( \dot{Q} \) replaced by \( \theta^{k_0} \dot{Q} \). The application of the lemma yields that

\[
\Phi(\theta^\ell \theta^{k_0} \dot{Q}) < \mu_* \quad \text{and} \quad \Psi_\alpha(\theta^\ell \theta^{k_0} \dot{Q}) < \kappa^* \quad (N_\ell)
\]

holds for every \( \ell \in \mathbb{N}_0 \).
Now, we come to the proof of (3.66). Contrary to the case $\mathbb{S} = \mathbb{N}$ we now have to take into account that the behavior switches from degenerate to non-degenerate at scale $\vartheta^{k_0} Q$. The crucial point thereby is that the involved constants must be independent of the integer $k_0$ which depends on the particular point $x_0$ and cannot be controlled. We consider an arbitrary radius $r \in (0, \varrho]$. If $r \in (\vartheta^{k_0} Q/2, \varrho]$ we find $0 \leq k \leq k_0$ such that $\vartheta^{k+1} Q < r \leq \vartheta^k Q$ (note that $\vartheta \leq \frac{1}{2}$) and therefore we can argue exactly as in the case $\mathbb{S} = \mathbb{N}$. In the second case when $r \in (0, \vartheta^{k_0} Q/2]$ we find $\ell \in \mathbb{N}_0$ such that $\vartheta^{\ell+1} \vartheta^{k_0} Q < r \leq \vartheta^\ell \vartheta^{k_0} Q$. Then, from the second inequality in $(N_k)$ we obtain

$$ \Psi_\alpha(r) \leq 2^p \vartheta^{-n-\alpha p} \Psi_\alpha(\vartheta^{\ell} \vartheta^{k_0} Q) < 2^p \vartheta^{-n-\alpha p} \kappa_*, $$

proving the second assertion in (3.66). Moreover, with the help of (2.17) and the second inequality in $(N_k)$ we infer as in (3.67) that

$$ |D \ell^{\ell} (\vartheta^{k_0} Q| \leq \frac{n+2}{(\vartheta^{\ell} \vartheta^{k_0} Q)^{1-\alpha} k_*^\frac{2}{p}}. $$

Combining this with the Caccioppoli inequality from (3.9) and $(N_k)$ we conclude that

$$ \int_{B_{\vartheta^{\ell} \vartheta^{k_0} Q/2}} |V_{\ell^{\ell} (\vartheta^{k_0} Q}(Du - D \ell^{\ell} (\vartheta^{k_0} Q)|^2 dx $$

$$ \leq c \left[ \Phi(\vartheta^{\ell} \vartheta^{k_0} Q) + |D \ell^{\ell} (\vartheta^{k_0} Q)|^p \left[ \omega(\Psi_\alpha(\vartheta^{\ell} \vartheta^{k_0} Q)^{\frac{1}{2}} p - \frac{p-1}{p}} \right] + \Psi_\alpha(\vartheta^{\ell} \vartheta^{k_0} Q)^{\frac{q-p}{q}} \right]^{\min(1-\frac{1}{p}, \frac{1}{p})} \right] $$

$$ \leq \frac{c(n+2)k_*}{(\vartheta^{\ell} \vartheta^{k_0} Q)^{1-\alpha} p} \left[ \mu_\alpha + \left( \omega(\kappa_*)^{\frac{q-p}{q}} + \Psi_\alpha(\vartheta^{\ell} \vartheta^{k_0} Q)^{\frac{q-p}{q}} \right) \right]^{\min(1-\frac{1}{p}, \frac{1}{p})}. $$

Now, for $r \in (0, \vartheta^{k_0} Q/2]$ we choose $\ell \in \mathbb{N}_0$ such that $\vartheta^{\ell+1} \vartheta^{k_0} Q/2 < r \leq \vartheta^\ell \vartheta^{k_0} Q/2$. Then (2.7) and the preceding estimate imply

$$ \int_{B_r} |Du - (Du)_r|^p dx \leq 2^p \vartheta^{-n} \int_{B_{\vartheta^{\ell} \vartheta^{k_0} Q/2}} |Du - D \ell^{\ell} (\vartheta^{k_0} Q)|^p dx $$

$$ \leq c \vartheta^{-n} \int_{B_{\vartheta^{\ell} \vartheta^{k_0} Q/2}} |V_{\ell^{\ell} (\vartheta^{k_0} Q}(Du - D \ell^{\ell} (\vartheta^{k_0} Q)|^2 dx $$

$$ + \chi_{p<2} |D \ell^{\ell} (\vartheta^{k_0} Q)|^p \left[ \omega(\vartheta^{\ell} \vartheta^{k_0} Q)^{\frac{q-p}{q}} \right]^p dx $$

$$ \leq c(\vartheta^{\ell} \vartheta^{k_0} Q)^p(\vartheta^{\ell} \vartheta^{k_0} Q)^p \leq cr^p(\vartheta^{\ell} \vartheta^{k_0} Q)^p, $$

where $c = c(n, N, \vartheta, \nu, \lambda, \alpha)$. This proves the second claim in (3.66) and therefore finishes the proof of the lemma. $\square$
3.6. Proof of Theorem 1.1

We fix an arbitrary $\alpha \in (0, 1)$. By $\varepsilon_2 = \varepsilon_2(n, N, p, \nu, L, \alpha, \omega(-), \tilde{\omega}(-), \eta(-)) \in (0, 1]$, $\kappa_* = \kappa_*(n, N, p, \nu, L, \alpha, \omega(-), \tilde{\omega}(-)) \in (0, 1]$ and $\varrho_1 = \varrho_1(n, N, p, \nu, L, \alpha, \omega(-), \tilde{\omega}(-), V(-)) \in (0, \varrho_0]$ we denote the corresponding constants from Lemma 3.15. Note that by Lebesgue's differentiation theorem, there holds $|\Sigma_1 \cup \Sigma_2| = 0$. Consequently, it suffices to show that every $x_0 \in \Omega \setminus (\Sigma_1 \cup \Sigma_2)$ is a regular point. For this we note first that for every $0 < \varrho < \text{dist}(x_0, \partial \Omega)$, the bound (2.17) and Poincaré's inequality imply

$$|D\ell_{x_0, q} - (Du)_{x_0, q}| \leq \frac{n + 2}{\varrho} \int_{B_{\varrho}(x_0)} |u - (u)_{x_0, q} - (Du)_{x_0, q}(x - x_0)| \, dx$$

$$\leq c(n) \int_{B_{\varrho}(x_0)} |Du - (Du)_{x_0, q}| \, dx.$$

Consequently, by Lemma 2.3 and (2.7) we obtain

$$\Phi(x_0, q) \leq c \int_{B_{\varrho}(x_0)} |V| |D\ell_{x_0, q}| |(Du - D\ell_{x_0, q})|^2 \, dx$$

$$\leq c \int_{B_{\varrho}(x_0)} |Du - D\ell_{x_0, q}|^p + \chi_{p>2} |D\ell_{x_0, q}|^{p-2} |Du - D\ell_{x_0, q}|^2 \, dx$$

$$\leq c \int_{B_{\varrho}(x_0)} |Du - (Du)_{x_0, q}|^p + \chi_{p>2} |(Du)_{x_0, q}|^{p-2} |Du - (Du)_{x_0, q}|^2 \, dx, \quad (3.68)$$

where $c = c(n, N, p)$. Moreover, from Poincaré's inequality we have

$$\Psi_{\alpha}(x_0, q) \leq q^{(1-\alpha)p} \int_{B_{\varrho}(x_0)} |Du|^p \, dx$$

$$\leq \int_{B_{\varrho}(x_0)} |Du - (Du)_{x_0, q}|^p \, dx + q^{(1-\alpha)p} |(Du)_{x_0, q}|^p. \quad (3.69)$$

Keeping in mind the definition of the sets $\Sigma_1$ and $\Sigma_2$, the estimates (3.68) and (3.69) imply the existence of a radius $0 < \varrho < \min\{\varrho_1, \text{dist}(x_0, \partial \Omega)\}$ with

$$\Phi(x_0, q) < \varepsilon_2 \quad \text{and} \quad \Psi_{\alpha}(x_0, q) < \kappa_*.$$

Using the absolute continuity of the integral, we can find a neighborhood $U \subseteq \Omega$ of $x_0$ such that

$$\Phi(x, q) < \varepsilon_2 \quad \text{and} \quad \Psi_{\alpha}(x, q) < \kappa_* \quad \text{for all} \quad x \in U.$$

Applying Lemma 3.15 in any point $x \in U$ thus yields

$$\sup_{x \in U, r \in (0, \varrho]} r^{p(1-\alpha)} \int_{B_r(x)} |Du - (Du)_{x, r}|^p \, dx < \infty \quad \text{and} \quad \sup_{x \in U, r \in (0, \varrho]} \Psi_{\alpha}(x, r) < \infty.$$
The first assertion ensures that $Du \in L^{p,\gamma}(U, \mathbb{R}^N)$, where $\gamma = n - p(1 - \alpha)$. Note that $\gamma$ can be chosen arbitrarily close to $n$, since $\alpha \in (0, 1)$ is arbitrary. Moreover, from the second one together with the definition of $\Psi(x, r)$ and Campanato’s characterization of Hölder continuous functions we deduce that $u \in C^{0,\alpha}_\text{loc}(U, \mathbb{R}^N)$. The proof of Theorem 1.1 is thus complete.

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References}


