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Almost alternating links

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Abstract

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We introduce the category of almost alternating links: nonalternating links which have a projection for which one crossing change yields an alternating projection. We extend this category to m-almost alternating links which require m crossing changes to yield an alternating projection. We show that all but five of the nonalternating knots up through eleven crossings and links up through ten crossings are almost alternating. We also prove that a prime almost alternating knot is either a hyperbolic knot or a torus knot. We then obtain a bound on the span of the bracket polynomial for m-almost alternating links and discuss applications.

Keywords: Alternating link, hyperbolic knot, bracket polynomial.

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1. Introduction

A projection P of a link L in the 3-sphere is *almost alternating* if one crossing change makes the projection alternating. A link L is *almost alternating* if L has an almost alternating projection and L does not have an alternating projection. Note that an alternating link does in fact always have an almost alternating projection since we can take an alternating projection and make it almost alternating by a single Type II Reidemeister move. In fact, the unknot has an almost alternating

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projection, as can be seen by changing one crossing in the standard projection of the trefoil knot, for instance.

Further, we define a projection P of a link L to be *m-almost alternating* if m crossing changes produce an alternating projection. A link L is *m-almost alternating* if it has an m-almost alternating projection and no $(m-1)$-almost alternating projection. Note that an m-almost alternating link has k-almost alternating projections for all $k \geq m$, which can be easily obtained by repeated Type II Reidemeister moves.

If a link L is m-almost alternating, we will say it has *dealternating number m*, where $m = 0$ corresponds to an alternating link. The dealternating number partitions all knots and links of n crossings into classes depending on how far they are from being alternating. Note that $0 \leq m \leq n/2$.

The dealternating number m of a knot can be thought of in analogy to the unknotting number. The unknotting number can be defined to be the minimum over all projections of the knot of the number of crossing changes necessary in order to turn the projection into a projection of the unknot. Similarly, we can define the dealternating number m to be the minimum over all projections of the knot of the number of crossing changes necessary to turn the projection into an alternating projection.

There have been numerous generalizations of the concept of alternating links, including among others, homogeneous links, pseudo-alternating links, adequate links, augmented alternating links and alternative links. Each of these generalizations was motivated by the desire to extend a particular property known for alternating links to a more general class of links. The category of almost alternating links is distinct from all of these other generalizations.

In Section 2, we prove that a prime almost alternating knot is either a torus knot or a hyperbolic knot (Theorem 2.3 and Corollary 2.6). This generalizes Menasco's proof of the same fact for alternating links [10]. We also demonstrate that the result does not extend to almost alternating links or to 2-almost alternating knots or links.

In Section 3 we show the surprising fact that all but three of the nonalternating knots up through eleven crossings and all but two of the nonalternating links up through ten crossings are almost alternating. We show this by illustrating a direct way which will often determine from Conway's notation for knots and links [2] whether a knot or link has an almost alternating projection. The Conway notation is particularly well adapted to telling directly whether a knot or link is alternating or almost alternating.

In Section 4 we generalize a result which has been proven independently by Kaufmann, Murasugi, and Thistlethwaite [6–8, 12, 14] concerning the span of the bracket polynomial of an alternating link. They prove that for an alternating link L in an n-crossing reduced connected alternating projection the bracket polynomial $\langle L \rangle$ has span equal to $4n$. The major theorem proven in this section is a generalization of this result to the case of links with an m-almost alternating projection. We will define two new conditions for links: *dealternator reduced* and *dealternator connected.*

These are extensions of the necessary conditions for the alternating case. The theorem states that if a link K has an m-almost alternating projection with n crossings which is dealternator connected and dealternator reduced, then $\mathrm{span}(\langle K\rangle) \leqslant 4(n-m-2)$. We are able to infer from this corollaries which are particularly useful in the almost alternating case, when $m=1$.

In Section 5 we discuss characteristics of alternating links which do not generalize to almost alternating links and give conjectures and possible avenues for further research.

This work was begun as an undergraduate thesis at Williams College by David Pesikoff under the direction of Colin Adams. Work was continued by the Knot Theory group of the SMALL Undergraduate Research Project at Williams College under the direction of Colin Adams and consisting of the remaining six authors. Jeff Brock is responsible for much of the results in Section 4. We would like to thank all of the agencies and foundations which provided support for this research.

2. Hyperbolicity

In this section, we generalize results of Menasco from alternating links to almost alternating knots, proving among other results, that a prime almost alternating knot is either a torus knot or a hyperbolic knot. We will assume the notation, definitions and results of [10]. In particular, a properly embedded surface F in $S^3 - L$ is called *pairwise incompressible* if for each disk $D \subseteq S^3$ meeting L transversely in one point, with $D \cap F = \partial D$, there is a disk $D' \subseteq F \cup L$ meeting L transversely in one point with $\partial D = \partial D'$.

Let L be an almost alternating link in an almost alternating projection π. The one crossing which makes the projection nonalternating is called the *dealternator.* Let $F \subseteq S^3 - L$ be a closed pairwise incompressible and incompressible surface. In addition, we assume that $S^3 - L$ is irreducible and L is prime.

As in [10], we place bubbles at each of the crossings. The bubble at the dealternator is denoted by B. All other bubbles at crossings are denoted A.

Following the arguments in [10], we can isotope F to a surface F' such that F' is in standard position with respect to the projection. This means that:

(1) If F' intersects a given bubble, it does so in a set of saddles.

(2) No word $w_{\pm}(C)$ associated to F' is empty.

(3) No loop of $F' \cap S^2_{\mp}$ meets a bubble in more than one arc.

Lemma 2.1. *All words $w_{\pm}(C)$ have the form BA^j with j a positive odd integer.*

Proof. The argument that Menasco utilizes in [10, Lemma 2] is unaffected by the presence of a B bubble and demonstrates that there are no words of the form A^j.

Thus all words contain a B and they can contain at most one B. The number of letters in a word must always be even. □

Lemma 2.2. *Let K be a knot such that $S^3 - K$ contains an incompressible pairwise incompressible surface F. Then every bubble must contain an even number of saddles.*

Proof. The set of all intersection curves in $F \cap S^3_+$ decomposes S^3_+ into distinct regions. These regions can each be colored either black or white so that every pair of adjacent regions are coloured distinctly.

Each particular bubble is crossed by an even number of intersection curves in $F \cap S^3_+$, the number of which is exactly twice the number of saddles in this bubble. Thus if a strand goes under a bubble, it will cross under an even number of curves and will thus come up on the other side of the bubble into a region with the same color as the region which the strand started from.

Hence we can consistently color the entire knot either black or white. Thus, all strands must have received the same color, say black. However, if any bubble had an odd number of crossings, the understrand at that bubble would receive a different color from the overstrand, contradicting the fact this is a knot of one color. □

Theorem 2.3. *If L is an almost alternating knot, and if F in $S^3 - L$ is a closed incompressible surface, then F contains a circle which is isotopic in $S^3 - L$ to a meridian of L.*

Proof. Let L be an almost alternating link containing an incompressible pairwise incompressible surface F. In the course of the proof, we will show that in fact the number of components of the link is greater than one, demonstrating that such a surface could not live in the complement of a knot of this type.

By the work of Menasco, we can isotope F to be in standard position. All of the intersection curves pass exactly once through the B bubble, and hence the intersection curves are all concentric on the projection sphere, with exactly two of them bounding disks containing no other intersection curves.

By Lemma 2.2, the number of saddles per bubble is even. Let t be the largest integer such that 2^t divides the number of saddles in each bubble that contains saddles. We will color the concentric regions between the intersection curves by coloring 2^t concentric regions in a row all black, followed by 2^t concentric regions in a row all white. Then we will repeat the process until all regions have been colored. It does not matter what color we start and end with on the two regions which are disks.

Since a strand which passes under a bubble will pass under $2^{t+1}j$ curves, the strand will have the same color when it exits as when it went under the bubble. Thus each component of the link is consistently colored a single color. Any bubble with exactly $2^t k$ saddles, where 2 does not divide k, will have its overstrand colored differently from its understrand. Hence, this link has at least two components, as we wished to show. □

Corollary 2.4. *If K is a prime almost alternating knot which is not a torus knot, then $S^3 - K$ has a complete hyperbolic structure of finite volume.*

Proof. If a prime knot K is not a torus knot, then it is either hyperbolic or $S^3 - K$ contains an incompressible pairwise incompressible torus by the work of Thurston. By Theorem 2.3, the second possibility cannot occur. □

Corollary 2.5. *An almost alternating knot which is hyperbolic cannot contain a closed embedded totally geodesic surface in its complement.*

Proof. In [11], the authors point out that a closed totally geodesic surface in a hyperbolic manifold must have a fundamental group realized as a group of isometries, none of which are parabolic. Hence, the surface cannot contain a simple closed curve isotopic to a meridian of the knot. Thus the surface will be both incompressible and pairwise incompressible, a possibility which is ruled out by Theorem 2.3. □

Corollary 2.6. *All prime knots of eleven or fewer crossings are torus knots or hyperbolic knots.*

Proof. In the next section, we will show that all but three of the prime knots of eleven or fewer crossings are alternating or almost alternating, and hence either torus knots or hyperbolic knots by [10] and Theorem 2.3. The three exceptions can be seen to be hyperbolic by applying Jeff Week's Hyperbolic Structures Computer Program (see [1] for a description). □

It is tempting to try to extend Theorem 2.3 and Corollary 2.4 to show that prime 2-almost alternating knots are either torus knots or hyperbolic knots. In fact, this is not the case. In Fig. 1, we show the usual projection of a Whitehead double of the trefoil knot and a 2-almost alternating projection of the same knot. This is a satellite knot and hence is neither a torus knot nor a hyperbolic knot.

This last example also demonstrates that the existence of an m-almost alternating projection of a knot which has minimal crossing number among all projections of the knot does not necessarily imply that the knot is m-almost alternating.

Theorem 2.3 also does not extend to almost alternating links. If we form a 3-component link by adding one parallel component to one of the two components in the Whitehead link, we obtain a prime nonsplittable link which is easily seen to be almost alternating, but which is neither a torus link nor a hyperbolic link.

3. Conway notation

In this section we show that all but at most five of the 393 nonalternating knots of eleven or fewer crossings and links of ten or fewer crossings in [2] are almost alternating. We will assume the notation, definitions and results of [2].

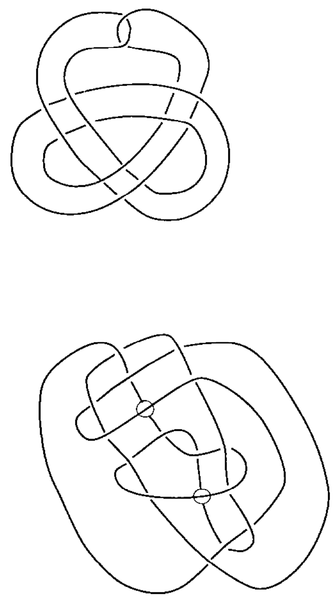


Fig. 1. The Whitehead double of the trefoil is 2-almost alternating.

One can check that any link whose Conway notation contains no negative signs is an alternating link. The presence of negative signs in the notation indicates the existence of almost alternating projections in some cases.

Theorem 3.1. *A link has an almost alternating projection provided either*:

(i) *its Conway notation includes only one negative*;

(ii) *its Conway notation includes two negative signs within parentheses and between periods.*

Proof. First we treat case (i). Using Conway's rules we can move a negative sign through the notation to the end of a parenthetical or to a period. If the last integer in the string after this process is a, changing the $a-$ to $a+$ will change exactly one crossing in the projection [3, p. 331]. The notation for the changed projection has no negative signs and is thus alternating. Therefore, the link has an almost alternating projection.

In the case where the negative sign lies outside and between parentheses, the notation $a-b$ stands for $a\bar{b}$. This reflection of b in the plane is simply a mirror image of b. As a tangle, $\bar{b}$ will remain alternating. The existing strands of $\bar{b}$ will have double over-crossings and double under-crossings. By pulling one strand as in Fig. 2 the resulting projection can be made almost alternating.

Now we deal with case (ii). Within the parentheses the notation can be made to have the form: $a_1, \ldots, a_j--$ where $a_1, \ldots, a_j$ are positive integers. By using Conway's rules we can show that this is equivalent to: $a, \ldots, (a_j)-(2)0$ (using the

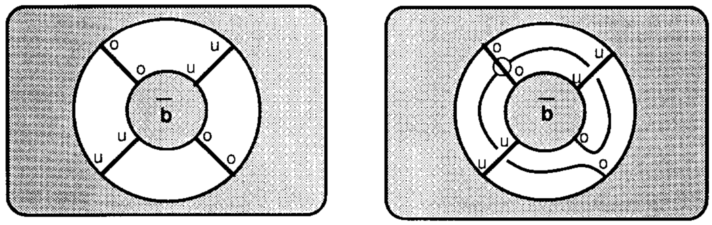


Fig. 2. Double over-crossings are indicated by "o o" and likewise for double under-crossings. Pulling one of the strands yields an almost alternating projection.

convention $a-bc=[a-b]c$). By case (i), we have that the tangle within parentheses has an almost alternating projection. Thus the link has an almost alternating projection. □

This theorem verifies that 368 of the nonalternating links up through ten crossings and the nonalternating knots up through eleven crossings have an almost alternating projection and are thus almost alternating.

We are left with the following knots and links to check.

- nine crossing knots: −20: −20: −20 (9_{49});
- nine crossing links: 2: −20: −20 (9^2_{61});
- ten crossing knots: 3: −20: −20 (10_{162}), −30: −20: −20 (10_{163});
- ten crossing 2-component links: −210: −20: −20, 2. −2. −20.20, **2. −20. −2.20**, 8* −20: −20;
- ten crossing links with three or more components: (2,2−),2,(2,2−),**20.−2.−20.20**, (2,2,2−) (2,2−);
- Nonalternating eleven crossing knots: −22:−20:−20, 22:−20:−20, −211:−20: −20, −40:−20:−20, −310:−20:−20, −2110:−20:−20, −30:−21:−20, −210:−30: −20, −210:−210:−20, 2.−3.−20.20, 2.−21.−20.20, **20.−3.−20.2, 20.−21.−20.2, 9*.−2:.−2**.

We have found almost alternating projections for all of the above nonalternating knots and links except those in boldface. This proves that all but at most five of the nonalternating ten crossing links and eleven crossing knots are almost alternating. There are 2-almost alternating projections of those in boldface.

4. The bracket polynomial

In this section we discuss the bracket polynomial of a link with an m-almost alternating projection. We extend the result that the span of the bracket polynomial for an alternating link with n crossings in a reduced, connected alternating projection equals four times the number of crossings.

We will assume the notation and results for the bracket polynomial given in the appendix of [6]. We will refer to the recursive process of taking A- and B-channel

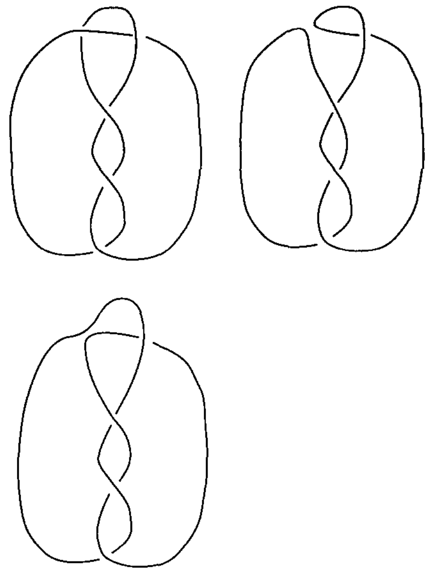


Fig. 3. Polynomial decomposition at a crossing yields two new projections.

splits at a crossing to develop the bracket polynomial of a link as *polynomial decomposition* at that crossing (Fig. 3).

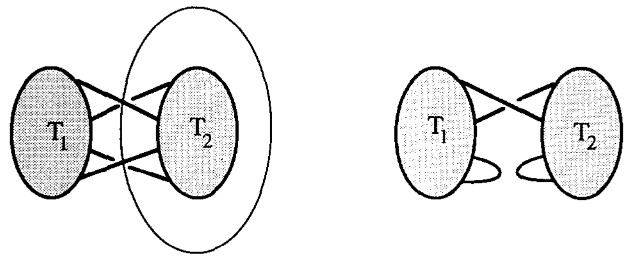
Let K_P be a reduced, connected m-almost alternating projection of a link K. Let $\boldsymbol{D}$ be the set of all 2^m alternating projections obtained by polynomial decomposition at the dealternators of K_P. Let $\boldsymbol{L}_i = \{L \in \boldsymbol{D} \mid L$ has i dealternators with A-channel splits$\}$. In general $|\boldsymbol{L}_i| = \binom{m}{i}$ and $\boldsymbol{D} = \bigcup_i \boldsymbol{L}_i$, $0 \leq i \leq m$. Let the A-channel state of a link L be written S_L^A and the B-channel state S_L^B [6–8].

Definition 4.1. K_P is said to be *dealternator reduced* if for all $L \in \boldsymbol{D}$, the projection L is reduced.

If a closed path π can be drawn through any subset of the dealternators and one other crossing of K_P such that it intersects K_P at no other point, call π a *dealternator reducibility path.* K_P is dealternator reduced if and only if it has no dealternator reducibility path (Fig. 4).

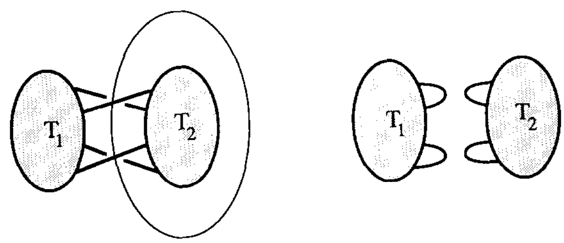
Definition 4.2. K_P is said to be *dealternator connected* if for each $L \in \boldsymbol{D}$, L is connected.

If a closed path π can be drawn through any subset of the dealternators of K_P such that π intersects K_P at dealternators and nowhere else, call this path a *dealternator severing path.* K_P is dealternator connected if and only if K_P has no dealternator severing path (Fig. 5).



(a) (b)

Fig. 4. (a) K_P has a dealternator reducibility path. (b) There exists $L \in \boldsymbol{D}$ such that L is not reduced.



(a) (b)

Fig. 5. (a) K has a dealternator severing path. (b) There exists $L \in \boldsymbol{D}$ such that L is disconnected.

Note that if an m-almost alternating projection is not connected and reduced, it cannot be dealternator connected and reduced. We are assuming, however, that K_P is reduced and connected.

The following lemma gives us the remaining tools needed to prove the theorem.

Lemma 4.3. *If K_P is dealternator connected and dealternator reduced, then for links $L, M \in \boldsymbol{L}_i$ and L' in $\boldsymbol{L}_{i+1}$, the following hold:*

(i) $\text{maxdeg}\langle L\rangle = \text{maxdeg}\langle M\rangle$ *and* $\text{mindeg}\langle L\rangle = \text{mindeg}\langle M\rangle$.

(ii) $\text{maxdeg}\langle L'\rangle = \text{maxdeg}\langle L\rangle - 2$ *and* $\text{mindeg}\langle L'\rangle = \text{mindeg}\langle L\rangle - 2$.

Proof. We prove this using the A- and B-channel states of projections $L \in \boldsymbol{D}$. We will begin by showing that for elements L and M of a given $\boldsymbol{L}_i$, the A-channel states S_L^A and S_M^A and the B-channel states S_L^B and S_M^B are such that $|S_L^A| = |S_M^A|$ and $|S_L^B| = |S_M^B|$. We must also show that for $L' \in \boldsymbol{L}_{i+1}$, $|S_{L'}^A| = |S_L^A| - 1$. These facts depend explicitly on whether K_P is dealternator connected and reduced. We argue by induction on i, the number of A-channel splits at the dealternators in a polynomial decomposition at the dealternators of K_P.

The case $i = 0$ refers to a set with one element $\boldsymbol{L}_0$ and is trivially true.

Assume that for $L, M \in \boldsymbol{L}_j$, $0 \leq j < m$, $|S_L^A| = |S_M^A|$. Now consider $L' \in \boldsymbol{L}_{j+1}$. L' has one more A-channel split at the dealternators than elements of $\boldsymbol{L}_j$. Each L' can be obtained from an element L in $\boldsymbol{L}_j$ by changing the split at one dealternator from a B-channel to an A-channel split.

The change in the split will either join two distinct components in S_L^A, resulting in $|S_{L'}^A| = |S_L^A| - 1$ or it will join a component to itself, creating a new component and resulting in $|S_{L'}^A| = |S_L^A| + 1$. We will show that the second possibility never occurs.

Note that the A-channel state of L_0 reflects the state of K_P with the greatest number of components. This follows from the fact that the A-channel state of L_0 is identical to the A-channel state of the alternating link obtained by changing all of the dealternators of K_P.

We obtain L' from L_0 by a sequence of dealternator switches. Originally, we can think of $S_{L_0}^A$ as a set of white A-islands in a shaded B-sea. Switching dealternators forms white bridges from island to island. A shaded lake is formed when a bridge joins a white region back to itself. This could occur only when there is a dealternator severing path π_0 as in Fig. 6. Since there are no such paths, each dealternator switch must join two distinct components and therefore, it must be the case that $|S_{L'}^A| = |S_L^A| - 1$.

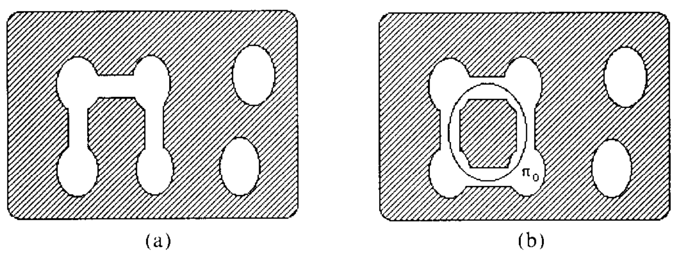


Fig. 6. (a) Dealternator switches form bridges between islands. (b) Shaded lakes occur when there is a dealternating severing path.

The induction is thus complete, yielding the fact that for any $0 \leq i \leq m$, and for all $L, M \in \boldsymbol{L}_i$, $|S_L^A| = |S_M^A|$. Furthermore, any $L' \in \boldsymbol{L}_{j+1}$ can be obtained from $L \in \boldsymbol{L}_j$ for some L so $|S_{L'}^A| = |S_L^A| - 1$.

We know that the maximum degree of L, a connected reduced alternating projection, is given by the term [7]

$$A^V d^{(|S|-1)}$$

where S is the A-channel state of L, V is the number of crossings in L and $d = -A^2 - A^{-2}$. Hence,

$$\operatorname{maxdeg}\langle L\rangle = \operatorname{maxdeg}\langle M\rangle.$$

Furthermore,

$$\operatorname{maxdeg}\langle L'\rangle = \operatorname{maxdeg}\langle L\rangle - 2,$$

where $L \in \boldsymbol{L}_i$ and $L' \in \boldsymbol{L}_{i+1}$.

The arguments that $\operatorname{mindeg}\langle L\rangle = \operatorname{mindeg}\langle M\rangle$ and that $\operatorname{mindeg}\langle L'\rangle = \operatorname{mindeg}\langle L\rangle - 2$ use exactly similar induction arguments concerning the changes from B-channel splits to A-channel splits at dealternators of K_P.

Because the number of components in the A-channel states of all links in $\boldsymbol{L}_i$ are the same, the coefficients of the maximum degree terms of the elements of $\boldsymbol{L}_i$ are equal for a given i and are either $+1$ or -1. Furthermore elements of $\boldsymbol{L}_{i+1}$ will have the opposite sign for the coefficients of their maximum degree terms since $|S_{L'}^A| = |S_L^A| - 1$. □

Theorem 4.4. *If a link K has n crossings in a dealternator reduced and dealternator connected m-almost alternating projection K_P, then* $\operatorname{span}(\langle K\rangle) \leqslant 4(n-m-2)$.

Proof. We begin by giving an explicit expression for the bracket polynomial of K_P using polynomial decomposition at the dealternators of K_P.

Let

$$H_i = \sum_{L \in \boldsymbol{L}_i} \langle L\rangle. \tag{1}$$

We can then rewrite the bracket polynomial of K as

$$\langle K\rangle = \sum_{i=0}^{m} A^{2i-m} H_i. \tag{2}$$

This sum represents the mth stage in the construction of the bracket polynomial of K by splitting at m crossings.

We know that for all $L, M \in \boldsymbol{L}_i$, $\operatorname{maxdeg}\langle L\rangle = \operatorname{maxdeg}\langle M\rangle$ and $\operatorname{mindeg}\langle L\rangle = \operatorname{mindeg}\langle M\rangle$ so

$$\operatorname{maxdeg}(H_i) = \operatorname{maxdeg}\langle L\rangle,$$

$$\operatorname{mindeg}(H_i) = \operatorname{mindeg}\langle L\rangle,$$

because H_i is simply a sum over the elements of $\boldsymbol{L}_i$.

Furthermore since for $L \in \boldsymbol{L}_i$ and $L' \in \boldsymbol{L}_{i+1}$, $\operatorname{maxdeg}\langle L'\rangle = \operatorname{maxdeg}\langle L\rangle - 2$ and $\operatorname{mindeg}\langle L'\rangle = \operatorname{mindeg}\langle L\rangle - 2$ we know:

$$\operatorname{maxdeg}(H_{i+1}) = \operatorname{maxdeg}(H_i) - 2,$$

$$\operatorname{mindeg}(H_{i+1}) = \operatorname{mindeg}(H_i) - 2.$$

By expanding (2) we see

$$\langle K\rangle = A^m H_m + A^{m-2} H_{m-1} + \cdots + A^{-m} H_0. \tag{3}$$

Thus,

$$\operatorname{maxdeg}(A^m H_m) = m + \operatorname{maxdeg}(H_m),$$

$$\operatorname{maxdeg}(A^{m-2} H_{m-1}) = m - 2 + \operatorname{maxdeg}(H_{m-1}) = m + \operatorname{maxdeg}(H_m),$$

$$\vdots$$

$$\operatorname{maxdeg}(A^{-m} H_0) = -m + \operatorname{maxdeg}(H_0) = m + \operatorname{maxdeg}(H_m).$$

So in general,

$$\operatorname{maxdeg}(A^{2i-m} H_i) = \operatorname{maxdeg}(A^{2j-m} H_j) \quad \text{for } 0 \leqslant i, j \leqslant m.$$

By an exactly similar argument,

$$\text{mindeg}(A^{2i-m}H_i) = \text{mindeg}(A^{2j-m}H_j) \quad \text{for } 0 \leq i, j \leq m.$$

By dealternator reducibility and dealternator connectedness, for all i, $L \in \boldsymbol{L}_i$ implies L is a reduced, connected alternating projection. Hence, $\text{span}(\langle L \rangle) = 4(n-m)$ by [6-8, 12, 14]. This yields $\text{span}(H_i) = \text{span}(\langle L \rangle)$, $L \in \boldsymbol{L}_i$, and $\text{span}(\langle K \rangle) \leq \text{span}(H_i)$.

Therefore, span $(\langle K \rangle) \leq 4(n-m)$.

We have seen that coefficients of the maximum and minimum degree terms of $\boldsymbol{L}_i$ alternate sign with i and are all equal to ± 1.

As before $|\boldsymbol{L}_i| = \binom{m}{i}$. We make note of the formula:

$$\sum_{i=1}^{m} (-1)^i \binom{m}{i} = 0. \tag{4}$$

Because $|\boldsymbol{L}_i| = \binom{m}{i}$, we know the coefficient of the maximum degree term of H_i is $\pm\binom{m}{i}$. From (4), we may conclude that the maximum degree terms of K_P sum to 0 and by a similar argument, the minimum degree terms of K_P sum to 0.

Therefore, $\text{span}(\langle K \rangle) \leq 4(n-m-2)$. □

Corollary 4.5. *If an almost alternating link L has n crossings in an almost alternating projection L_P, then* $\text{span}(\langle L \rangle) \leq 4(n-3)$.

Proof. If the projection L_P is not reduced, we can replace it with a reduced almost alternating projection of fewer crossings. Since L_P is almost alternating, it must in fact be dealternator connected. If it were not dealternator connected, the dealternator would be the only connection between two alternating tangles. Flipping one of the tangles will untwist the dealternator and yield an alternating projection, contradicting that part of the definition of almost alternating which excludes alternating links.

Suppose now that L_P is not dealternator reduced. It must have a dealternator reducibility path. If this path passes through the A-channel of the dealternator, it must pass through the B-channel of the other crossing and vice versa. Hence, it must take the form of Fig. 7. By untwisting the dealternator we can see that L must be an alternating link, again contradicting the assumption that L is almost alternating.

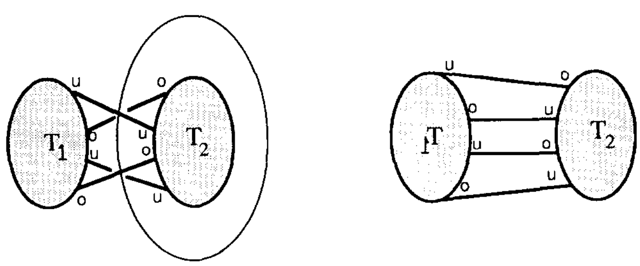


Fig. 7. (a) L_P has a dealternator reducibility path. (b) Untwisting the dealternator shows L is alternating.

Thus L_P must be dealternator connected and dealternator reduced. Applying the theorem gives span$(\langle L\rangle) \leqslant 4(n-3)$. □

Corollary 4.6. *If L is an alternating link in an almost alternating projection of n crossings and if* span$(\langle L\rangle) = 4(n-3)$, *then the following hold*:

(i) *n is the least number of crossings in any almost alternating projection of the link.*

(ii) *The crossing number of L is either n, $n-1$ or $n-2$.*

Proof. Suppose there is an almost alternating projection with fewer than n crossings. Corollary 4.5 would then imply that span$(\langle L\rangle) < 4(n-3)$, contradicting our hypothesis.

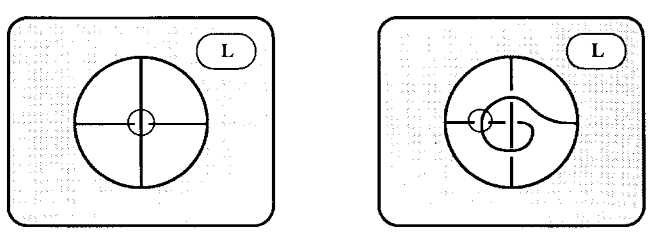
To see (ii), we use the fact (due to Murasugi and Thistlethwaite, see [7]) that a link in a nonalternating projection of k crossings has span$(\langle L\rangle) < 4k$. Thus it must be the case $4(n-3) < 4k$ where k can be taken to be the minimal crossing number of this nonalternating link. Thus $k \leqslant n < k+3$, yielding the result. □

In fact, span$(\langle K\rangle)$ does equal $4(n-3)$ for all but three of the almost alternating knots of nine or fewer crossings.

Remark. We conjecture that if an m-almost alternating projection does have some number of dealternator reducibility paths, then span$(\langle K\rangle) \leqslant 4(n-m-1)$. This bound is higher but still useful in some cases. It has also been realized by certain projections with dealternator reducibility paths.

5. Conclusions

Not all results about alternating knots and links can be generalized to the class of almost alternating knots. The result of Crowell [4] (see also Gabai [5]) that Seifert's algorithm applied to an alternating link always yields a Seifert surface of minimal genus cannot be extended to almost alternating links. For any particular almost alternating link, one can find projections for which Seifert's algorithm will yield Seifert surfaces of arbitrarily high genus. Starting with a given almost alternating projection, the under-strand at the dealternator can be isotoped as in Fig. 8, resulting



(a) (b)

Fig. 8. (a) The dealternator crossing of the projection L. (b) L' is a new almost alternative projection.

in a projection which is still almost alternating, but for which the corresponding surface obtained by Seifert's algorithm has genus one greater than the genus of the surface obtained from the original projection. Iterating this process will generate surfaces of arbitrarily high genus.

There have been a few other generalizations of alternating links. In particular, there is a hierarchy among the classes of alternate, homogeneous, and pseudo-alternating, where each class contains the former. It is important to note that the class of almost alternating links does not fit into this hierarchy and is indeed a class of independent importance. The appendix to [3] shows that not all of the nonalternating knots of ten or fewer crossings are homogeneous, while we have shown these knots to be almost alternating. Also, Cromwell states that all torus links are homogeneous, while $(2, q)$ torus knots are not almost alternating.

There are many relevant open questions relating to almost alternating knots and links.

(1) When is an almost alternating link splittable or prime? As in the case of alternating links, we would like to be able to determine from the almost alternating projection whether a link is splittable or prime.

(2) What torus knots are almost alternating? We conjecture that the $(3, 4)$ and $(3, 5)$ torus knots are the only ones. In general, what is the almost alternating number m of a (p, q) torus knot?

(3) What almost alternating projections are projections of the unknot?

(4) How can it be shown that a knot other than a satellite knot is at least 2-almost alternating? We do not know if the three eleven crossing knots which have not been seen to be almost alternating are in fact not almost alternating. These three knots do have 2-almost alternating projections.

(5) What percentage of knots of n crossings are m-almost alternating for each integer m from 0 to $n/2$? Does some m depending on n predominate as n goes to ∞?

(6) Show that there exist m-almost alternating projections which realize the minimal crossing number of that link. Intuitively, dealternators which are spread far apart in a large alternating grid should leave the crossing number the same as for the corresponding alternating link.

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