Set Systems with Finite Chromatic Number

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1. INTRODUCTION

In this paper we consider the chromatic number of set systems, i.e. the minimal cardinality $k$ for which the ground set can be colored by $k$ colors with no monocolored member of the system. We are especially interested in the case when $k$ is finite (but the system may be infinite). Another property widely investigated is the transversal property, i.e. when the system has a one-to-one choice function. If the system consists of finitely many finite sets, a very satisfactory characterization is known; such an $\mathcal{H}$ has the transversal property if the union of any $t$ members of $\mathcal{H}$ has at least $t$ elements [9]. Compactness arguments extend this to the case when an arbitrary number of finite sets is given. For systems consisting of both finite and infinite sets a (much more complex) characterization has recently been proved by Aharoni, Nash-Williams, and Shelah [3], successfully using and extending the tools elaborated in [13]. For applications, see also [1].

Property B, which is the set theory nickname for being 2-chromatic, was introduced in [4,12] and discovered again by P. Erdős and A. Hajnal in [5], where it was observed that it is very similar to the transversal property: a number of results are true for both properties, at least for systems of infinite sets. One of the reasons for this similarity was disclosed in [10]: if a system of infinite sets has the transversal property, it has property B as well. Other recent results show some more asymmetry [2]. One can meditate about the possibility of an Aharoni, Nash-Williams, and Shelah type characterization of property B. Nevertheless, even for finite families, there is no real hope of finding a nice characterization, as property B is known to be NP-complete (see [8]). One can prove the folklore observation that a $\geq 3$-chromatic finite system contains two members intersecting in a singleton, and this was extended to a surprising (but exponential) characterization by Linial and Tarsi (see [11]).

Our suggestion is that, modulo finite systems, there must exist a certain characterization for $k$-chromatic systems, if the system is not too large (if it is countable, say). In this paper we try to collect some evidence supporting this belief. In the first part we construct some infinite, minimal $k$-chromatic systems; of finite sets and one infinite set (Statement 1) and of infinite sets (Theorem 1). In Section 3 we define an excluded-subfamily characterization of being $k$-chromatic, which works for small subsystems of $P(\omega)$, and has at least one interesting consequence (see Corollary 1). In Section 4 we show that it is consistent that there is a system of $\aleph_1$ countable sets without property B, but not in $P(\omega)$.

2. MINIMAL $k$-CHROMATIC FAMILIES

A by-product of the work on characterizing the transversal property is the result that a countable family without the transversal property has a minimal such subfamily (Podewski and Steffens [13]). We think that the same is true for countable non-$k$-chromatic families as well ($k < \omega$). It is not clear at all what minimal non-$k$-chromatic families look like. There are, of course, the finite families of finite sets; we think of them as ‘known’. By compactness, all minimal families with just finite sets are finite. An infinite family with exactly one infinite member which is minimal...
3-chromatic is the following \{\{0, 1\}, \{0, 2\}, \ldots, \{0, n\}, \ldots, \{1, 2, \ldots\}\}. P. Erdős suggested the following generalization of this construction:

**Statement 1.** For \(2 \leq r < \omega\), \(2 \leq k < \omega\) there is a minimal \((k + 1)\)-chromatic system containing exactly one infinite set and infinitely many \(r\)-element sets.

**Proof.** Choose \(k\) disjoint \(r\)-element sets, \(A_1, \ldots, A_k\). Put \(\mathcal{H} = \{B \subseteq A_1 \cup \cdots \cup A_k : |B| = r, B \neq A_i\) for \(1 \leq i \leq k\}\). \(\mathcal{H}\) is clearly \(k\)-chromatic. We show that it has only one good \(k\)-coloration. Assume that \(C_1, \ldots, C_k\) are the color classes of a good coloring. By the construction of \(\mathcal{H}\), for every \(j\), either \(C_i \subseteq A_i\), for some \(i\), or \(|C_i| \leq r - 1\). If the first possibility occurs \(t\) times, we obtain \(t \cdot r + (k - t) \cdot (r - 1) \geq k \cdot r\), i.e. \(t = k\). This is only possible if \(C_i = A_j\), i.e. the coloring is a permutation of \(A_1, \ldots, A_k\).

Select \(x, y \in A_1\). Our system \(\mathcal{H}\) has the property that there is a good \(k\)-coloring of it, and in every good coloring with \(k\) colors \(x\) and \(y\) obtain the same color. Now choose a minimal \(\mathcal{H} \subseteq \mathcal{K}\) with these properties. Next, choose disjoint sets \(X, S_i (1 \leq i < \omega)\) with \(|X| = \omega\), \(|S_i| = kr - 2\), \(X = \{x_0, x_1, \ldots\}\). For \(1 \leq i < \omega\) copy our system \(\mathcal{K}\) on \(\{x_0, x_i\} \cup S_i\), with \(x_0, x_i\) as the distinguished elements. We claim that the system \(\mathcal{Y} = \{X\} \cup \{\mathcal{H}_i : 1 \leq i < \omega\}\) is a minimal \((k + 1)\)-chromatic family. If \(f\) is a good \(k\)-coloring of \(\bigcup \{\mathcal{H}_i : 1 \leq i < \omega\}\), then \(f(x_0) = f(x_i)\) for \(1 \leq i < \omega\), i.e. \(X\) is monocolored. Removing \(X\), there is a good coloring with \(k\) colors. If we remove \(A \in \mathcal{H}_i\), there is a good coloring \(f\) of \(\mathcal{H}_i - \{A\}\) with \(f(x_0) \neq f(x_i)\), and this can be extended to \(\mathcal{H}_j\) (\(j \neq i\)) with \(f(x_0) = f(x_i)\). Also, \(X\) is not monocolored by this extension.

Notice the following (folklore?) observation.

**Statement 2.** If \(\mathcal{H}\) is a minimal \(k\)-chromatic family, \(A \in \mathcal{H}\), \(x \in A\), then there exist members \(B_1, \ldots, B_{k-2}\) such that \(\{x\} = A \cap B_1 = B_1 \cap B_j\) for \(1 \leq i < j \leq k - 2\).

**Proof.** By minimality, \(\mathcal{H} - \{A\}\) admits a good \((k - 1)\)-coloring, say, \(f : \bigcup \mathcal{H} \to \{0, 1, \ldots, k - 2\}\). \(A\) is necessarily monocolored by \(f\) (as \(\mathcal{H}\) is \(k\)-chromatic), say \(A \subseteq f^{-1}(0)\). If we re-color \(x\) as \(f(x) = i\), by minimality this must create an \(i\)-colored set, \(B_i\). Now the sets \(A, B_i (1 \leq i \leq k - 2)\) are as wanted.

To show that there exists a minimal 3-chromatic family of infinite sets we give the following construction of R. Aharoni (Haifa), which is included with his kind permission. This is a result of [6] (misleadingly called there Conjecture \(G^*\)), but the proof is simpler than those given there.

**Statement 3.** There exists a minimal 3-chromatic family of infinite, countable sets.

**Proof.** Take a countable family \(\mathcal{F} = \{A_i : i < \omega\} \subseteq P(\omega)\) with \(A_i \cap A_j \neq \emptyset\), \(A_i \neq A_j\) (\(i \neq j\)), \(A_i \cap \{0, 1, \ldots, i\} = \emptyset\). Choose an \(\mathcal{H} \subseteq P(\omega)\), \(\mathcal{H} \supseteq \mathcal{F}\), with the following properties: if \(A, B \in \mathcal{H}\), \(A \neq B\), then \(A \cap B \neq \emptyset\), \(A \neq B\); moreover, \(\mathcal{H}\) is maximal with respect to these properties. This is possible: by Zorn’s lemma. If \(A \in \mathcal{H}\), and \(A \neq A_i\) (\(i < \omega\) then, as \(A\) meets all the \(A_i\)'s, \(A \cap (\omega - \{0, 1, \ldots, i\})\) is non-empty, so \(A\) is infinite. We show that \(\mathcal{H}\) is 3-chromatic. If it were 2-chromatic, let \(X\) be one of the color classes; then \(\mathcal{H} \cup \{X\}\) would be a proper extension of \(\mathcal{H}\). If we remove \(A\) from \(\mathcal{H}\), then the coloring \(A, \omega - A\) is a good 2-coloring.

It is not clear how large a minimal 3-chromatic family can be.
Conjecture 1. There is a minimal 3-chromatic family of countable sets, \( \mathcal{H} \), with \( \bigcup \mathcal{H} = \omega_1 \).

One would think that every \( k \)-chromatic family contains a minimal \( k \)-chromatic subfamily. However, this is not the case.

Statement 4. (a) There is a \( k \)-chromatic system \( \mathcal{H} \subseteq P(\omega) \), with \( A \cap B \) either empty or infinite for \( A, B \in \mathcal{H} \).

(b) There is a \( k \)-chromatic family \( \mathcal{H} \subseteq P(\omega) \) such that no \( \mathcal{H}' \subseteq \mathcal{H} \) is minimal \( k \)-chromatic.

Proof. (a) Let \( A_0, \ldots, A_{k-1} \) be disjoint, infinite sets, and let \( D_i \) be a non-trivial ultrafilter on \( A_i \). Put \( \mathcal{H} = \{ H \subseteq A_0 \cup \ldots \cup A_{k-1} : \text{there are } i < j < k \text{ such that } H \subseteq A_i \cup A_j, H \cap A_i \in D_i, H \cap A_j \in D_j \} \). Part (b) follows from (a) and Statement 2.

Surprisingly, the following problem seems to be very hard:

Conjecture 2. There is an almost-disjoint family \( \mathcal{H} \subseteq [X]^{\omega} \) without property B such that if \( A, B \in \mathcal{H} \) then \( |A \cap B| \neq 1 \).

Notice the following easy result:

Statement 5. If \( \mathcal{H} \subseteq [X]^{\omega} \) has \( |A \cap B| \neq 1 \) for \( A, B \in \mathcal{H} \) then \( \mathcal{H} \) is at most \( \omega \)-chromatic.

Proof. Let \( \{ A_i : i \in I \} \) be a maximal family of pairwise disjoint sets from \( \mathcal{H} \). Let \( f: \bigcup A_i \to \omega \) be a coloring with \( |f^{-1}(n) \cap A_i| = 1 \) for \( i \in I, n < \omega \). This \( f \) is a good coloring for \( \mathcal{H} \).

Conjecture 3. For \( 2 \leq k < \omega \) there are arbitrarily large chromatic systems \( \mathcal{H} \) with \( |A| = \omega, |A \cap B| \neq k \) for \( A, B \in \mathcal{H} \).

Next we extend Statement 3.

Theorem 1. If CH holds, then for every \( 2 \leq k < \omega \) there exists a minimal \((k + 1)\)-chromatic subfamily of \([\omega]^{\omega}\).

Proof. Let \( A \subseteq^* B, A =^* B \) denote that \( |A - B| < \omega, |A \triangle B| < \omega \) hold, respectively (\( \triangle \) is the symmetric difference). Two functions \( f_1, f_2: \omega \to k \) are equivalent (in short, \( f_1 \sim f_2 \)), if \( f_1^{-1}(i) = f_2^{-1}(\pi(i)) \) for \( i < k \) (\( \pi \) a permutation on \( k \)). Let a maximal family of pairwise non-equivalent functions be enumerated as \( \{ f_\alpha : \alpha < \omega_1 \} \) (this is the point where CH is used). Our strategy is the following. By transfinite recursion we are going to construct for every \( \alpha < \omega_1 \) a collection \( \{ A_{\alpha, n} : n < \omega \} \subseteq [\omega]^{\omega} \), and functions \( g_{\alpha, n}: \omega \to k \), whenever there is an \( f \sim f_\alpha \) which is a good coloring of \( \{ A_{\beta, m} : \beta < \alpha, m < \omega \} \). If no such \( f \) exists, we do not do anything in the \( \alpha \)th step.

If such an \( f \) does exist, we may assume, for notational simplicity, that \( f = f_\alpha \). When constructing our sets and functions, the following conditions will be satisfied:

1. \( A_{\alpha, n} \subseteq f_\alpha^{-1}(0), (\alpha < \omega_1, n < \omega) \);
2. \( g_{\alpha, n} \) is a good coloring of \( \{ A_{\beta, m} : \beta < \omega_1, m < \omega, (\beta, m) \neq (\alpha, n) \} \);
3. \( A_{\alpha, n} \subseteq g_{\alpha, n}^{-1}(0) (\alpha < \omega_1, n < \omega) \);
4. for \( 0 < i < k, n < \omega, x \in A_{\alpha, n} \), \( s \in [\omega]^{< \omega} \), there is an \( m < \omega \) with \( A_{\alpha, m} \cap s = \emptyset, x \in A_{\alpha, m}, A_{\alpha, m} - \{ x \} \subseteq g_{\alpha, n}(i) \);
(5) if \( l < k \), and
\[
g^{-1}_{\beta_0 n_0}(i_0) \cdots \cup g^{-1}_{\beta_t n_t}(i_t) = ^* \omega
\]
then \( l = k - 1 \), \( \beta_0 = \cdots = \beta_t \), \( n_0 = \cdots = n_t \), and \( \{i_0, \ldots, i_t\} = k \).

We can prove some statements on systems satisfying (1)–(5).

CLAIM 1. If \( f \sim g_{\alpha,n} \), then \( f \) is not a good coloring of \( \{A_{\alpha,m} : m < \omega \} \).

PROOF. We can assume that \( f^{-1}(i) = g^{-1}_{\alpha,n}(i) \) for \( i < k \). If \( A_{\alpha,n} \subseteq f^{-1}(0) \), we are done. Otherwise, choose \( x \in A_{\alpha,n} - f^{-1}(0) \), \( i = f(x) \), \( s = g^{-1}_{\alpha,n}(i) - f^{-1}(i) \). Clearly, \( i \neq 0 \), and \( s \) is a finite set. Select \( m \) according to (4) and we obtain \( A_{\alpha,m} \subseteq f^{-1}(i) \), a contradiction.

Assume now that the construction is given up to the \( a \)th step and \( f_{\alpha} \) is a good coloring of \( \{A_{\alpha,m} : m < \omega \} \).

CLAIM 2. There is an \( i < k \) such that for no \( \beta < \alpha \), \( n < \omega \), \( j < k \), \( f_{\alpha}^{-1}(i) \subseteq * g_{\beta,n}(j) \) holds.

PROOF. Assume otherwise. Choosing for every \( i < k \) a triplet \( \langle \beta, n, j \rangle \), we obtain an equality as in (5), i.e. there are \( \beta < \alpha \), \( n < \omega \) such that for \( i < k \) there is a \( j < k \) with \( f_{\alpha}^{-1}(i) \subseteq * g_{\beta,n}(j) \). As \( f_{\alpha} \), \( g_{\beta,n} \) both are \( k \)-member partitions, \( f_{\alpha} \sim g_{\beta,n} \), a contradiction to Claim 1.

Again, by slightly changing \( f_{\alpha} \), we assume that Claim 2 holds with \( i = 0 \). This means that for every \( \beta < \alpha \), \( n < \omega \) there are at least two \( j \)’s with \( |f_{\alpha}^{-1}(0) \cap g_{\beta,n}(j)| = \omega \).

Next we notice that:

(6) for \( l \leq k \), if \( \{\langle \beta_t, n_t \rangle : t < l \} \) are different, \( i_0, \ldots, i_{l-1} < k \), then
\[
|g^{-1}_{\beta_0 n_0}(i_0) \cap \cdots \cap g^{-1}_{\beta_{l-1},n_{l-1}}(i_{l-1})| = \omega
\]
implies condition (5). We will, therefore, construct our objects satisfying (1)–(4) and (6).

If \( \{A_{\beta,n}, g_{\beta,n} : \beta < \alpha \), \( n < \omega \} \) are given with (1)–(4) and (6), it is easy to add \( \{g_{\alpha,n} : n < \omega \} \) satisfying (6). We only need to ensure that certain (countably many) intersections are infinite, and this can be done by a diagonalization. We can even simultaneously satisfy (2), i.e. make sure that the new \( g_{\alpha,n} \) functions are good colorings for \( \{A_{\beta,m} : \beta < \alpha \), \( m < \omega \} \).

A similarly easy task is to construct a system \( \{A_{\alpha,n} : n < \omega \} \) satisfying (1) and
(4') if \( x \in A_{\alpha,n} \), \( s \in [\omega]^{< \omega} \), then there is an \( m < \omega \) with \( A_{\alpha,m} \cap s = \emptyset \), \( A_{\alpha,n} \cap A_{\alpha,m} = \{x\} \).

The best way of doing this is to associate an \( m < \omega \) to every instance \( \langle n, x, s \rangle \) of (4'), and then to put every \( y \in f^{-1}(0) \) into at most one \( A_{\alpha,m} \) with \( m \leq y \). We can also ensure (2), i.e. that the old \( g_{\beta,m} \) functions are good colorings of the newly created sets, by Claim 2.

We can combine these tasks into one diagonal argument solving all problems. Here, the only problem may be caused by (4), i.e. if we put a \( y \in f^{-1}(0) \) into \( A_{\alpha,n} \) then this forces, for some \( m \)’s, that \( g_{\alpha,m}(y) = i \) and also \( g_{\alpha,n}(y) = 0 \). May \( g_{\alpha,m} \) still satisfy (2)? Yes, as if \( A_{\alpha,m} \cap A_{\alpha,n} = \{x\} \), then \( g_{\alpha,m}(x) = 0 \), \( g_{\alpha,m}(y) = i \neq 0 \), so \( A_{\alpha,n} \) is well colored.

Finally we show that our system is minimal \((k + 1)\)-chromatic. By (2), if we remove a set from our system, there is a good \( k \)-coloring of the remaining system. Now assume
that $f$ is a good $k$-coloring. $f \sim f_\alpha$ for some $\alpha$, and by the construction for this $\alpha$, $A_{\alpha,n}$, $g_{\alpha,n}$ are defined, although not necessarily $f = f_\alpha$. By (1) and (4) there is an $m < \omega$ with $A_{\alpha,m} \subseteq f^{-1}(0)$, a contradiction.

For Martin's axiom see [7].

**Theorem 2 (MA).** Assume $\mathcal{H} \subseteq P(\omega)$, $|\mathcal{H}| \leq \kappa$ is a family with
\[(7) |A| \geq 2 \text{ for } A \in \mathcal{H};
\]
\[(8) \text{there are no } A_1, \ldots, A_k \in \mathcal{H}, x \in \omega \text{ with } A_i \cap A_j = \{x\} \text{ (1} \leq i < j \leq k); \text{ then } \mathcal{H} \text{ is } 

\leq k\text{-chromatic.}

**Proof.** First we define a partially ordered set, $P$. An element of $P$ is a function $p$, with $\text{Dom} p$ a finite subset of $\omega$, $\text{Rng} p \sim k$, and, if $F \in \mathcal{H}$, then $F \notin p^{-1}(i) (i < k)$. We order $P$ with $p \prec q$ if $p \preceq q$. Enumerate $\mathcal{H}$ as $\{H_\alpha : \alpha < \omega_1\}$, and for $\alpha < \omega_1$ put $D_\alpha = \{p \in P : |p''H_\alpha| \geq 2\}$. We show that every $D_\alpha$ is dense. Let $p \in P$. Assume that $p''(H_\alpha \cap \text{Dom} p)$ is an one-element set, say, consisting of $0$. Choose $x \in H_\alpha - \text{Dom} p$, by hypothesis, and try to extend $p$ to a $q$, by $q_i(x) = i (1 \leq i < k - 1)$. If $q_i$ is not a condition, there is a $B_i \subseteq q_i^{-1}(i)$, so $\{A, B_i : 1 \leq i < k - 1\}$ form a forbidden subsystem described in (8).

As $|P| = \omega$, $P$ is c.c.c. By MA, there is a generic $G \subseteq P$, $D_\alpha \cap G = \emptyset$ for $\alpha < \omega_1$, so $f = \bigcup G$ partially colors $\omega$, each $H$ obtaining at least two colors.

3. Toward a Characterization of $k$-Colorability

In this section we try to give a characterization of countable $k$-chromatic systems. This 'characterization' is similar in spirit to those for the transversal property as it gives obstructions; but it is different, as the structure of the obstructions is not clear.

**Definition.** A $k$-$\alpha$-obstruction is a minimal $(k + \alpha)$-chromatic family of finite sets. For $\alpha > 0$, a $k$-$\alpha$-obstruction is a family $\mathcal{F} = \{A\} \cup \bigcup \{\mathcal{F}_i : i < \omega\}$, where $A$ is a countable infinite set, $A = \bigcup \{B_i : i < \omega\}$ $B_0 \subseteq B_1 \subseteq \ldots$, $|B_i| < \omega$, and $|B_i| \cup \mathcal{F}_i$ contains a $k$-$\beta_i$-obstruction for some $\beta_i < \alpha$. A $k$-obstruction is a $k$-$\alpha$-obstruction for some $\alpha < \omega_1$.

**Statement 6.** A $k$-obstruction is not $\leq k$-chromatic.

**Proof.** By induction on $\alpha < \omega_1$ we show that a $k$-$\alpha$-obstruction is not $k$-colorable. For $\alpha = 0$ this is obvious from the definition. Assume $\mathcal{F} = \{A\} \cup \bigcup \{\mathcal{F}_i : i < \omega\}$ is a $k$-$\alpha$-obstruction and the statement holds for $\beta < \alpha$. If $f$ is a good $k$-coloring of $\mathcal{F}$, then $f$ cannot color any of the systems $\{B_i\} \cup \mathcal{F}_i$, so $f$ is always a monocoloring on $B_i$ and hence $f$ monocolours $A$.

**Theorem 3.** (MA). If $\mathcal{F} \subseteq P(\omega)$ does not contain a $k$-obstruction, $|\mathcal{F}| \leq \kappa$, then $\mathcal{F}$ is $\leq k$-chromatic.

**Proof.** Put $p \in P$ if $p$ is a function, $\text{Dom} p = [\mathcal{F}]^{<\omega}$, $p(F) \in [F]^{<\omega}$ for $F \in \text{Dom} (p)$ and the system $\mathcal{F}_p = \mathcal{F} - \text{Dom} p \cup \text{Rng} p$ does not contain an obstruction. The idea is that we try carefully to change every $F \in \mathcal{F}$ to a finite subset of it while keeping the condition alive. We put $p \leq q$ just in case $p \preceq q$.

First we check c.c.c.. If $p_1$, $p_2$ are two members of $P$, with $\text{Rng} p_1 = \text{Rng} p_2$, $R_{p_1} = R_{p_2}$, $R_{p_1} | S = R_{p_2} | S$, where $S = \text{Dom} p_1 \cap \text{Dom} p_2$ (any uncountable set of conditions contains
such a pair), then taking \( q = p_1 \cup p_2, \) \( \mathcal{F}_q \subseteq \mathcal{F}_{p_1}, \mathcal{F}_{p_2}, \) therefore \( \mathcal{F}_q \) cannot contain an obstruction; so \( q \) is a condition, and therefore a common extension of \( p_1, p_2. \)

Next we show that if \( F \in \mathcal{F} \), then \( D_F = \{ p \in P : F \in \text{Dom} \ p \} \) is dense. This is obvious if \( F \) is finite. If \( |F| = \omega \), take an increasing union \( F = \bigcup \{ B_i : i < \omega \} \) with \( |B_i| < \omega. \) If \( p \in P, F \notin \text{Dom} \ p \) and the function \( p' = p \cup \langle F, B_i \rangle \) is not a member of \( P, \) then the system \( \mathcal{F} - \{ F \} \cup \{ B_i \} \) contains a \( k \)-obstruction, while \( \mathcal{F}' \) does not: here \( \mathcal{F}' = \mathcal{F} - \text{Dom} \ p \cup \text{Rng} \ p. \) But this gives the result that there is a \( k \)-obstruction in \( \mathcal{F}' \), a contradiction.

By \( \text{MA}_k, \) there is a generic subset \( G \subseteq P \) with \( G \cap D_F \neq \emptyset \) for \( F \in \mathcal{F}. \) Then \( \bigcup G \) gives a function \( h \) with \( \text{Dom} \ h = \mathcal{F}, \text{Rng} \ h \subseteq [\omega]^{<\omega}. \) \( \text{Rng} \ h \) is \( \leq_k \)-chromatic, since for every finite subsystem \( h(F_1), \ldots, h(F_n) \) there is a \( p \in G \) with \( F_i \in \text{Dom} \ p, \ p(F_i) = h(F_i), \) \( \mathcal{F} - \{ F_i \} \cup \{ h(F_i) : i \leq r \} \) not containing a \( k \)-obstruction; in particular, \( \{ h(F_i) : i \leq r \} \) is \( \leq_k \)-chromatic.

**Corollary 1.** (\( \text{MA}_k \)). If \( \mathcal{H} \subseteq P(\omega), \) \( |\mathcal{H}| \leq \kappa, \) then \( \mathcal{H} \) is \( \leq_k \)-chromatic if every countable subfamily of \( \mathcal{H} \) is \( \leq_k \)-chromatic.

**4. AN INCOMPACTNESS RESULT**

In this section we prove that there may be a difference between \( \omega \) and \( \omega_1 \) as ground sets. It is consistent that \( \aleph_1 \) sets in \([\omega]^{\omega} \) always have property \( B, \) but not necessarily on \( \omega_1. \) To this end, we show that Ostaszewski's principle, \( \clubsuit \) is consistent with a weak form of \( \text{MA}_{\omega_1}. \) Let \( \clubsuit \) denote the following statement: there are sets \( \{ T_\alpha : \alpha < \omega_1, \alpha \text{ limit} \} \) with \( T_\alpha \) cofinal in \( \alpha \) with order type \( \omega, \) and, if \( X \) is cofinal in \( \omega_1, \) there is an \( \alpha \) with \( T_\alpha \subseteq X. \) Let \( M \) denote \( \text{MA}_{\omega_1} \) restricted to the case when the partial order is countable (and \( \aleph_1 \) dense sets are given to meet). For our next theorem we give, instead of the original, a much shorter proof given by J. E. Baumgartner, which is included here with his kind permission.

**Theorem 4.** If \( \text{ZF} \) is consistent, then so is \( \text{ZFC} + \clubsuit + M \) (and \( 2^{\aleph_0} = \aleph_2 \)).

**Proof.** We use Shelah's original model for the consistency of \( \text{ZFC} + \clubsuit + 2^{\aleph_0} = \aleph_2 \) and show that \( M \) also holds in it [14]. Let \( V \) model \( \text{ZFC} + \text{GCH}. \) The applied notion of forcing is \( P \ast Q \) where \( P \) adds \( \aleph_3 \) Cohen subsets of \( \omega_1 \) with countable conditions, say \( \{ c_\alpha : \alpha < \omega_3 \}. \) \( Q \) collapses \( \omega_1 \) onto \( \omega \) with finite conditions. Notice that, as \( Q \) is the same as defined in \( V \) or in \( V^P, \) \( P \ast Q \) is in fact a direct product, \( P \diamond Q. \) Assume that \( \{ D_\xi : \xi < \omega_2 \} \) are dense sets of a countable partial ordering, as \( P \ast Q \) has the \( \aleph_2 \)-c.c., there is an \( \alpha < \omega_3 \) with \( \{ D_\xi : \xi < \omega_2 \} \subseteq V[\{ c_\beta : \beta < \alpha \}][f], \) where \( f \) is the generic \( \omega \rightarrow \omega_1^{<\omega} \) collapsing map. If \( R \) is the countable condition order for adding a subset of \( \omega_1 \) over \( V, \) \( R \) is countable in \( V[f], \) so it contains a dense set that is isomorphic to the universal countable partial ordering; therefore \( c_\alpha \) adds a Cohen-real over \( V[\{ c_\beta : \beta < \alpha \}][f], \) i.e. meeting all the \( D_\xi \)’s. 

**Corollary 2.** It is consistent that there is a system of \( \aleph_1 \) countably infinite sets without property \( B \) but not on \( \omega. \)

**Proof.** The statements can be deduced from \( \clubsuit \) and \( M; \) see Chapter 21 in [7].

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