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Dimension zero at all scales

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Abstract

We consider the notion of dimension in four categories: the category of (unbounded) separable metric spaces and (metrically proper) Lipschitz maps, and the category of (unbounded) separable metric spaces and (metrically proper) uniform maps. A unified treatment is given to the large scale dimension and the small scale dimension. We show that in all categories a space has dimension zero if and only if it is equivalent to an ultrametric space. Also, 0-dimensional spaces are characterized by means of retractions to subspaces. There is a universal zero-dimensional space in all categories. In the Lipschitz Category spaces of dimension zero are characterized by means of extensions of maps to the unit 0-sphere. Any countable group of asymptotic dimension zero is coarsely equivalent to a direct sum of cyclic groups. We construct uncountably many examples of coarsely inequivalent ultrametric spaces.

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1. Introduction

Asymptotic dimension is one of the most important asymptotic invariants of metric spaces introduced by Gromov [12]. There are several notions of large scale dimension introduced later [4,9,10]. The asymptotic dimension of Gromov is known to be the largest and in case it is finite all dimensions coincide. These dimensions also coincide when one of them is zero, but it is still unknown if an example of space exists with one of these dimensions finite but the asymptotic dimension of Gromov infinite. The notion of asymptotic dimension can be introduced for any set with coarse structure [21] (or a ballean [1,20]) but in this paper we consider separable metric spaces only.

Our attempts to find the small scale analogs of large scale dimensions brought us to an idea of macroscopic and microscopic functors on a category of metric spaces: given a metric space (X, d) and $\varepsilon > 0$ we consider the (ε-discrete) metric $\min(d, \varepsilon)$ on X and (ε-bounded) metric $\max(d, \varepsilon)$ [5]. Therefore we can define and work with all-scales notions and then obtain the large scale (or small scale) results as corollaries after applying the macroscopic (or microscopic) functor.

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In this paper we consider five categories of separable metric spaces: Lipschitz, Uniform, the corresponding Metrically Proper subcategories (see the definitions at the end of Introduction), and the Coarse category defined by Roe [21].

The concept of dimension appropriate for the Lipschitz category is the Assouad–Nagata dimension [15]. For discrete metric spaces the notion of Assouad–Nagata dimension is equivalent to the notion of asymptotic dimension of linear type considered by Gromov [12] and Roe [21] (Dranishnikov and Zarichnyi call it "asymptotic dimension with Higson property" [11]). For bounded metric spaces the notion of Assouad–Nagata dimension is equivalent to the notion of capacity dimension introduced recently by Buyalo [6,7].

In Section 4 we introduce the concept of dimension appropriate for the uniform category. For discrete metric spaces the notion of uniform dimension is equivalent to the notion of asymptotic dimension introduced by Gromov. For a bounded metric space X the uniform dimension $\dim_u(X)$ coincides with the large dimension $\Delta d X$ from the book [14].

Ultrametric spaces play the central role in this paper. We show that in (Proper) Lipschitz and (Proper) Uniform categories a metric space (X, d) has dimension 0 if and only if there is an ultrametric ρ on X such that the identity map $(X, d) \to (X, \rho)$ is an equivalence (for separable metric spaces and continuous maps this result was proved by de Groot [13] and Nagata [18]; for metric spaces and Lipschitz maps it is proved in [8, Chapter 15]; for discrete spaces and coarse maps this result belongs to M. Zarichnyi [24]). We also exhibit an ultrametric space which is universal (in all categories) for all 0-dimensional spaces. Notice that there is an ultrametric space containing isometric copy of any ultrametric space [3,16,17].

In (Proper) Lipschitz and (Proper) Uniform categories we characterize 0-dimensional spaces by means of retractions to subspaces. In the Lipschitz category we prove that the following conditions are equivalent:

(1) X has dimension 0;
(2) the unit 0-sphere S^0 is an absolute extensor for X;
(3) every metric space is an absolute extensor for X.

We failed to find the analogous characterization in the Uniform category.

In Sections 5 and 6 we consider discrete metric spaces in the Coarse category. It is easy to see that a finitely generated group of asymptotic dimension 0 is finite and therefore all such groups are coarsely equivalent. To define asymptotic dimension for an infinitely generated countable group one should consider a left invariant proper metric on it. We describe a natural way to introduce such a metric and prove that any group of asymptotic dimension 0 is coarsely equivalent to an abelian group. It is known that a countable group has asymptotic dimension 0 if and only if it is locally finite [22] but we are not aware of any characterization of locally finite countable groups up to coarse equivalence. In Section 6 we construct uncountably many examples of coarsely inequivalent metric spaces of asymptotic dimension 0. The idea of the construction does not work for groups.

Definition 1.1. A map $f : X \to Y$ of metric spaces is called *Lipschitz* if there is a constant $\lambda > 0$ such that the inequality $d_Y(f(x), f(x')) \leqslant \lambda \cdot d_X(x, x')$ holds for all points $x, x' \in X$. f is called λ-*Lipschitz* if we need to specify the constant λ. f is called λ-*bi-Lipschitz* if both f and f^{-1} are λ-Lipschitz.

For any Lipschitz map f we denote

$$Lip(f) = \inf\{\lambda \mid f \text{ is } \lambda\text{-Lipschitz}\}$$

Notice that a Lipschitz map f is $Lip(f)$-Lipschitz.

Definition 1.2. A metric space X is called a *Lipschitz extensor for a metric space* Y if there exists a constant $m > 0$ such that for any closed subspace $A \subset Y$ any Lipschitz map $f : A \to X$ extends to a Lipschitz map $F : Y \to X$ with $Lip(F) = m \times Lip(f)$. We call the space X an $m\times$-*Lipschitz extensor for* Y if we need to specify the constant m.

A map $f : X \to Y$ is called *metrically proper* if for any bounded subset A of the space Y its preimage $f^{-1}(A)$ is bounded.

Definition 1.3. The *Lipschitz category* consists of separable metric spaces with morphisms being Lipschitz maps. Its subcategory of unbounded spaces and metrically proper maps is called the *Proper Lipschitz category*.

We call a map $f : X \to Y$ *uniform* if there is a function $\delta_f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to 0} \delta_f(t) = 0$ such that $d_Y(f(x), f(x')) \leqslant \delta_f(d_X(x, x'))$ for all points $x, x' \in X$. To specify the function δ_f we sometimes say that the map f is δ_f-uniform. A map f is called *bi-uniform* if both f and f^{-1} are uniform.

Definition 1.4. The *Uniform category* consists of separable metric spaces with morphisms being uniform maps. Its subcategory of unbounded spaces and metrically proper maps is called the *Proper Uniform category*.

We call a metric space X *discrete* if there is $\varepsilon > 0$ such that X is ε-discrete.

We call a map $f : X \to Y$ *large scale uniform* if there is a function $\delta_f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d_Y(f(x), f(x')) \leqslant \delta_f(d_X(x, x'))$ for all points $x, x' \in X$. A map is called *coarse* if it is large scale uniform and metrically proper. Metric spaces X and Y are *coarsely equivalent* if there exist a constant $C > 0$ and two coarse maps $f : X \to Y$ and $g : Y \to X$ such that the maps $g \circ f$ and $f \circ g$ are C-close to the identity.

2. Ultrametric spaces

Definition 2.1. A metric space (X, d) is called *ultrametric* if for all $x, y, z \in X$ we have $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$.

An ultrametric space X can be characterized by the following very useful property:

Ultrametric property of a triangle

If a triangle in a space X has sides (distances between vertices) $a \leqslant b \leqslant c$, then $b = c$.

The following properties of ultrametric space are easy to check. A ball of radius D in an ultrametric space has diameter D. Two balls of radius D in an ultrametric space are either D-disjoint or identical.

Proposition 2.2. *Let (X, d) be a metric space. The metric d is an ultrametric if and only if $f(d)$ is a metric for every nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$.*

Proof. If d is ultrametric and $a \leqslant b = c$ are sides of a triangle in (X, d) then $f(a) \leqslant f(b) = f(c)$ are sides of the corresponding triangle in $(X, f(d))$ and therefore $f(d)$ is an ultrametric.

If d is not an ultrametric then there is a triangle in (X, d) with sides $a \leqslant b < c$. Consider the function

$$f(t) \begin{cases} t, & \text{if } t \leqslant b, \\ \frac{2b}{c-b}t + \frac{bc-3b^2}{c-b}, & \text{if } t \geqslant b. \end{cases}$$

The sides of the corresponding triangle in $(X, f(d))$ are $f(a) \leqslant f(b) = b < 3b = f(c)$ which contradicts the triangle inequality. □

Definition 2.3. A metric is said to be 3^n*-valued* if the only values assumed by the metric are 3^n, $n \in \mathbb{Z}$.

The triangle inequality for a metric d implies the following:

Lemma 2.4. *Any 3^n-valued metric is an ultrametric.*

Lemma 2.5. *Any ultrametric space is 3-bi-Lipschitz equivalent to a 3^n-valued ultrametric space.*

Proof. Given an ultrametric space (X, d) we define a new metric ρ on X as follows:

$$\rho(x, y) = 3^n \quad \text{if } 3^{n-1} < d(x, y) \leqslant 3^n.$$

Clearly, the identity map $\mathrm{id} : (X, d) \to (X, \rho)$ is expanding and 3-Lipschitz. □

Let us describe an ultrametric space (L_ω, μ) which is universal for all separable ultrametric spaces with 3^n-valued metrics. This space appeared naturally in different areas of mathematics (see, for example, [16] and references therein).

Let us fix a countable set S with a distinguished element $s_0 \in S$. The set L_ω is a subset of the set of infinite sequences $\bar{x} = \{x_n\}_{n\in\mathbb{Z}}$ with all elements x_n from the set S. A sequence $\bar{x}$ belongs to L_ω if there exists an index $k \in \mathbb{Z}$ such that $x_n = s_0$ for all $n < k$. The metric μ is defined as $\mu(\bar{x}, \bar{y}) = 3^{-m}$ where $m \in \mathbb{Z}$ is the minimal index such that $x_m \neq y_m$. Clearly, the space L_ω is a complete separable ultrametric (by Lemma 2.4) space.

To prove that any separable ultrametric space with 3^n-valued metric embeds isometrically into (L_ω, μ) we follow the idea of P.S. Urysohn [23] and show that the space L_ω is *finitely injective*:

Lemma 2.6. *Let (X, d) be a finite metric space with 3^n-valued metric d. For any subspace $A \subset X$, any isometric map $f : A \to L_\omega$ admits an isometric extension $\tilde{f} : X \to L_\omega$.*

Proof. It is sufficient to prove lemma in case $X \setminus A$ consists of one point x. In such case we have to find a point $\bar{z} \in L_\omega$ such that $\mu(\bar{z}, f(a)) = d(x, a)$ for every point $a \in A$. Let $A_x = \{a \in A \mid d(x, a) = d(x, A)\}$ be the set of all points in A closest to x and let $d(x, A) = 3^{-n}$. Fix a point $b \in A_x$ and define $\bar{z} = \{z_n\}_{n\in\mathbb{Z}}$ as follows: $z_m = f(b)_m$ if $m < n$; $z_m = s_0$ if $m > n$; z_n is any element of the set S other than $f(c)_n$ for any point $c \in A_x$.

Clearly, $\mu(\bar{z}, f(c)) = 3^{-n} = d(x, c)$ for any point $c \in A_x$. For any point $a \in A \setminus A_x$ we have $d(a, x) = d(a, b) = 3^{-m} > 3^{-n}$ which means that $f(a)_m \neq f(b)_m = z_m$ and therefore $\mu(\bar{z}, f(a)) = 3^{-m} = d(x, a)$. □

Theorem 2.7. *Any separable metric space (X, d) equipped with 3^n-valued metric d embeds isometrically into the space (L_ω, μ).*

Proof. Since X is separable, it is sufficient to embed isometrically a countable dense subspace A of X. One can embed such a subspace by induction using Lemma 2.6. □

Corollary 2.8. *Any separable ultrametric space admits 3-bi-Lipschitz embedding into the space (L_ω, μ).*

Proof. Combine Lemma 2.5 and Theorem 2.7. □

Theorem 2.9. *Every closed subset A of an ultrametric space X is a λ-Lipschitz retract of X for any $\lambda > 1$. If the subset A is unbounded, the retraction can be chosen to be metrically proper.*

Proof. Suppose that X is an ultrametric space and $A \subset X$ is a closed subspace. If $\lambda > 1$ is given, choose a number $\delta > 1$ such that $\delta^2 < \lambda$.

Let us fix a base point $x_0 \in X$. Take an arbitrary well-order $<_k$ on each non empty Annulus $A_k = \{x \mid k \leqslant d(x, x_0) < k + 1\}$ of X for every $k \in \mathbb{N} \cup \{0\}$. Now we say $z \prec z'$ for any two points $z, z' \in X$ if $z \in A_k$, $z' \in A_{k'}$ and $k > k'$ or if $z, z' \in A_k$ and $z <_k z'$. Notice that $\prec$ is an order in X such that for every non empty bounded subset C of X the restricted order $\prec|_C$ is a well-order.

We define a retraction $r : X \to A$ as follows. For a point $x \in X$ we look at the nonempty bounded set

$$A_x = \{a \in A \mid d(x, a) \leqslant \delta \cdot \operatorname{dist}(x, A)\}$$

and put $r(x)$ to be the minimal point in the set A_x with respect to the order $\prec$.

Let us show that the retraction r is λ-Lipschitz. Assume that for some points $x, y \in X$ we have $d(r(x), r(y)) > \lambda \cdot d(x, y)$. Without loss of generality we may assume that $r(x) \prec r(y)$.

If $d(y, r(x)) \leq d(y, r(y))$, then $r(x) \in A_y$ and $r(x) \prec r(y)$ contradicts the choice of $r(y)$ to be the minimal point in the set A_y.

In case $d(y, r(x)) > d(y, r(y))$ we denote by D the distance between $r(x)$ and $r(y)$ and notice that $d(y, r(x)) = d(r(x), r(y)) = D$ in the isosceles triangle $\{y, r(x), r(y)\}$. Since $D > d(x, y)$, we have $d(x, r(x)) = d(y, r(x)) = D$ in the isosceles triangle $\{x, y, r(x)\}$.

$$d(x, r(y)) \geqslant \operatorname{dist}(x, A) \geqslant \frac{1}{\delta} \cdot d(x, r(x)) = \frac{D}{\delta} > \frac{D}{\lambda} > d(x, y).$$

Therefore $d(x, r(y)) = d(y, r(y))$ in the isosceles triangle $\{x, y, r(y)\}$. The point $r(x)$ does not belong to A_y since $r(x) \prec r(y)$, thus $d(y, r(x)) = D > \delta \cdot \operatorname{dist}(y, A)$. Then there exists a point $z \in A$ with $d(y, z) < \frac{D}{\delta}$.

$$d(y, z) \geqslant \operatorname{dist}(y, A) \geqslant \frac{d(y, r(y))}{\delta} = \frac{d(x, r(y))}{\delta} \geqslant \frac{D}{\delta^2} > \frac{D}{\lambda} > d(x, y).$$

Therefore $d(x, z) = d(y, z)$ in the isosceles triangle $\{x, y, z\}$. Since $d(x, z) < d(x, r(x))$, we have $z \in A_x$, but $d(x, z) < \frac{D}{\delta} = \frac{d(x, r(x))}{\delta}$ contradicts the definition of A_x (two points $a, a' \in A_x$ cannot satisfy $d(x, a) < \frac{d(x,a')}{\delta}$).

If the subset A is unbounded, we prove that the retraction r is metrically proper. Let B be any bounded subset of A. Choose a point $a \in A$ which is in an annulus greater than any annulus that has non-empty intersection with B (therefore, $a \prec B$). Given any point $x \in r^{-1}(B)$ we have $a \notin A_x$, therefore $d(x, r(x)) \leqslant \delta \cdot d(x, A) < d(x, a)$. The ultrametric property of the triangle $\{x, a, r(x)\}$ implies $d(r(x), a) = d(x, a)$ therefore:

$$d(x, B) \leqslant d(x, r(x)) < d(r(x), a) \leqslant \operatorname{diam}(B) + d(a, B) \qquad \square$$

Example 2.10. Let $X = \{x_n\}_{n=1}^{\infty}$ be a sequence of points. Define $d(x_1, x_n) = 1 + \frac{1}{n}$ and $d(x_m, x_n) = \max\{1 + \frac{1}{m}, 1 + \frac{1}{n}\}$ for any $m, n > 1$. Then d is an ultrametric on X and there is no 1-Lipschitz retraction of X onto $A = \{x_n\}_{n=2}^{\infty}$.

3. Assouad–Nagata dimension

Definition 3.1. Let X be a metric space, A be a subspace of X, and S be a positive number.

A is *S-bounded* if for any points $x, x' \in A$ we have $d_X(x, x') \leqslant S$.

An *S-chain* in A is a sequence of points $x_1, \ldots, x_k$ in A such that for every $i < k$ the set $\{x_i, x_{i+1}\}$ is S-bounded.

A is *S-connected* if for any points $x, x' \in A$ can be connected in A by an S-chain.

Notice that any subset A of X is a union of its *S-components* (the maximal S-connected subsets of A). If B and B' are two S-components of the set A then B and B' are S-disjoint. Intuitively, a metric space X has dimension 0 at scale $S > 0$ if all S-components of X are uniformly bounded.

Definition 3.2. A metric space X has Assouad–Nagata dimension zero (notation $\dim_{AN}(X) \leqslant 0$) if there exists a constant $m \geqslant 1$, such that for any $S > 0$ all S-components of X are mS-bounded.

It is easy to see that bi-Lipschitz maps preserve Assouad–Nagata dimension.

Ultrametric spaces are the best examples of metric spaces of Assouad–Nagata dimension zero. Indeed, for any positive number D any D-component of an ultrametric space is a D-ball and therefore is D-bounded. Let us characterize spaces of Assouad–Nagata dimension 0 using ultrametrics.

The following theorem is proved in [8, Proposition 15.7]. We provide a proof for completeness.

Theorem 3.3. *If a metric space (X, d) has Assouad–Nagata dimension $\dim_{AN}(X) \leqslant 0$, then there is an ultrametric ρ on X such that the identity map $\mathrm{id} : (X, d) \to (X, \rho)$ is bi-Lipschitz.*

Proof. Suppose that for a number $m > 1$, all S-components of X are mS-bounded. Consider two points $x, z \in X$ and put

$$S = \frac{d(x, z)}{2m}.$$

Then the points x and z belong to different S-components of X. Thus for any chain $x = x_0, x_1, \ldots, x_{k-1}, x_k = z$ we have

$$d(x, z) \leqslant 2m \cdot \max_{0 \leqslant i < k} \bigl\{d(x_i, x_{i+1})\bigr\}.$$

Now define $\rho(x, z)$ to be the infimum of $\max_{0 \leqslant i < k}\{d(x_i, x_{i+1})\}$ over all finite chains $x_0, x_1, \ldots, x_{k-1}, x_k$ with $x = x_0$ and $x_k = z$. Clearly

$$\frac{1}{2m} \cdot d(x, z) \leqslant \rho(x, z) \leqslant d(x, z).$$

To see that ρ is an ultrametric, take three points x, y, z in X and let s be the infimum of all positive numbers S such that all three points belong to one S-component of X. If all three points belong to one s-component or all three belong to different s-components, then $\rho(x, y) = \rho(x, z) = \rho(y, z) = s$. If the points x and y belong to one s-component which does not contain z, then $\rho(x, y) \leqslant s = \rho(x, z) = \rho(y, z)$. □

Theorem 3.4. *Any separable metric space of Assouad–Nagata dimension* 0 *admits a bi-Lipschitz embedding into the space* (L_ω, μ).

Proof. Apply Theorems 3.3 and 2.8. □

Theorem 3.5. *In the Lipschitz category the following conditions are equivalent*:

(1) $\dim_{AN}(X) \leqslant 0$;
(2) *there exists a number* λ *such that every closed subset of* X *is a* λ*-Lipschitz retract of* X;
(3) *there exists a number* λ *such that every metric space is a* $\lambda\times$*-Lipschitz extensor for* X;
(4) *the unit* 0*-sphere* S^0 *is an extensor for* X.

Conditions (1), (2), *and* (3) *are equivalent in the Proper Lipschitz category.*

Proof. (1) ⇒ (2) in both Lipschitz and Proper Lipschitz categories. Theorem 3.3 allows us to find an ultrametric ρ on X which is bi-Lipschitz equivalent to d. Application of Theorem 2.9 completes the proof.

(2) ⇒ (3) in both Lipschitz and Proper Lipschitz categories. Given a closed subspace $A \subset X$ and a Lipschitz map $f : A \to Y$ to some metric space Y we fix a λ-Lipschitz retraction $r : X \to A$. Then the composition $f \circ r : X \to K$ has the Lipschitz constant bounded by $\lambda \cdot Lip(f)$.

(3) ⇒ (4) Obvious.

(4) ⇒ (1) Let $m \geqslant 1$ be a number such that any λ-Lipschitz map from any closed subspace $A \subset X$ to S^0 can be extended to $m\lambda$-Lipschitz map of X. If an S-component of X is not mS-bounded, there are points z_0 and z_1 with $d(z_0, z_1) > mS$ and an S-chain of points $z_0 = x_0, x_1, \ldots, x_k = z_1$. Notice that the map $f : \{z_0\} \cup \{z_1\} \to S^0$ defined as $f(z_0) = 0$ and $f(z_1) = 1$ is $\frac{1}{d(z_0,z_1)}$-Lipschitz but any extension of this map to the chain is at least $\frac{1}{S}$-Lipschitz and cannot be $\frac{m}{d(z_0,z_1)}$-Lipschitz (since $\frac{1}{S} > \frac{m}{d(z_0,z_1)}$).

(3) ⇒ (1) in the Proper Lipschitz category. If an S-component of X is not λS-bounded, there are points z_0 and z_1 with $d(z_0, z_1) > \lambda S$ and an S-chain of points $z_0 = x_0, x_1, \ldots, x_k = z_1$. Let A be any unbounded λS-discrete subspace of X containing the points z_0 and z_1. Notice that the identity map id_A is 1-Lipschitz but any extension of this map to the chain is not λS-Lipschitz. □

Problem 3.6. Is there an analog of condition (4) from Theorem 3.5 in the Proper Lipschitz category?

4. Uniform dimension

Definition 4.1. A metric space X has *uniform* dimension zero (notation $\dim_u(X) \leqslant 0$) if there exists a continuous increasing function $\mathcal{D} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mathcal{D}(0) = 0$ and $\lim_{t\to\infty} \mathcal{D}(t) = \infty$, such that for any positive number S every S-component of X is $\mathcal{D}(S)$-bounded.

To specify the function $\mathcal{D}$ we sometimes say that the space X has *uniform dimension zero of type* $\mathcal{D}$.

If the function $\mathcal{D}$ does not exceed some linear function $\mathcal{D}(t) \leqslant k \cdot t$ for all $t \geqslant 0$, then the space X has Assouad–Nagata dimension 0. We want the dimension control function to be increasing and continuous to guarantee the existence of the inverse function $\mathcal{D}^{(-1)}$.

It is easy to check that the uniform dimension is preserved under the bi-uniform maps:

Lemma 4.2. *Let* $f : X \to Y$ *be a bi-uniform map. Then* $\dim_u(X) = \dim_u(f(X))$.

Theorem 4.3. *If a metric space* (X, d) *has uniform dimension* $\dim_u(X) \leqslant 0$*, then there is an ultrametric* ρ *on* X *such that the identity map* $\mathrm{id} : (X, d) \to (X, \rho)$ *is bi-uniform.*

Proof. Suppose that the space X has uniform dimension zero of type $\mathcal{D}$. Consider two points $x, z \in X$ and put

$$S = \frac{1}{2}\mathcal{D}^{-1}\bigl(d(x, z)\bigr).$$

Then the points x and z belong to different S-components of X. Thus for any chain $x = x_0, x_1, \ldots, x_{k-1}, x_k = z$ we have

$$\mathcal{D}^{-1}\bigl(d(x, z)\bigr) \leqslant 2 \cdot \max_{0 \leqslant i < k}\bigl\{d(x_i, x_{i+1})\bigr\}.$$

Now define $\rho(x, z)$ to be the infimum of $\max_{0 \leqslant i < k}\{d(x_i, x_{i+1})\}$ over all finite chains $x_0, x_1, \ldots, x_{k-1}, x_k$ with $x = x_0$ and $x_k = z$. Clearly

$$\frac{1}{2} \cdot \mathcal{D}^{-1}\bigl(d(x, z)\bigr) \leqslant \rho(x, z) \leqslant d(x, z).$$

To see that ρ is an ultrametric, take three points x, y, z in X and let s be the infimum of all positive numbers S such that all three points belong to one S-component of X. If all three points belong to one s-component or all three belong to different s-components, then $\rho(x, y) = \rho(x, z) = \rho(y, z) = s$. If the points x and y belong to one s-component which does not contain z, then $\rho(x, y) \leqslant s = \rho(x, z) = \rho(y, z)$. □

Corollary 4.4. *A separable metric space* X *has uniform dimension zero if and only if it admits a bi-uniform embedding into* L_ω*.*

Proof. If $\dim_u(X) \leqslant 0$ we can change the metric on X bi-uniformly to get an ultrametric space and then embed it in a bi-Lipschitz way into L_ω using Theorem 2.8.

If X embeds bi-uniformly into L_ω, its image has uniform dimension zero as a subspace of L_ω. Then X has uniform dimension zero by Lemma 4.2. □

Theorem 4.5. *In both Uniform and Proper Uniform categories the following conditions are equivalent*:

(1) $\dim_u X \leqslant 0$;
(2) *there exists a continuous increasing function* $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\mu(0) = 0$ *and* $\lim_{t\to\infty} \mu(t) = \infty$*, such that every closed subspace of* X *is* μ*-uniform retract of* X*.*

Proof. (1) ⇒ (2) Theorem 4.3 allows us to find an ultrametric ρ on X which is bi-uniformly equivalent to d. Application of Theorem 2.9 completes the proof.

(2) ⇒ (1) If an S-component of X is not $\mu(S)$-bounded, there are points z_0 and z_1 with $d(z_0, z_1) > \mu(S)$ and an S-chain of points $z_0 = x_0, x_1, \ldots, x_k = z_1$.

In the Uniform category let $A = \{z_0\} \cup \{z_1\}$. In the Proper Uniform category we consider any unbounded closed subspace A of X containing the points z_0 and z_1 and such that the distance from $\{z_0\} \cup \{z_1\}$ to the rest of A is greater than $d(z_0, z_1)$.

Notice that any retraction of X onto A restricted to the chain takes some S-closed points to two points of distance greater than $d(z_0, z_1) > \mu(S)$. Thus such a retraction cannot be μ-uniform. □

Problem 4.6. Are there analogs of conditions (3) and (4) from Theorem 3.5 in the Proper Uniform category?

5. Locally finite countable groups

It is proved in [22] that a countable group G (equipped with any proper metric) has asymptotic dimension zero if and only if G is locally finite (i.e. every finitely generated subgroup of G is finite). The purpose of this section is to show that such a group is bi-uniformly equivalent to a locally finite abelian group. Also we classify locally finite countable groups up to bi-uniform equivalence. The problem of classification of locally finite countable groups up to coarse equivalence remains open. Notice that for discrete metric spaces the notions of bi-uniform equivalence and bijective coarse equivalence coincide.

A left invariant metric d on a countable group G is *proper* if and only if every bounded subset of (G, d) is finite. Thus a left invariant proper metric d on G is bounded from below and therefore the asymptotic dimension of (G, d) is equal to its uniform dimension. There is only one way (up to bi-uniform equivalence) to introduce a proper left-invariant metric on G [22, Proposition 1]. Thus the asymptotic dimension of a countable group does not depend on the choice of a proper left-invariant metric.

Let G be a locally finite countable group. Let us describe a particularly simple way to define a proper left-invariant metric on G. Consider a filtration $\mathcal{L}$ of G by finite subgroups $\mathcal{L} = \{1 \subset G_1 \subset G_2 \subset G_3 \cdots\}$ and define the metric $d_{\mathcal{L}}$ associated to this filtration as:

$$d_{\mathcal{L}}(x, y) = \min\{i \mid x^{-1}y \in G_i\}.$$

Clearly, $d_{\mathcal{L}}$ is an ultrametric (therefore, the asymptotic dimension of $(G, d_{\mathcal{L}})$ is zero).

Lemma 5.1. *Suppose two groups G and H have filtrations by finite subgroups: $\mathcal{L} = \{1 \subset G_1 \subset G_2 \subset G_3 \cdots\}$ of G and $\mathcal{K} = \{1 \subset H_1 \subset H_2 \subset H_3 \cdots\}$ of H. If the index $[G_i : G_{i-1}]$ is less than or equal to the index $[H_i : H_{i-1}]$ for all i, then $(G, d_{\mathcal{L}})$ admits an isometric embedding into $(H, d_{\mathcal{H}})$. Moreover, if $[G_i : G_{i-1}] = [H_i : H_{i-1}]$ for all i (equivalently, the cardinality of G_i equals cardinality of H_i for all i), then the groups $(G, d_{\mathcal{L}})$ and $(H, d_{\mathcal{H}})$ are isometric.*

Proof. Put $a_i = [G_i : G_{i-1}]$ and $b_i = [H_i : H_{i-1}]$. Fix an injection $f_1 : G_1 \to H_1$ and assume injections $f_k : G_k \to H_k$ are known for $k \leqslant n$ such that the following two properties hold:

(1) $f_i(x) = f_j(x)$ for $i < j$ and $x \in G_i$,
(2) the injection $f_k : G_k \to H_k$ is isometric.

Pick an injection of the set of cosets $\{x \cdot G_n\}$ of G_n in G_{n+1} into the set of cosets $\{y \cdot H_n\}$ of H_n in H_{n+1}. That amounts to picking representatives $1, x_1, \ldots, x_m$ $(m = a_{n+1} - 1)$ of cosets of G_n in G_{n+1} and picking representatives $1, y_1, \ldots, y_l$ $(l = b_{n+1} - 1)$ of cosets of H_n in H_{n+1}. Make sure the injection takes $\{1 \cdot G_n\}$ to $\{1 \cdot H_n\}$. Now we extend f_n to $f_{n+1} : G_{n+1} \to H_{n+1}$ as follows: if $x \in G_{n+1} \setminus G_n$, we represent x as $x_k \cdot x'$ for some unique $k \leqslant m$ and we define $f_{n+1}(x)$ as $y_k \cdot f_n(x')$.

If x and z belong to different cosets of G_n in G_{n+1}, then $f_{n+1}(x)$ and $f_{n+1}(z)$ belong to different cosets of H_n in H_{n+1} and $d_{\mathcal{L}}(x, z) = n + 1 = d_{\mathcal{H}}(f_{n+1}(x), f_{n+1}(z))$. If x and z belong to the same coset $x_k \cdot G_n$ of G_n in G_{n+1}, then $x = x_k \cdot x'$, $z = x_k \cdot z'$. Since $f_{n+1}(x) = y_k \cdot f_n(x')$, $f_{n+1}(z) = y_k \cdot f_n(z')$, and the map f_n is isometry, then

$$d_{\mathcal{L}}(x, z) = d_{\mathcal{L}}(x', z')d_{\mathcal{H}}\big(f_n(x'), f_n(z')\big)d_{\mathcal{H}}\big(f_{n+1}(x), f_{n+1}(z)\big).$$

By pasting all f_n we get an isometric injection $f : G \to H$. Notice that in case $[G_i : G_{i-1}] = [H_i : H_{i-1}]$ for all i, the map f is bijective and establishes an isometry between $(G, d_{\mathcal{L}})$ and $(H, d_{\mathcal{H}})$. □

Lemma 5.2. *Given two locally finite groups G and H the following conditions are equivalent:*

(1) *There are left-invariant proper metrics d_G on G and d_H on H such that (G, d_G) is isometric to (H, d_H).*
(2) *There are filtrations by finite subgroups: $\mathcal{L} = \{1 \subset G_1 \subset G_2 \subset G_3 \cdots\}$ of G and $\mathcal{K} = \{1 \subset H_1 \subset H_2 \subset H_3 \cdots\}$ of H such that the cardinality of G_i equals cardinality of H_i for all i.*

Proof. In view of Lemma 5.1, it suffices to show (1) ⇒ (2). Obviously, we may pick an isometry $f : G \to H$ such that $f(1_G) = 1_H$ (replace any f by $f(1_G)^{-1} \cdot f$). Notice f establishes bijectivity between m-component of G containing 1_G and the m-component of H containing 1_H. Also, those components are subgroups of G and H. Thus, define G_1 as 1-component of G containing 1_G and, inductively, G_{i+1} as $(\operatorname{diam}(G_i) + i)$-component of G containing 1_G. □

Main example. If G is a direct sum of cyclic groups $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{a_i}$ we consider the metric on G associated to the filtration

$$\mathcal{L} = \{1 \subset \mathbb{Z}_{a_1} \subset \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \subset \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus \mathbb{Z}_{a_3} \subset \cdots\}.$$

If we write elements of the group $\bigoplus_{i=1}^{\infty}\mathbb{Z}_{a_i}$ as $p = p_1 p_2 \dots p_n$ where $p_j \in \mathbb{Z}_{a_j}$ and denote $|p| = n$ then the ultrametric $d_{\mathcal{L}}$ can be defined explicitly as

$$d_{\mathcal{L}}(p,q) = \begin{cases} \max\{|p|,|q|\}, & \text{if } |p| \neq |q|, \\ \max\{i \mid p_i \neq q_i\}, & \text{if } |p| = |q|. \end{cases}$$

Theorem 5.3. *A locally finite countable group G with a proper left invariant metric d is bi-uniformly equivalent to a direct sum of cyclic groups.*

Proof. Fix a filtration $\mathcal{L}$ of G by finite subgroups $\mathcal{L} = \{1 \subset G_1 \subset G_2 \subset G_3 \cdots\}$. Then (G,d) is bi-uniformly equivalent to $(G, d_{\mathcal{L}})$ [22, Proposition 1]. By Lemma 5.1, $(G, d_{\mathcal{L}})$ is isometric to $\bigoplus_{i=1}^{\infty}\mathbb{Z}_{a_i}$ where $a_i = [G_i : G_{i-1}]$. □

Definition 5.4. Let G be a countable locally finite group and p be a prime number. We define a *p-Sylow number of G* (finite or infinite) as follows:

$$|p\text{-Syl}|(G) = \sup\{p^n \mid p^n \text{ divides } |F|,\ F \text{ a finite subgroup of } G, n \in \mathbb{Z}\}.$$

Notice that if the p-Sylow number of G is finite, it is equal to the order of a p-Sylow subgroup of some finite subgroup of G. For an abelian torsion group G the p-Sylow number of G is equal to the order of the p-torsion subgroup of G.

We are going to use the following theorem of Protasov:

Theorem 5.5. *(See [19, Theorem 5].) Two countable locally finite groups G and H with proper left invariant metrics are bi-uniformly equivalent if and only if, for every finite subgroup F of G, there exists a finite subgroup E of H such that $|F|$ is a divisor of $|E|$, and, for every finite subgroup E of H, there exists a finite subgroup F of G such that $|E|$ is a divisor of $|F|$.*

Corollary 5.6. *Let G and H be countable direct sums of finite prime cyclic groups. Let d_G and d_H be proper left invariant metrics on G and H. Then the metric spaces (G, d_G) and (H, d_H) are bi-uniformly equivalent if and only if the groups G and H are isomorphic.*

Theorem 5.7. *Let G and H be locally finite countable groups with proper left invariant metrics d_G and d_H. The metric spaces (G, d_G) and (H, d_H) are bi-uniformly equivalent if and only if for every prime p we have $|p\text{-Syl}|(G) = |p\text{-Syl}|(H)$.*

Proof. Assume the metric spaces (G, d_G) and (H, d_H) are bi-uniformly equivalent. Our goal is to show that if $|p\text{-Syl}|(G) \geqslant p^n$, then $|p\text{-Syl}|(H) \geqslant p^n$. If there is a finite subgroup F of G such that p^n divides $|F|$, then by Theorem 5.5 there is a subgroup E of H such that p^n divides $|E|$. Thus $|p\text{-Syl}|(H) \geqslant p^n$.

Now suppose $|p\text{-Syl}|(G) = |p\text{-Syl}|(H)$ for every prime p. By Theorem 5.5, it is enough to show that for every finite subgroup F of G, there exists a finite subgroup E of H such that $|F|$ is a divisor of $|E|$. If $|F| = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ then $p_i^{\alpha_i} \leqslant |p_i\text{-Syl}|(H)$ for every i. For every i find a subgroup E_i of H such that $p_i^{\alpha_i}$ divides $|E_i|$. Let E be a finite subgroup of H containing all the groups E_i. Clearly, $|F|$ divides $|E|$. □

Definition 5.8. A metric space is of *bounded geometry* if there is a number $r > 0$ and a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that the r-capacity (the maximal cardinality of r-discrete subset) of every ε-ball does not exceed $c(\varepsilon)$.

Notice that any countable group with proper left invariant metric has bounded geometry.

A large scale analog $\mathcal{M}^0$ of 0-dimensional Cantor set is introduced in [11]: it is the set of all positive integers with ternary expression containing 0's and 2's only (with the metric from $\mathbb{R}_+$): $\mathcal{M}^0 = \{\sum_{i=0}^{\infty} a_i 3^i \mid a_i = 0, 2\}$.

Proposition 5.9. *(See [11, Theorem 3.11].) The space $\mathcal{M}^0$ is universal for proper metric spaces of bounded geometry and of asymptotic dimension zero.*

Proposition 5.10. *The space $\mathcal{M}^0$ is coarsely equivalent to $\bigoplus_{i=1}^{\infty} \mathbb{Z}_2$.*

Proof. To define a map $f : \bigoplus_{i=1}^{\infty} \mathbb{Z}_2 \to \mathcal{M}^0$ we consider an element $p = p_1 p_2 \ldots p_n$ of the group $\bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ where $p_j \in \{0, 1\} = \mathbb{Z}_2$ and put

$$f(p) = \sum_{i=1}^{\infty} 2p_i \cdot 3^{i-1}.$$

It is easy to check that the map f is a coarse equivalence: for any elements $p, q \in \bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ we have

$$3^{d_{\mathcal{L}}(p,q)} \leqslant d_{\mathcal{M}^0}\big(f(p), f(q)\big) \leqslant 3 \cdot 3^{d_{\mathcal{L}}(p,q)}. \qquad \square$$

Remark 5.11. (Cf. Proposition 2.2.) The proof above shows that the group $\bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ with the ultrametric $3^{d_{\mathcal{L}}}$ is bi-Lipschitz equivalent to the space $\mathcal{M}^0$.

Proposition 5.12. *(Cf. [19, Theorem 4].) Let G and H be locally finite countable groups with proper left invariant metrics. Then the metric space G can be coarsely embedded in the metric space H (this map is not a homomorphism).*

Proof. By Propositions 5.9 and 5.10 the group G can be coarsely embedded in the group $\bigoplus \mathbb{Z}_2$. By Lemma 5.1 the group $(\bigoplus \mathbb{Z}_2, d_{\mathcal{L}})$ embeds isometrically into any group $(\bigoplus_{i=1}^{\infty} \mathbb{Z}_{a_i}, d_{\mathcal{L}})$. Finally, the group H is bi-uniformly equivalent to a direct sum of cyclic groups by Theorem 5.3. $\square$

Let G and H be countable locally finite groups. Using Theorem 5.3 one can show that if

$$\sum_{p\text{-prime}} \big| |p\text{-Syl}|(G) - |p\text{-Syl}|(H) \big| < \infty$$

then the groups G and H are coarsely equivalent. Is the converse true?

Problem 5.13. Classify countable abelian torsion groups up to coarse equivalence.

Let us suggest a program to answer 5.13. Notice that any abelian torsion group is coarsely equivalent to a direct sum of groups $\mathbb{Z}_p$ with p being prime. Therefore the following groups are of importance: $\mathbb{Z}_p^{\infty}$ (the infinite direct sum of copies of $\mathbb{Z}_p$) and $\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_{p^{n(p)}}$, where $n(p) \geqslant 1$ for each $p \in \mathcal{P}$, $\mathcal{P}$ being a subset of primes.

Problem 5.14. Suppose $\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_{p^{n(p)}}$ and $\bigoplus_{q \in \mathcal{Q}} \mathbb{Z}_{q^{m(q)}}$ are coarsely equivalent. Is the symmetric difference of $\mathcal{P}$ and $\mathcal{Q}$ finite? If so, does $n(p)$ equal $m(p)$ for all but finitely many p?

Problem 5.15. Suppose $\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_p^{\infty}$ and $\bigoplus_{q \in \mathcal{Q}} \mathbb{Z}_q^{\infty}$ are coarsely equivalent. Is $\mathcal{P}$ equal $\mathcal{Q}$?

Call two countable abelian torsion groups G and H *virtually isometric* if there are subgroups of finite index G' of G and H' of H such that G' is isometric to H' for some choice of proper and invariant metrics on G' and H'. Notice virtually isometric groups are coarsely equivalent.

Problem 5.16. Suppose two countable abelian torsion groups G and H are coarsely equivalent. Are G and H virtually isometric?

6. Examples of coarsely inequivalent ultrametric spaces

In this section we construct uncountably many coarsely inequivalent ultrametric spaces. Notice that any ultrametric space has asymptotic dimension zero.

Definition 6.1. Let (X, x_0) and (Y, y_0) be pointed metric spaces. We define a *metric wedge* $X \vee Y$ as the topological wedge of these spaces with the following metric:

$$d(z, z') \begin{cases} d_X(z, z'), & \text{if } z, z' \in X, \\ d_Y(z, z'), & \text{if } z, z' \in Y, \\ \max\{d_X(z, x_0), d_Y(z', y_0)\}, & \text{if } z \in X \setminus \{x_0\} \text{ and } z' \in Y \setminus \{y_0\}. \end{cases}$$

Similarly, one can define metric wedge of an arbitrary family of pointed metric spaces (cf. [2, Example 2] or [3, Theorem 2.2]).

The following lemma is easy to prove.

Lemma 6.2. *The metric wedge of any family of pointed ultrametric spaces is a pointed ultrametric space.*

If X is a bounded ultrametric space of diameter less than M, then *the cone* $\mathrm{Cone}(X, M)$ is obtained from X by adding a vertex v and declaring $d(v, x) = M$ for all $x \in X$. $\mathrm{Cone}(X, M)$ is a pointed ultrametric space with the vertex v being its base point.

Our examples will be obtained by wedging cones over basic ultrametric spaces, scaled copies of 0-skeleta of simplices.

Given a set λ of integers bigger than 1, we create a list X_i, $i \geqslant 1$, of spaces (called *islands*) satisfying the following conditions:

(1) The cardinality n_i of X_i belongs to λ.
(2) There is an integer $m_i \geqslant n_i$ such that $d(x, y) = m_i$ for all $x \neq y \in X_i$. Notice $m_i = \mathrm{diam}(X_i)$.
(3) For each $m \geqslant n$ and $n \in \lambda$ the set of islands X_i such that $m = \mathrm{diam}(X_i)$ and $n = |X_i|$ is infinite.

The wedge X_λ of all $\mathrm{Cone}(X_i, k_i)$, where $k_i = \sum_{j \leqslant i} m_j$ (put $m_j = 0$ for $j \leqslant 0$), is the λ-*archipelago*. k_i is the *separation* of island X_i in the λ-archipelago.

Proposition 6.3. *If $\lambda_1 \neq \lambda_2$, then the λ_1-archipelago is not coarsely equivalent to the λ_2-archipelago.*

Proof. Let X_1 be a λ_1-archipelago, X_2 be a λ_2-archipelago, and suppose that $f : X_1 \to X_2$ and $g : X_2 \to X_1$ are coarse equivalences such that the maps $g \circ f$ and $f \circ g$ are C-close to the identity and do not move the base points. Assume that the set $\lambda_1 \setminus \lambda_2$ is not empty and fix a number n in it.

There are three parameters associated to an island in any archipelago: the size, the diameter, and the separation. For simplicity, an (n, N, S)-island contains n points, is of diameter N, and separation S. Notice $n \leqslant N \leqslant S$.

Let us explain the idea of the proof. Since the space X_1 contains a lot of n-point islands, we are going to choose an (n, N, S)-island $P \subset X$ such that $f(P)$ is also an n-point island in X_2. Since the archipelago X_2 has no n-point islands, we get a contradiction. First we choose the size N of the island P to be so large that the map f is injective on P and the map g is injective on $f(P)$. Then we choose the separation S of the island P to be so large that $f(P)$ is contained in some island Q in X_2 and $g(Q)$ is contained in some island in X_1 (in fact, $g(Q) \subset P$).

Let us introduce some notations that we use in the rest of the proof. Given a coarse equivalence $h : Y \to Z$ of metric spaces we denote by ρ_h and δ_h two real functions such that $\rho_h(d_Y(y, y')) \leqslant d_Z(h(y), h(y')) \leqslant \delta_h(d_Y(y, y'))$ for any $y, y' \in Y$. If one of the spaces Y, Z is unbounded then the other is also unbounded and $\lim_{t\to\infty} \rho_h(t) = \infty = \lim_{t\to\infty} \delta_h(t)$.

Fix an integer $N > C$ such that $\rho_f(N) > C$. Notice that since $N > C$, any (n, N, S)-island $P \subset X_1$ is C-discrete and C-separated from the rest of X_1. Therefore the map $g \circ f$ is identity on P and the map f is injective on P.

Clearly, the image $f(P)$ of any (n, N, S)-island $P \subset X_1$ is $\delta_f(N)$-bounded in X_2 and therefore is contained in one $\delta_f(N)$-component Q of X_2. If the island P is S-separated in X_1, then its image $f(P)$ is at least $\rho_f(S)$-far from the base point of X_2. We choose S large enough to satisfy $\rho_f(S) > \delta_f(N)$ and thus to make sure that the $\delta_f(N)$-component Q containing $f(P)$ is an island. Assume Q is (k, m, S')-island where $m \leqslant \delta_f(N)$ and $k > n$ (recall that f is injective on P).

Since $\rho_f(N) > C$, the image $f(P)$ is C-discrete and therefore $m > C$. But then the map $f \circ g$ is identity on Q and the map g is injective on Q.

The image $g(Q)$ is $\delta_g(m)$-bounded and contains P. By choosing S to be greater than $\delta_g(\delta_f(N))$ we guarantee that the island P is more than $\delta_g(m)$-separated from the rest of X_1, therefore the set $g(Q)$ is entirely in P. Since g is injective on Q, we must have $n \geqslant k$. Contradiction. □

Corollary 6.4. *There are uncountably many coarsely inequivalent asymptotically* 0*-dimensional subspaces of the ray* $\mathbb{R}_+$.

Proof. Due to Proposition 5.9 it is sufficient to check that every λ-archipelago X is proper and has bounded geometry.

Given $R > 0$, a ball $\bar{B}(x, R)$ either coincides with $\bar{B}(x_0, R)$, where x_0 is the center of the archipelago X, consists of x only, or is the island containing x which has at most R points in that case. Thus the number of points in any ball $B(x, R)$ is bounded by some number depending on R only. This shows both X being proper and of bounded geometry. □

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